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WAR AND PEACE: RECOVERING THE MARKET'S PROBABILITY DISTRIBUTION OF CRUDE OIL FUTURES PRICES DURING THE GULF CRISIS

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ABSTRACT

This paper investigates the market's expectations for oil prices during the Persian Gulf crisis. To do so a general method for using options markets to recover the implied distribution for futures prices is developed. The method applies to a wide class of distributions. In particular, it is not limited to those distributions arising from diffusion or jump-diffusion processes.

War and Peace:
Recovering the Market's Probability Distribution of Crude Oil Futures Prices During the Gulf Crisis

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This paper investigates the market's expectations for oil prices during the Persian Gulf crisis. To do so a general method for using options markets to recover the implied distribution for futures prices is developed. The method applies to a wide class of distributions. In particular, it is not limited to those distributions arising from diffusion or jump-diffusion processes.[2]

During the Persian Gulf crisis there was wide interest in the likely movement in oil prices in the event of a substantial disruption in oil supplies, and the perceived probability of such a disruption. Given their information content, futures and options prices are a natural source for insight on these issues. Analysts typically use Black's

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1. The authors are staff economists in the Division of International Finance. This paper represents the views of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or other members of its staff. We would like to especially thank David Bates, Jon Faust, and Ed Green for their extensive comments. We also thank Allan Brunner, Neil Ericsson, Jeff Fuhrer, Ludger Hentschel, and George Moore, as well as participants in the International Finance Monday Workshop. Elizabeth Vrankovich and Dara Akbarian provided valuable research assistance.

2. Most option pricing models follow Black and Scholes (1973) and assume that the price of the underlying asset follows a geometric Brownian process described as follows: $dS/S = \mu \cdot dt + \sigma \cdot dZ$ where S is the price of the asset, μ is a deterministic drift parameter and dZ is i.i.d. normal with mean zero. This generates a lognormal distribution for the asset price.

Under a jump diffusion process, the motion of the underlying asset price is described as follows: $dS/S = (\mu - \lambda \cdot \bar{\kappa}) \cdot dt + \sigma \cdot dZ + \kappa \cdot dq$ where μ and dZ are the same as above; $\text{Prob}(dq = 1) = \lambda \cdot dt$; and $(1+\kappa)$ is lognormally distributed. The asset follows geometric Brownian motion until a moment where $dq = 1$. At that moment the asset price jumps by the random amount $\kappa \cdot S$. After the jump the process resumes Brownian motion from the new level until the next jump.

(1976) option pricing model to "back-out" expected future prices and/or volatilities (e.g. Overdahl and Matthews (1988))[3]. However, the lognormality assumption of Black's model can render it ill-suited for periods such as the crisis when several distinct and identifiable regimes were possible. For example, market participants may have perceived a tri-modal distribution for future oil prices, where the three peaks correspond to a return to normalcy, the continuation of unsettled conditions, and a large disruption in Saudi Arabian production in the event of war.

To investigate unsettled times in other markets, methods have been developed to estimate jump diffusion processes with options prices (see Bates (1990) for an application to the stock market prior to October 1987). The jump diffusion process has two advantages. First, combined with a fairly standard specification for the utility function of the aggregate investor, it yields tractable formulae for pricing the nondiversifiable risk associated with the jumps. Second, it yields manageable formulae for close approximations to the value of American options--the type most commonly traded. However, the jump diffusion process has the drawback of imposing a great deal of structure on the stochastic process underlying the futures prices.

The method developed here was designed to apply to a wide class of distributions and to impose minimal structure on the underlying stochastic process for futures prices. In exchange for this generality we give up some scope in terms of where the method can be applied. In particular, our method is only directly applicable to those markets where

3. Black (1976) is a straightforward modification of Black and Scholes (1973), to account for the zero cost of carry for futures contracts.

changes in the futures price are uncorrelated with changes in the market portfolio, i.e., where the market beta is zero.[4] Our method also gives up some accuracy in the formulae for approximating the American option prices. Our method is based on finding bounds for the option prices given a distribution for futures prices. We find that for the estimated distributions the bounds are quite close together and argue that the approximations based on them will be quite accurate.

The rest of the paper is organized as follows: Section 2 derives bounds for the price of an American option on a futures contract under very general assumptions concerning the distribution of the futures price at the option's expiration. The third section illustrates how the bounds can be used with option prices to recover the distribution for futures prices. Section 4 discusses the particulars of an application to the oil market, while Section 5 presents the results of that application. A summary and concluding remarks are found in Section 6.

II. Bounds on American Options' Prices

With European style options the relationship between the distribution of futures prices and the option price is very direct. For calls (puts), the value of the option is simply the value of the portion of the distribution above (below) the strike discounted back to the present using an appropriate interest rate. For American style options the

4. The method developed here depends on the risk neutral valuation of the options. As such it could be applied to markets with a positive beta, provided the underlying stochastic process is smooth enough to permit the dynamic replication of the option. In this case, however, the estimated distributions represent the risk-neutral, or martingale equivalent, probabilities rather than the true, or actuarial, probabilities. For a discussion of this issue see Cox and Ross (1976) and the references therin.

relationship between the distribution and the option price is less direct owing to the premium associated with the right to exercise before expiration. In general the option's value will depend on the entire stochastic process for futures prices, not just the distribution for futures prices at the option's expiration. To deal with this early exercise premium we develop bounds for the maximum and minimum value of an option given that the futures price is taken from a particular distribution at the option's expiration. That is, for all stochastic processes that imply a given distribution for the futures at the option's expiration, there are bounds for the option's value which can be expressed in terms of that distribution alone. In the estimation routine these bounds are weighted to arrive at a predicted value for the option.

The bounds are derived in detail below, but a short preview will clarify where we are headed. The lower bound corresponds to a stochastic process whereby no uncertainty is resolved until the last day of the option's life (i.e. futures prices remain constant until the day of expiration). The upper bound corresponds to a stochastic process whereby all uncertainty in the futures price is resolved tomorrow (i.e. any price move is made tomorrow and between tomorrow and expiry prices do not change further). The value of the option under these two processes can be expressed in terms of the distribution at the end point alone.

In the next two subsections we introduce some notation and review the standard option pricing problem. The last subsection presents formulae for the maximal and minimal values with some intuition. The proofs are relegated to the appendix. The derivations in this section are in terms of futures prices that follow a discrete state Markov process. In the actual implementation we switch to a continuous distribution formulation.

II.1 Notation

Time is indexed by days from the option's expiration. Today is denoted by T, tomorrow by T-1, etc., until date 0 which corresponds to the option's expiry. All dating and references to expiry are relative to the option's expiration--not the futures'. The futures price t periods prior to expiry is a discrete random variable denoted by F^t which can take on a finite number of values indexed by m = {1,2,..M}[5]. Let $\underline{F}^t$ denote the Mx1 vector of possible realizations of the futures price in period t. This support is the same for all periods and the time superscript will be suppressed. A particular realization for the futures price at time t is denoted F^t_m.

The process by which futures prices evolve is described by a length T sequence of MxM transition probability matrices $\{\Psi^T, \Psi^{T-1}, \ldots, \Psi^1\}$ where for t < T a typical element of Ψ^t is $\psi^t_{ij} = \text{Prob}[F^{t-1} = F^{t-1}_j \mid F^t = F^t_i]$. In period T, the period T futures price is known so Ψ^T collapses to a single row $(\psi^T_{m.})$ representing the probability of moving from today's futures price to any point on the support tomorrow. We will denote the product of all elements in the sequence of transition matrices by $\Gamma^T \equiv \psi^T_{m.} \cdot \prod_{t=T,..,2} \Psi^{t-1}$. Γ^T is the 1xM vector product of the Markov transition matrices. As such, it is the period T distribution of futures prices at the option's expiration given that the futures price in period T is F^T_m.

5. In fact futures prices can only take on discrete values since they are quoted to the penny. Considering only a finite number of values is not a constraint in practice.

This is the distribution of interest that we estimate using the options prices.

II.2 Value of American Options on Futures

We now derive the price of an American option in terms of an arbitrary sequence of transition probabilities. The derivation assumes that changes in the price of oil futures are uncorrelated with changes in the market portfolio, i.e., that oil price risk is diversifiable.[6] Under this assumption, oil futures and the options on them will be priced as though traders were risk neutral using the true probabilities. We also assume that the cost of carry for futures is zero.[7]

With no cost of carry, futures prices will martingale, i.e., $F_m^T = E^T[F^{T-1}]$, where E^T denotes expectations taken at time T. By iterated expectations

6. To confirm that the market beta on crude oil futures is near zero, we ran the following regressions: $\Delta F = \alpha + \beta \cdot (R_m - R_f)$ and $\Delta F = \alpha + \beta \cdot R_m$, where ΔF is the weekly dollar change in the price of the nearby crude oil futures contract, R_m is the weekly gross rate of return on the S&P 500 index, and R_f is the gross rate of return on a T-bill maturing in one week. The results for the two specifications were qualitatively the same. Between June 1983 and July 1990 and between February 1991 and December 1991 the estimated betas were insignificantly different from zero and the corrected R squared of the regressions were near zero or negative. Within the crisis period (August 1990 and January 1991) there were five weeks when large declines in oil prices were accompanied by large increases in the S&P index and four weeks when large increases in oil prices were accompanied by large declines in the index. Including these nine observations in the sample results in a small (negative) though statistically significant beta. Including the crisis period raises the corrected R squared, though it remained below .06. For a discussion of this specification see Black (1976) or Dusak (1973).

French (1990) obtained similar results for the pre-crisis period. He estimated the covariance between daily changes in West Texas Intermediate spot prices and the S&P 500 between January 1984 and December 1988 and found it nearly zero.

7. For large traders, margin requirements can be met with liquid, interest bearing securities such as T-bills.

$$(1) \quad F_m^T = E^T[F^0] = \Gamma^T \cdot \underline{F} = \psi_{m.}^T \cdot \prod_{t=T,\ldots,2} \Psi^{t-1} \cdot \underline{F} \ .$$

To value the American call option, we begin with its value at expiry and work backwards (using notation similar to Smith (1979)). At expiry, time 0, the call option is worth

$$(2) \quad C[F_m^0, 0, X] = \mathrm{Max}[F_m^0 - X, 0] \ ,$$

where X is the strike price of the option.

One day prior to expiry, the option can be exercised for $F_m^1 - X$ or it can be held until expiry. If held, it is worth its expected value at time 0 discounted back to period 1 or $\rho^1 \cdot E^1[C[F^0, 0, X]]$ where $\rho^1 \equiv 1/(1+r^1)$ is the one period discount factor given r^1 is the one day interest rate at period 1. Given this choice, the value of the option is

$$(3) \quad C[F_m^1, 1, X] = \mathrm{Max}[F_m^1 - X, \ \rho^1 \cdot E^1[C[F^0, 0, X]]]$$

where the futures price is known to be F_m^1.

Similarly, two days before expiry the value of the option is given by

$$(4) \quad C[F_m^2, 2, X] = \mathrm{Max}[F_m^2 - X, \ \rho^2 \cdot E^2[C[F^1, 1, X]]].$$

Substituting from (3) yields

$$(5) \quad C[F_m^2, 2, X] = \mathrm{Max}[F_m^2 - X, \ \rho^2 \cdot E^2[\mathrm{Max}[F^1 - X, \ \rho^1 \cdot E^1[C[F^0, 0, X]]]]].$$

Similar substitutions give the value of the option an arbitrary number of days prior to expiry. Let $C[\underline{F}^2, 2, X]$ denote the M vector of option values--one for each possible realization of the futures price.

In terms of the stochastic processes discussed above, for a particular realization of the futures price in period t, F_m^t, the value of the option is given by the following recursive relation:

$$(6)\quad C[F_m^t, t, X] = \text{Max}[F_m^t - X,\ \rho^t \cdot \psi_{m.}^t \cdot C[\underline{F}^{t-1}, t-1, X]], \qquad \forall t > 0;$$

$$C[F_m^t, t, X] = \text{Max}[F_m^t - X,\ 0], \qquad t = 0.$$

From (1), we can replace F_m^t with $\psi_{m.}^t \cdot \prod_{j=t,..,2} \Psi^{j-1} \cdot \underline{F}$ which yields an expression for the option's value in terms of the interest rates, the strike price, and the sequence of Ψs alone.

While we might like to estimate all of the elements of the Ψs freely, this is clearly not feasible given the information in the market. An approach used by Bates (1990) and others is to specify a stochastic process which generates all of the Ψs from relatively few parameters. The approach we take is somewhat different and focuses on the Γ alone. Our approach begins with upper and lower bounds on the option's price.

II.3 Bounds on the Option Price

To compute the bounds, we start with the following thought experiment: Suppose we were in period T and knew the true Γ^T, but did not know the sequence of Ψs that generated it. What can we say about the option's price? We find that given a Γ^T there is a sequence of Ψs, that yields a value for the option lower than any other sequence of Ψs whose product is

Γ^T. Similarly there is a sequence of Ψs that yields a value for the option higher than any other sequence whose product is Γ^T. The value of the option under these sequences are therefore bounds for the option price consistent with Γ^T.

The derivations of the minimal and maximal values are relegated to the Appendix. Below we give the formulae and intuition for the bounds, which are straightforward. The sequence for the minimal value corresponds to a stochastic process whereby the futures price does not change between period T and period 1. Between period 1 and period 0 it moves with probabilities given by Γ^T. To see why this sequence generates a minimum value consistent with Γ^T, note that the option can be exercised in period T to yield $F_m^T - X$. If it is not exercised in period T, then it will not be exercised until period 0. Under the assumed sequence, the futures price does not move between period T and period 1 so waiting to exercise for a fixed futures price just delays the receipt of a fixed revenue, which is suboptimal since revenues are discounted in time. The option may be held until period 0, however, because between period 1 and period 0 the futures price does move--with transition probabilities given by Γ^T.

In period 0 the option is worth $\text{Max}[F_m^0 - X, 0]$. Its expected value at time T, if exercised in period 0, is $\Gamma^T \cdot \text{Max}[\underline{F} - X, 0]$ which in terms of period T dollars is $\bar{\rho}^T \cdot \Gamma^T \cdot \text{Max}[\underline{F} - X, 0]$, where $\bar{\rho}^T \equiv \prod_{t=T,..,1} \rho^t$.

Therefore, to value the option under this sequence we need only compare two expectations, each of which uses only Γ^T. The option's value under this sequence is the following:

$$(7) \quad \underset{\smile}{C}[\Gamma^T; T, X, \bar{\rho}^T] = \text{Max}[\Gamma^T \cdot \underline{F} - X,\ \bar{\rho}^T \cdot \Gamma^T \cdot \text{Max}[\underline{F} - X, 0]].$$

In effect this process has removed any premium from the possibility of early exercise between periods T-1 and period 1. We note that this is the value of a European option with the added value of being able to exercise today.

The intuition behind the process that yields the maximal value is similar. Suppose that the futures price will move between periods T and T-1 and not move again between periods T-1 and period 0. For those futures prices for which it is worth exercising after period T-1, it is also worth exercising in period T-1. Moreover, since revenues are discounted it is always better to exercise earlier than later if revenues are the same. Thus, since exercising in period T-1 means the revenues are discounted least, the option will never be exercised after T-1. Therefore, if the option is exercised it will be exercised in period T or period T-1, but not thereafter.

For such a process, the value of exercising the option both today and tomorrow can be written in terms of Γ^T. Hence the maximum value of the option can be expressed in terms of the Γ^T:

$$(8) \quad \overset{>}{C}[\Gamma^T;T,X,\rho] = \mathrm{Max}[\Gamma^T\cdot \underline{F} - X,\ \rho^T\cdot\Gamma^T\cdot \mathrm{Max}[\underline{F} - X,\ 0]].$$

American put options are valued analogously with minimal and maximal values in terms of Γ^T given by

$$(9) \quad \overset{<}{P}[\Gamma^T;T,X\ \bar{\rho}^T] = \mathrm{Max}[X - \Gamma^T\cdot \underline{F},\ \bar{\rho}^T\cdot\Gamma^T\cdot \mathrm{Max}[X - \underline{F},\ 0]] \quad \text{and}$$

$$(10) \quad \overset{>}{P}[\Gamma^T;T,X,\rho^T] = \mathrm{Max}[X - \Gamma^T\cdot \underline{F},\ \rho^T\cdot\Gamma^T\cdot \mathrm{Max}[X - \underline{F},\ 0]].$$

Comparing (9) with (10), or (7) with (8), we note that the maximal and minimal values differ only by the discount factor used in the second item in the outside Max[]. In (10), $\Gamma^T \cdot$ Max[X - $\underline{F}$, 0] is discounted by the one period factor ρ^T where in (9) the same sum is discounted by the T period discount factor $\bar{\rho}^T$. For reasonable values of the interest rates, say 7%, the bounds on an option with six months to expiry will differ by at most about 3-1/2%. For options deep in the money the first item in the Max[] of (9) will be larger than the second and the difference between the bounds will be less than this 3-1/2%.

III. Recovering the Distribution

Equations (7) - (10) give bounds for American option prices T days before expiration in terms of the interest rates, strike prices, and the period T distribution for futures prices in period 0. To recover this distribution from actual option prices we need to clarify exactly what information the option prices contain, impose some structure on the distribution, and construct a point estimate for the option's value from the bounds.

III.1 Weighting Scheme

To apply standard estimation techniques requires a point estimate for the option price conditional on the estimated distribution. To generate such an estimate we weight the upper and lower bounds computed above. There are many possible weighting schemes. To find a reasonable one, we compared the true value of an American option with its bounds for several known processes. Where the option's value lay between the bounds depended on how far the option was out of the money and on how quickly uncertainty about the futures price was resolved. Based on these comparisons, we chose to use two weights. The first weight, w1, is used for all call options that are in the money and for all put options that are out of the money ($F_m^T \geq X$). The second weight, w2, is used for the calls that are out of the money and the puts that are in the money ($F_m^T < X$).

In terms of the above, the estimated options prices are given as follows where estimated parameters are denoted by ^:

$$(11)\quad \hat{C} = \hat{w}1\cdot\overset{>}{C}[\hat{\Gamma}^T;.] + (1-\hat{w}1)\cdot\overset{<}{C}[\hat{\Gamma}^T;.] \qquad \text{if} \quad \hat{\Gamma}^T\cdot\,\underline{F} \geq X$$

$$(12)\quad \hat{C} = \hat{w}2\cdot\overset{>}{C}[\hat{\Gamma}^T;.] + (1-\hat{w}2)\cdot\overset{<}{C}[\hat{\Gamma}^T;.] \qquad \text{if} \quad \hat{\Gamma}^T\cdot\,\underline{F} < X$$

$$(13)\quad \hat{P} = \hat{w}1\cdot\overset{>}{P}[\hat{\Gamma}^T;.] + (1-\hat{w}1)\cdot\overset{<}{P}[\hat{\Gamma}^T;.] \qquad \text{if} \quad \hat{\Gamma}^T\cdot\,\underline{F} < X$$

$$(14)\quad \hat{P} = \hat{w}2\cdot\overset{>}{P}[\hat{\Gamma}^T;.] + (1-\hat{w}2)\cdot\overset{<}{P}[\hat{\Gamma}^T;.] \qquad \text{if} \quad \hat{\Gamma}^T\cdot\,\underline{F} \geq X$$

In the previous section, we claimed that for reasonable discount factors, the bounds are quite close, so a weighted average of them is a good approximation to the option's value. Chart One gives a feel for how close the bounds are using the estimated distribution (discussed below) for a typical day. The top two panels plot the bounds for calls and puts for various strike prices (solid lines for upper bounds and dashed lines for lower bounds). We note that the lines are almost indistinguishable. The bottom two panels plot the difference between the bounds across strikes.

For the day plotted, the options had 38 days to expiration and the relevant T-bill rate was about 7 percent. Thus the discount factors in the formulae for the bounds differed by about 0.8 percent. On this day the futures price was about \$29, so none of the puts was more than \$5 in the money and the difference between the bounds for each put was the full 0.8 percent of the option price. For the calls, the story is the same for those with strikes above \$20. For those with strikes below \$20, the first item in the expression for the lower bound is greater than the second, and the bounds differ by less than 0.8 percent of the option price.

III.2 Data Limitations vis-a-vis the Distribution

It is clear that the logic behind the bounds does not depend in any crucial way on the assumption of a discrete distribution. The bounds can be used for an arbitrary distribution by making the following substitutions:

$\Gamma^T \cdot \underline{F}$ becomes $E^T[F^0]$;

$\Gamma^T \cdot \mathrm{Max}[\underline{F} - X, 0]$ becomes $(E^T[F^0|F^0 \geq X] - X) \cdot Pr^T[F^0 \geq X]$; and

$\Gamma^T \cdot \mathrm{Max}[X - \underline{F}, 0]$ becomes $(X - E^T[F^0|F^0 \leq X]) \cdot Pr^T[F^0 \leq X]$.

With these substitutions and (11)-(14) the actual option prices can be written in terms of estimated conditional expectations, estimated probabilities, and an error term as follows:

$$(15)\quad C[X] = \hat{wi} \cdot \mathrm{Max}[\hat{E}^T[F^0] - X,\ \rho^T \cdot (\hat{E}^T[F^0|F^0 \geq X] - X) \cdot \hat{Pr}^T[F^0 \geq X]] + (1-\hat{wi}) \cdot \mathrm{Max}[\hat{E}^T[F^0] - X,\ \bar{\rho}^T \cdot (\hat{E}^T[F^0|F^0 \geq X] - X) \cdot \hat{Pr}^T[F^0 \geq X]] + \epsilon$$

where $i = 1$ if $\hat{E}^T[F^0] \geq X$ and $i = 2$ otherwise

$$(16)\quad P[X] = \hat{wi} \cdot \mathrm{Max}[X - \hat{E}^T[F^0],\ \rho^T \cdot (X - \hat{E}^T[F^0|F^0 \leq X]) \cdot \hat{Pr}^T[F^0 \leq X]] +$$

$$(1-\hat{w}i)\cdot \mathrm{Max}[X - \hat{E}^T[F^0],\ \bar{\rho}^T\cdot(X - \hat{E}^T[F^0|F^0 \leq X])\cdot \hat{Pr}^T[F^0 \leq X]] + \epsilon$$

where $i = 1$ if $\hat{E}^T[F^0] < X$ and $i = 2$ otherwise

The error, ϵ, will be the result of any noise in the system, two examples of which are immediately obvious. First, as the same weights are applied across all options for a given contract/day, there will be a pricing error induced by weighting the two bounds. Second, actual option prices are rounded to the nearest penny, also creating errors in the equation.

From (15) and (16) it is clear that even if there were no errors in the pricing relations, the fact that strikes are at discrete intervals and, more importantly, that they do not span the entire support of futures prices places an important limitation on what the option prices can reveal about the distribution. The recorded option prices only contain information about the conditional expectation and probability mass in the following segments of the support: 1) the segment below the lowest strike, 2) the segments between each strike, and 3) the segment above the highest strike. In particular, if X_L and X_H are the lowest and highest strikes, then all the information revealed by the options will be in terms of the following:

(17) $\quad E^T[F^0|F^0 < X_L] \quad , \quad Pr^T[F^0 < X_L]$

(18) $\quad E^T[F^0|\ X_i < F^0 < X_j] \ , \ Pr^T[\ X_i < F^0 < X_j] \qquad X_L \leq X_i < X_H \ , \ X_i < X_j \leq X_H$

(19) $\quad E^T[F^0|F^0 \geq X_H] \quad , \quad Pr^T[F^0 \geq X_H].$

Any number of distributions could generate the same results for the conditional expectations and probabilities in (17)-(19). For example, for any given distribution we can construct a second distribution out of a series of non-overlapping uniform densities which will be observationally equivalent to the given distribution relative to the data described by (17)-(19).

Thus, it is clear that any estimated distribution requires careful interpretation, especially in the regions below the lowest strike and above the highest strike. For crude oil, strikes are almost always $1.00 apart (in a few instances $5.00), allowing a fine demarkation of the distribution within the range of strikes. In the tails beyond the strikes, however, we have information only on the conditional expectations and the probabilities. Thus the shape of the distribution in the tails will depend importantly on the functional form assumed for the distribution. Chart Two illustrates this point with three observationally equivalent distributions. The solid line is a mixture of three lognormals, while the dashed lines replace the upper tail with uniform densities that yield the same results for (19).

III.3 Functional Form of Distribution

In choosing a functional form for the estimated distribution we tried to balance flexibility, parsimony, and ease of interpretation. For reasons explained in Section IV, we specify that the futures price at the option's expiration is drawn from a mixture of three lognormal distributions. More formally, the distribution function for futures prices g[] is given by

$$(20)\quad g[F^0] = \pi_1 g_1[F^0] + \pi_2 g_2[F^0] + \pi_3 g_3[F^0]$$

where

$$(21)\quad g_i[F^0] = \left(\frac{1}{\mathrm{sqrt}[2\pi]\sigma_i F^0}\right)*\exp\left[\left(\frac{\ln(F^0)-u_i}{\sigma_i}\right)^2/2\right].$$

The components of (15) and (16), using the properties of the lognormal distribution, can be expressed as:

$$(22)\quad E^T[F^0] = \sum_{i=1}^{3} \pi_i \exp\left[u_i+\frac{\sigma_i^2}{2}\right]$$

$$(23)\quad Pr^T[F^0 \geq X] = \sum_{i=1}^{3} \pi_i\left(1-\Phi\left[\frac{\ln(X)-u_i}{\sigma_i}\right]\right)$$

$$(24)\quad Pr^T[F^0 \leq X] = \sum_{i=1}^{3} \pi_i \Phi\left[\frac{\ln(X)-u_i}{\sigma_i}\right]$$

$$(24)\quad E^T[F^0|F^0 \geq X] = \sum_{i=1}^{3} \pi_i \exp\left[\frac{\sigma_i^2+2u_i}{2}\right]\left(\Phi\left[\frac{\ln(X)-u_i-\sigma_i^2}{\sigma_i}\right]-\frac{1}{2}\right) \Big/ Pr_t[F^0 \geq X]$$

$$(25)\quad E^T[F^0|F^0 \leq X] = \left(E_t[F^0] - \sum_{i=1}^{3} \pi_i \exp\left[\frac{\sigma_i^2+2u_i}{2}\right]\left(\Phi\left[\frac{\ln(X)-u_i-\sigma_i^2}{\sigma_i}\right]-\frac{1}{2}\right)/Pr_t[F^0 \geq X\right],$$

where Φ represents the cumulative normal distribution function.[8]

Using equations (15),(16) and (20)-(25) a pricing equation can be written for any option in terms of eleven parameters $((\pi_i, u_i, \sigma_i, w1, w2)$ $i = 1,2,3)$ and two observables (X and ρ).

The parameters of the model, exemplified by equations (15) and (16), are estimated by minimizing the sum of squared errors for all options on a given contract/day, imposing the following constraints:

$$(26)\quad \sum_{i=1}^{3} \pi_i = 1 \quad ; \quad 0 \leq w_i \leq 1 \ , \ i=1,2.$$

8. The mean of the lognormal distribution is $\exp(u+\sigma^2/2)$ (Mood, Graybill, and Boes (1974)). Calculation of (20) and (21) used integral 3.322 from Gradshteyn and Ryzhik (1980).

The restriction on the sum of the πs reduces the number of parameters from eleven to ten. Details on the data and estimation are found in Section IV. Note that given our martingale assumption for futures prices, (20) also provides an estimate of the current futures price, F_m^T.

III.4 Benchmark Distribution

In order to gauge the results from our model, the distribution for futures prices was also recovered using a "standard" option pricing model. The most common assumption in the option pricing literature is that the underlying commodity price follows a geometric Brownian process, which implies that the futures price at expiration will be drawn from a lognormal distribution. Given the lognormal distribution, and the assumption that it is possible to form a riskless hedge between the option and the underlying commodity, a partial differential equation governing the movement of the option price through time can be generated. For European options the partial differential equation can be solved when the terminal boundary condition $\mathrm{Max}[0,F^0-X]$ is applied. For American options the partial differential equation cannot be solved due to the possibility of early exercise. Several approximations have been developed to price American options under these "standard" assumptions, with the quadratic approximation of Barone-Adesi and Whaley (1987) (hereafter BAW) being the most easily calculated and most commonly used. We constructed a "standard" model by assuming that prices will be drawn from a single lognormal distribution and used the BAW approximation to generate option pricing equations. We recovered the 2 parameters (u_b and σ_b) of the single lognormal (hereafter SLN) distribution by minimizing the sum of squared deviations of predicted from actual option prices, again, using the BAW approximation to price each option.

IV. Application to the Oil Market

IV.1 Data Sources

Options on crude oil futures have been traded on the New York Mercantile Exchange (NYMEX) since November 14, 1986. Data on settle prices for all crude oil options on futures for all trading days over the period July 2, 1990 through March 30, 1991 were purchased from NYMEX. For each options contract and each trading day (contract/day) NYMEX records nine pieces of information about the option:

1. Strike Price
2. Open Interest
3. Opening Price
4. High Price
5. Low Price
6. Last Price
7. Settle Price
8. Volume
9. Exercises

We used the settle price as the value of the option in equations (15) and (16). The settle price is determined at the end of each day by a settlement committee made up of roughly 20 options market participants. The committee frequently relies on the average of bid and ask prices during the last minutes of trading as starting points for the settlement prices. Heavily traded options are priced first, with put-call parity used to price low volume options at the same strike when the futures market has settled. In the event of a limit move on the futures market[9]

9. There were no price limits in the options market. Over the entire sample there were limits on crude oil futures price changes for all contracts except for the one closest to expiration. In December of 1990 the limits on crude oil futures price movements were widened substantially.

the settlement committee relies on options on the unconstrained spot or nearby contract and spread trading.[10]

During July 1990 through March of 1991, trading was concentrated in seven contracts. For each contract/day, all option prices that were recorded with no open interest, no volume, and no settlements were excluded from the data set. In addition, trading days within five working days of the contract's expiration were also excluded from the data set.[11] Table 1 lists summary information for each of the contracts after the exclusions.

Table 1

Contract	Estimation Range	Total Days	Total Options	Number of Options per Day		Range of Strikes per Day		Range of Futures Price	
				Min	Max	Min $	Max $	Min $	Max $
Oct90	7/02/90-08/29/90	41	1254	17	42	14	37	17.74	31.93
Nov90	7/02/90-10/04/90	66	2335	15	57	15	45	18.11	40.42
Dec90	7/02/90-11/01/90	84	3150	13	55	15	44	18.38	38.80
Jan91	7/02/90-11/29/90	104	3889	13	48	16	42	18.55	37.30
Feb91	8/02/90-01/03/91	104	3866	11	43	5	51	23.27	35.95
Mar91	9/10/90-01/31/91	99	3588	12	51	10	50	18.99	34.53
Apr91	8/01/90-02/28/91	144	4827	11	44	10	45	17.91	33.21

Daily prices for the seven Treasury bills that matured as close as possible after the options contracts expired were used to calculate ρ. For each contract/day there are N (# of options) equations like (15) and (16) that form a constrained, nonlinear minimization problem. Among the seven contracts there are a total of 642 trading days; each trading/contract day was treated separately, therefore, 642 minimizations

10. We are grateful to NYMEX Board of Directors member Jim Zamora of ZAHR Trading and former NYMEX employee Brad Horne for their descriptions of the settlement process.

11. One day's worth of data for the December contract was also excluded due to an obvious error in data entry on the part of NYMEX.

were performed. Each day yielded two sets of parameter estimates, the set of ten parameters from MLN and the set of two parameters from SLN.

IV.2 Estimation

Throughout the Persian Gulf crisis, market commentary focused on three distinct outcomes: 1) a return to pre-Crisis conditions (e.g. Iraq would peacefully withdraw from Kuwait), 2) a severe disruption to Persian Gulf oil supplies (e.g. damage to Saudi Arabian facilities during a war), and 3) a continuation of unsettled conditions over the relevant horizon (e.g. a prolonged stalemate in which outcome 1 or 2 might eventually occur). Given these three possibilities, we chose a mixture of three lognormals as the form of the distribution to be estimated. If in fact market participants felt that prices were likely to be drawn from a tri-modal distribution this could be easily captured by the mixture. Moreover, the mixture could also easily accommodate a uni-modal distribution if that would best fit the data (e.g. $\pi_1=\pi_2=0$). Ex ante, we expected that as news hit the market, the relative weighting of the three lognormals might change, as well as the parameters of each of the three lognormals. For example, news of an Iraqi rocket attack on a Saudi Arabian oil field might increase the weighting on the lognormal distribution with the highest mode, as well as increase the relevant range encompassed by this lognormal distribution. Section V presents estimated distributions for selected events during the Persian Gulf crisis.

Estimation of MLN was performed with the Numerical Algorithms Group (NAG) FORTRAN algorithm E04UPF on an IBM RS-6000, with an average minimization taking approximately 10 minutes per contract/day. SLN was also estimated using E04UPF. Starting values for SLN were taken from a minimization in which the options were priced as if they were European. The SLN estimates were then used as starting values for MLN according to

$$u_1^0 = \hat{u}_b - .25,\ u_2^0 = \hat{u}_b,\ u_3^0 = \hat{u}_b + .5\ ,$$
$$\sigma_1^0 = \hat{\sigma}_b - .001,\ \sigma_2^0 = \hat{\sigma}_b,\ \sigma_3^0 = \hat{\sigma}_b + .001\ ,$$

where u_i^0 and σ_i^0 represent starting values for MLN. Bounds for the parameters were set so that $0<\hat{u}_i<\infty$, $.0001<\sigma_i<\infty$.[12] Analytic derivatives were provided for both estimations. The derivatives were calculated using Mathematica and they were numerically verified within the E04UPF algorithm prior to estimation.

The estimation procedures are illustrated in Chart Three. The top panel plots the estimated density function using both a mixture of lognormals (MLN) and a single lognormal (SLN). Given the density from the single lognormal, the BAW formulae give predicted values for the option prices. The triangles in the lower panels plot the difference between the BAW predictions and the actual prices.

Given the MLN estimated distribution for the futures price, we can compute upper and lower bounds for the option prices. The solid (dashed) line in the lower panels plots the difference between the upper (lower) bound for the option price and the actual option price. The predicted option price is a weighted average of these bounds, where the weights are

12. Moreover, each (u, σ) pair was restricted such that the probability of the futures price reaching $150 per barrel was less than 5 percent, under each of the lognormal distributions. These bounds prevented the algorithm from wandering off in nonsensical directions.

determined in the minimization routine. The hollow boxes plot the difference between the option prices predicted from the MLN distribution and the actual prices. We note that this same MLN distribution was used to draw the plots in Chart One.[13]

V. Results

V.1 Summary Measures

The first moments of the estimated distributions from MLN and SLN were very similar and were extremely close to the actual futures price. (The actual futures price can be viewed as an independent estimate of the mean of the distribution.) The percentage mean absolute difference (PMAD) between the mean from MLN and the mean from SLN was 0.1 percent. The PMAD between the mean from MLN and the actual futures price across all contract/days amounted to 0.45 percent, while the PMAD between the mean from SLN and the actual futures price amounted to 0.51 percent. The relationship between the mean from MLN and the actual futures price is shown in Chart Four. Note that the contract/days in which there was a substantial discrepancy between MLN's mean and the futures price (points off the horizontal line through zero) were the contract/days on which actual futures prices moved exactly \$1.00, \$1.50, \$2.00, \$3.00, or \$4.00,

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13. The bounds and errors plotted in charts One and Two differ slightly from those used by the minimization routine for the following technical reason: The minimization routine behaves significantly better if the objective function is differentiable and if analytic derivatives are supplied. The derived formulae for the bounds are not differentiable since they include the Max operator. In estimation, we used a logit weighting scheme to construct a differentiable approximation to the Max operator where the weights on the two items in the Max move to zero and one as the items move farther apart. The data plotted on the charts were constructed using the actual formulae for the bounds, rather than the differentiable approximation, together with the estimated distribution and weights.

that is, contract/days on which there was a limit move on the futures contract. As discussed above, there were no limits in the options market, hence the mean from MLN on these days should not be expected to equal the futures price.

The similarity between the MLN and SLN distributions does not carry over to higher order moments. Chart Five depicts representative probability density functions taken from the two methods; the top panel uses estimates from the October contract on July 10th (3 weeks before the crisis) and the bottom panel uses estimates from the January contract on October 10th (in the midst of the crisis). Prior to the outbreak of the crisis there is little qualitative difference in the two estimates, while during the crisis the estimates from SLN cannot as easily accommodate the significant probability mass above $50 per barrel. The SLN estimate is forced to increase σ, significantly extending the right-hand tail of the distribution. Clearly, the oil market saw a significant chance of a major disruption that had the capability of pushing prices to levels not seen before.

Chart Six attempts to shed some light on the differences between the right hand tails of the two estimates, using the April contract as an example. The top panel plots $1.25*F_m^T$ along with $E^T[F^0|F^0>1.25*F_m^T]$ from MLN and SLN. The bottom panel plots $Pr^T[F^0>1.25*F_m^T]$. As can be seen from the chart, the conditional expectation from MLN is generally above that of SLN, while the probability from MLN is generally below that of SLN. The reason for this result is visually apparent in the bottom panel of Chart Five. The large σ estimated via SLN forces relatively more of the probability mass to the right but, since the distribution must remain unimodal, leaves the bulk of the right-hand mass nearer the unconditional mean. These results hold across all the contracts. For 574 out of the 642 contract/days the conditional expectation from MLN is above that of SLN. For 476 out of the 642 contract/days the probability of being above $1.25*F_m^T$ from MLN is below that of SLN. To make this more concrete: If a policy maker or analyst were using the SLN estimates when the MLN were closer to the truth, she would tend to overestimate the market's assessment of the probability of a major disruption while underestimating the impact on prices of such a disruption.

These differences in the right-hand tails of the distributions are also apparent when examining the pricing errors generated by SLN and MLN. The right-hand tail of the distribution will be more important for pricing out of the money calls and in the money puts. For out of the money calls, across all contract days, SLN had a mean error (actual - predicted) of \$0.0865 compared to \$0.0005 for MLN. For in the money puts SLN had a mean error of \$0.0445 compared to -\$0.0004 for MLN. For these options, SLN, on average, underpredicted the prices, again indicating that SLN did not allocate enough probability mass to the right-hand tail. As might be expected, SLN tended to overpredict the prices for in the money calls (mean error of -\$0.0430) and out of the money puts (mean error of -\$0.0388), an overallocation of probability mass to the left-hand tail of the distribution.

Although the differences between the estimates from MLN and those from SLN are apparent, it is not obvious that these differences are significant in a statistical sense. This issue is complicated since the SLN model cannot be nested within MLN.[14] Since the models are not nested, the standard, asymptotic-chi-square assumption cannot be used when forming a likelihood ratio test (or its F-test analog). Short of a Monte Carlo simulation, little can be done to get around the non-nesting problem. However, goodness-of-fit measures can shed some light on the performance of the two models. The table below presents summary RMSE calculations across all options and all days for each contract. These measures are in dollars, presenting the average error across all the options for that contract. Using the October contract as an example, the average error for MLN was a bit less than $0.02 while for SLN it was a bit less than $0.07. (Both of these errors are relatively small, as prices are recorded only to the nearest penny). As can be seen, the errors from MLN are well below those for SLN. Additionally, the table presents the number of days for which an F-test could not reject at the five percent level the restrictions involved in moving from MLN to SLN, acting as if the models were nested.[15] This last column is not statisti-

14. The problem is as follows: We have two competing non-linear models that explain a vector of option prices (y) on any given day. Denote the two models by

(1) $y = g[\pi_1, \pi_2, \pi_3, u_1, u_2, u_3, \sigma_1, \sigma_2, \sigma_3, w_1, w_2 | Z]$

(2) $y = h[u_b, \sigma_b | Z]$,

where Z represents the data matrix containing strikes and interest rates.

Model 2 can almost be nested within Model 1 ($\pi_1 = \pi_3 = 0$ or $u_1 = u_2 = u_3$ and $\sigma_1 = \sigma_2 = \sigma_3$), except that g[] and h[] represent different functional forms. In particular, g[] uses the weighted bounds where h[] uses the BAW approximations.

15. The test statistic is given by

$$\frac{(sse1 - sse2)/df1}{sse1/(df2 - df1)}$$

(Footnote continues on next page)

cally appropriate, given the non-nesting problem, but is presented as informal evidence for model-selection.

Table 2

Contract	MLN RMSE	SLN RMSE	Non-rejections/ Total Days
October	.0156	.0667	2/41
November	.0184	.0724	2/66
December	.0322	.0849	2/84
January	.0367	.0832	6/104
February	.0369	.1071	3/104
March	.0341	.1339	3/99
April	.0305	.1211	14/149

In sum, for most all of the contract/days considered, the formulation of MLN appears to be preferred to that of SLN.

V.2 Selected Events

Throughout the Persian Gulf crisis, the oil market often experienced large movements in price when market participants' expectations concerning likely crisis outcomes were revised as "news" hit the market. It makes for interesting storytelling, and further highlights the differences between MLN and SLN, to compare estimated oil futures PDFs from the two models immediately before and after receipt of the "news".

(Footnote continued from previous page)
where df1 equals eight (the ten parameters of MLN minus the 2 parameters of SLN) and df2 equals the number of options for the particular contract/day. This a conservative approach to the degrees of freedom; were the models truly nested, MLN would reduce to SLN with either 2 ($\pi_1=\pi_2=0$) or 4 ($u_1=u_2=u_3$, $\sigma_1=\sigma_2=\sigma_3$) restrictions. The ambiguity in restrictions is symptomatic of a lack of identification for some of the parameters of MLN under the null hypothesis that SLN is the true model. See Breusch (1986).

On Thursday October 25, 1990 the London Financial Times carried a report that Iraqi forces had attached explosives to 300 of Kuwait's 1000 oil wells, quoting a senior Kuwaiti engineer who had left Kuwait one week earlier. This revelation pushed oil prices up sharply, with the futures contract nearest to expiration (December) rising $3.17 per barrel. Chart Seven plots the PDFs from MLN and SLN for October 22 (top panel) and October 25 (bottom panel) using the January contract. On October 22, market expectations for futures prices were centered quite tightly around $24 per barrel. The news of the mining widened each model's distribution significantly, with MLN allowing for a sizeable probability mass between $60 and $70 per barrel.

The largest one-day change in oil prices in NYMEX history occurred on Thursday January 17, 1991 when 1) several governments announced a coordinated release of oil from their emergency inventories and 2) it became clear that the coalition forces had total air supremacy. On January 17 the settle price for the March contract fell $9.66 while the settle price for the April contract fell $7.82. The six panels of Chart Eight trace the evolution of expected PDFs on the days surrounding January 17. Prior to the first air strike (as can be seen in the first two panels), the market was still expecting a fairly significant chance of a major oil market disruption (perhaps Iraqi damage of Saudi Arabian oil facilities) that could push prices to the $40-$60 per barrel range. On January 17th these PDFs tightened dramatically, and on ensuing days the PDF generated from MLN moved closer and closer to that from SLN. By January 23, there was little difference between the two PDFs, as the market returned to almost a pre-crisis distribution (compare panel 6 with the top panel of Chart 2). Through March of 1991 (the end of our data sample) there was little difference between the two models, as evidenced by the relatively large number of non-rejections for the April contract in the Table 2 above.

VI Conclusion

This paper develops a method for using option prices to estimate the market's probability distribution for commodity prices. The method is quite general, allowing the standard lognormal distribution to be replaced by any from within a wide class of distributions. The particular assumption of a mixture of three lognormal distributions used here was driven by conditions in the oil market during the Persian Gulf crisis. As the focus is only on the commodity's probability distribution, no structure is placed on the stochastic process governing movements in the commodity price over time, unlike jump-diffusion methods. This lack of structure is appealing, although the method is silent on the evolution of prices which yield the distribution at the option's expiration. Our methodology should be useful to researchers who wish to impose a minimum of structure and are 1) examining other markets during unsettled times, or 2) investigating asset price distributions that are not adequately described by the lognormal distribution (e.g. leptokurtotic distributions). The major limitation of our method is that it is only directly applicable to those markets where price changes are uncorrelated with changes in the market portfolio.

In the application to the oil market we find that the options markets were consistent with the market commentary at the time, in that they reflected a significant probability of a major disruption in oil prices. We find that the estimated price of oil conditional on a major disruption was often in the \$50-\$60 per barrel range, which is also consistent with market commentary. We also find that the standard lognormal assumption did a poorer job of characterizing the data than did our method. In particular, we find that if policy makers or analysts had used the distribution from the lognormal model where our model was closer to the truth, they would have generally overestimated the market's assessment of the probability of a major disruption and underestimated the impact on prices of such a disruption.

Finally, examination of particular days confirmed the large shift in market expectations that occurred when significant crisis-related news reached the oil market.

References

Barone-Adesi, Giovanni and Robert Whaley, "Efficient Analytic Approximation of American Option Values", Journal of Finance 42(2):301-320 (1987).

Bates, David S., "The Crash of '87: Was It Expected? The Evidence From Options Markets", Journal of Finance 46(3):1009-1044 (July 1990).

Black, Fisher, "The Pricing of Commodity Contracts", Journal of Financial Economics 3:167-179 (1976).

Black, Fisher and Myron Scholes, "The Pricing of Options and Corporate Liabilities", Journal of Political Economy 81:637-659 (1973).

Breusch, Trevor S., "Hypothesis Testing in Unidentified Models", Review of Economic Studies 53:635-651 (1986),

Cox, John C. and Stephan A. Ross, "The Valuation of Options for Alternative Stochastic Processes", Journal of Financial Economics 3:145-166 (1976)

Dusak, Katherine, "Futures Trading and Investor Returns: An Investigation of Commodity Market Risk Premiums", Journal of Political Economy 81:1387-1406 (1973).

French, Mark W., "Oil Markets and Asset Pricing Models", Unpublished mimeo, Board of Governors of the Federal Reserve System (1990).

Gradshteyn, I.S. and I.M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press (1980).

Mood, Alexander, Franklin Graybill and Duane Boes, Introduction to the Theory of Statistics, Third Edition McGraw Hill (1974).

Overdahl, James A. and H. Lee Matthews, "The Use of NYMEX Options to Forecast Crude Oil Prices", The Energy Journal 9(4):135-147 (1988).

Smith, Clifford W. Jr., "Option Pricing a Review", Journal of Financial Economics 3:3-51 (1976).

Appendix

Theorem One:

Given a period T futures price of F^T_m, let the true sequence of transition probabilities be given by $\{\psi^T_{m.}, \Psi^{T-1}, \ldots, \Psi^1\}$, implying that the period-T distribution for futures in period 0 is given by $\Gamma^T = \psi^T_{m.} \cdot \prod_{t=T,\ldots,2} \Psi^{t-1}$. Of all sequences that yield Γ^T, the value of an option (put or call) is greatest for the following: $\{\Gamma^T, I, I, \ldots, I\}$. For this sequence, the value of the call option is given by $C^T[F_m] = \mathrm{Max}[F_m - x,\ \rho^T\ \Gamma^T \cdot \mathrm{Max}[F_m - x,\ 0]]$.

Proof:

The proof has three steps. In step one we take an arbitrary sequence of transition probabilities that yields Γ and replace it with an alternative sequence where Ψ^1 is replaced with the identity matrix and Ψ^2 is replaced with $\bar{\Psi}^2 = \Psi^2 \cdot \Psi^1$ (Γ^T is obviously unchanged). We show that in period 2 the value of the option under the original sequence is no greater than under the alternative sequence. In step two we show that given a sequence of transition probabilities of the form $\{\psi^T_{m.}, \Psi^{T-1}, \ldots, \Psi^{T-h}, I, I, \ldots, I\}$, the option will never be exercised after T-h-1, so it can be treated as though it expired in period T-h-1 and had transition probabilities given by $\{\psi^T_{m.}, \Psi^{T-1}, \ldots, \Psi^{T-h}\}$. In step three we note that repeated application of step one to the ever shorter sequence of transition probabilities produced in step two eventually gives us $\{\Gamma^T, I, I, \ldots, I\}$ as the sequence which yields the maximum value for the option for all sequences consistent with Γ^T. For this sequence the value of the option is as given in the statement of the theorem.

Step One: We want to show that the value of the option in period 2 under the original process is no greater than the value of the option in period 2 under the alternative process. We make the argument for call options, but the analogous argument for puts is transparent.

Under the original process the period-2 value of the American call option for any state m, is given by the following:

$$(1)\qquad C^2[F^2_m] = \mathrm{Max}[F^2_m\text{-}x,\ \rho^2\sum_j\Psi^2_{mj}\mathrm{Max}[F^1_j\text{-}x,\ \rho^1\sum_k\Psi^1_{jk}\mathrm{Max}[F^0_k\text{-}x,\ 0]]].$$

Its value under the alternative process is given by

$$(2)\qquad \bar{C}^2[F^2_m] = \mathrm{Max}[F^2_m\text{-}x,\ \rho^2\sum_k\bar{\Psi}^2_{mk}\mathrm{Max}[F^1_k\text{-}x,\ \rho^1\sum_n I_{kn}\mathrm{Max}[F^0_n\text{-}x,\ 0]]].$$

To prove that $\bar{C}^2[F^2_m] \geq C^2[F^2_m]$ for all m, it is sufficient to prove

$$(3)\qquad \sum_k\bar{\Psi}^2_{mk}\mathrm{Max}[F^1_k\text{-}x,\ \rho^1\sum_n I_{kn}\mathrm{Max}[F^0_n\text{-}x,\ 0]] \geq$$

$$\sum_j\Psi^2_{mj}\mathrm{Max}[F^1_j\text{-}x,\ \rho^1\sum_k\Psi^1_{jk}\mathrm{Max}[F^0_k\text{-}x,\ 0]]$$

To prove (3), we assume it does not hold and show that a contradiction results. Thus we assume

$$(4)\qquad \sum_k\bar{\Psi}^2_{mk}\mathrm{Max}[F^1_k\text{-}x,\ \rho\sum_n I_{kn}\mathrm{Max}[F^0_n\text{-}x,\ 0]] < \sum_j\Psi^2_{mj}\mathrm{Max}[F^1_j\text{-}x,\ \rho\sum_k\Psi^1_{jk}\mathrm{Max}[F^0_k\text{-}x,\ 0]]$$

where the superscript on ρ^1 is suppressed. The identity matrix weights in (4) collapse to yield

$$(5)\qquad \sum_k\bar{\Psi}^2_{mk}\mathrm{Max}[F^1_k\text{-}x,\ \rho\mathrm{Max}[F^0_k\text{-}x,\ 0]] < \sum_j\Psi^2_{mj}\mathrm{Max}[F^1_j\text{-}x,\ \rho\sum_k\Psi^1_{jk}\mathrm{Max}[F^0_k\text{-}x,\ 0]].$$

Since $F^1_k = F^0_k$ and $0 < \rho < 1$, $((F^1_k\text{-}x) > \rho(F^0_k\text{-}x)) \Leftrightarrow ((F^0_k\text{-}x) > 0)$. Thus the interior Max[] on the LHS of (5) can be eliminated to yield

$$(6) \qquad \sum_k \bar{\Psi}^2_{mk}\mathrm{Max}[F^0_k - x,\ 0] < \sum_j \Psi^2_{mj}\mathrm{Max}[F^1_j - x,\ \rho\sum_k \Psi^1_{jk}\mathrm{Max}[F^0_k - x,\ 0]]$$

Let $\Omega = \{j : (F^1_j - x) > \rho\sum_k \Psi^1_{jk}\mathrm{Max}[F^0_k - x,\ 0]\}$ and let $\backslash\Omega$ denote its complement. Then (6) can be written as

$$(7) \qquad \sum_k \bar{\Psi}^2_{mk}\mathrm{Max}[F^0_k - x,\ 0] < \sum_{j\in\Omega} \Psi^2_{mj}(F^1_j - x) + \rho\sum_{j\in\backslash\Omega} \Psi^2_{mj}\sum_k \Psi^1_{jk}\mathrm{Max}[F^0_k - x,\ 0]$$

Using the martingale assumption, $F^1_j = \sum_k \Psi^1_{jk}F^0_k$, and the fact that for all Markov transition matrices $\sum_k \Psi_{jk} = 1\ \forall j$, (7) yields

$$(8) \qquad \sum_k \bar{\Psi}^2_{mk}\mathrm{Max}[F^0_k - x,\ 0] < \sum_{j\in\Omega} \Psi^2_{mj}\sum_k \Psi^1_{jk}(F^0_k - x) + \rho\sum_{j\in\backslash\Omega} \Psi^2_{mj}\sum_k \Psi^1_{jk}\mathrm{Max}[F^0_k - x,\ 0]$$

Let $A \equiv \sum_{j\in\backslash\Omega} \Psi^2_{mj}\sum_k \Psi^1_{jk}\mathrm{Max}[F^0_k - x,\ 0]$. Adding and subtracting A from the RHS of (8) gives

$$(9) \qquad \sum_k \bar{\Psi}^2_{mk}\mathrm{Max}[F^0_k - x,\ 0]] < \sum_{j\in\Omega} \Psi^2_{mj}\sum_k \Psi^1_{jk}(F^0_k - x) +$$

$$\sum_{j\in\backslash\Omega} \Psi^2_{mj}\sum_k \Psi^1_{jk}\mathrm{Max}[F^0_k - x,\ 0] + (\rho - 1)A.$$

Let $\Delta = \{k : F_k^0 - x > 0\}$. Then the remaining Max[] operators in (9) can be eliminated to yield

$$(10) \qquad \sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x) < \sum_{j\in\Omega} \Psi^2_{mj} \sum_k \Psi^1_{jk}(F_k^0-x) + \sum_{j\in\backslash\Omega} \Psi^2_{mj} \sum_{k\in\Delta} \Psi^1_{jk}(F_k^0-x) + (\rho-1)A.$$

Separating terms by Δ on the RHS, we have

$$(11) \qquad \sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x) < \sum_{j\in\Omega} \Psi^2_{mj} \sum_{k\in\Delta} \Psi^1_{jk} (F_k^0-x) + \sum_{j\in\Omega} \Psi^2_{mj} \sum_{k\in\backslash\Delta} \Psi^1_{jk} (F_k^0-x) +$$

$$\sum_{j\in\backslash\Omega} \Psi^2_{mj} \sum_{k\in\Delta} \Psi^1_{jk}(F_k^0-x) + (\rho-1)A.$$

Combining terms for Ω and $\backslash\Omega$ yields

$$(12) \qquad \sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x) < \sum_{j} \Psi^2_{mj} \sum_{k\in\Delta} \Psi^1_{jk} (F_k^0-x) + \sum_{j\in\Omega} \Psi^2_{mj} \sum_{k\in\backslash\Delta} \Psi^1_{jk} (F_k^0-x) + (\rho-1)A.$$

Reversing the order of summation in the first term on the RHS gives us

$$(13) \qquad \sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x) < \sum_{k\in\Delta} \sum_{j} \Psi^2_{mj} \Psi^1_{jk} (F_k^0-x) + \sum_{j\in\Omega} \Psi^2_{mj} \sum_{k\in\backslash\Delta} \Psi^1_{jk} (F_k^0-x) + (\rho-1)A.$$

By construction $\bar{\Psi}^2_{mk} = \sum_j \Psi^2_{mj}\Psi^1_{jk}$, so (13) yields

$$(14) \qquad \sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x) < \sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x) + \sum_{j\in\Omega} \Psi^2_{mj} \sum_{k\in\backslash\Delta} \Psi^1_{jk} (F_k^0-x) + (\rho-1)A.$$

Subtracting $\sum_{k\in\Delta} \bar{\Psi}^2_{mk}(F_k^0-x)$ from both sides yields

$$(15) \qquad 0 < \sum_{j \in \Omega} \Psi^2_{mj} \sum_{k \in \backslash\Delta} \Psi^1_{jk} \; (F^0_k - x) \; + \; (\rho - 1)A.$$

By construction of Δ, $\forall k \in \backslash\Delta$, $(F^0_k - x) \le 0$, so $\sum_{j \in \Omega} \Psi^2_{mj} \sum_{k \in \backslash\Delta} \Psi^1_{jk} (F^0_k - x) \le 0$.

Since A is the sum of terms of the form $\mathrm{Max}[F^0_k - x,\ 0]$, A must be non-negative, which implies $(\rho-1)A$ is non-positive. Thus the RHS of (15) cannot be positive and we have a contradiction. This completes step one.

<u>Step Two:</u>

Given a sequence of transition probabilities of the form $\{\psi^T_{m.}, \Psi^{T-1}, \ldots, \Psi^{T-h}, I, I, \ldots, I\}$, the option will never be exercised after T-h-1, and it can be treated as though it expired in period T-h-1 and had transition probabilities given by $\{\psi^T_{m.}, \Psi^{T-1}, \ldots, \Psi^{T-h}\}$.

<u>Proof:</u>

By assumption, $\forall t$: $0 < t < T-h$, $\Psi^t = I$ so the value of the option is given by

$$(16) \qquad C^t[F_j] = \mathrm{Max}[F_j - x,\ \rho^t \sum_n I_{jn} C^{t-1}[F_n]] = \mathrm{Max}[F_j - x,\ \rho^t C^{t-1}[F_j]]$$

Let $\Omega = \{j\colon F_j - x > 0\}$. For all $j \in \Omega$, $C^0[F_j] = \mathrm{Max}[F_j - x,\ 0] = F_j - x$. Since $0 < \rho < 1$, by induction over (16) we have

$$(17) \qquad C^t[F_j] = (F_j - x), \quad \forall j \in \Omega, \quad \forall t\colon 0 < t < T-h.$$

By construction, $\forall j \in \backslash\Omega$, $F_j - x \le 0$, so $C^0[F_j] = \mathrm{Max}[F_j - x,\ 0] = 0$. Again by induction over (16) we have

(18) $C^t[F_j] = 0 \quad \forall j \in \backslash\Omega, \quad \forall t: 0 < t < T\text{-}h.$

Combining (17) and (18) we have

(19) $C^t[F_j] = \mathrm{Max}[F_j\text{-}x,\ 0] \quad \forall t: 0 < t < T\text{-}h.$

In particular, under the assumed sequence of transition probabilities

(20) $C^{T\text{-}h\text{-}1}[F_j] = \mathrm{Max}[F_j\text{-}x,\ 0].$

But (20) is simply the value of an option that expires in period T-h-1. Thus the value of the option under the assumed sequence is the same as the value of an option that expires in period T-h-1 with transition probabilities given by $\{\psi^T_{m.}, \Psi^{T\text{-}1}, \ldots, \Psi^{T\text{-}h}\}$. This completes step two.

Step Three:

Whatever form this shorter sequence from step two takes, from step one, we know it can be no greater than the sequence where $\Psi^{T\text{-}h}$ is replaced by I and $\Psi^{T\text{-}h+1}$ is replaced by $\bar{\Psi}^{T\text{-}h+1} = \Psi^{T\text{-}h+1} \cdot \Psi^{T\text{-}h}$.

The repeated application of steps one and two eventually leads to a sequence of transition probabilities of the form $\{\Gamma, I, I, \ldots, I\}$.

With these transition probabilities, and the knowledge that the option will never be exercised after period T-1, the value of the option under this sequence can be expressed as follows

(21) $C^T[F_m] = \mathrm{Max}[F_m\text{-}x,\ \rho^T\Gamma \cdot \mathrm{Max}[F_m\text{-}x,\ 0]]$

This completes the proof.

Theorem Two:

Given a period T futures price of F_m^T, let the true sequence of transition probabilities be given by $\{\psi_{m.}^T, \Psi^{T-1}, \ldots, \Psi^1\}$, implying that the period-T distribution for futures in period 0 is given by $\Gamma^T = \psi_{m.}^T \cdot \prod_{t=T,\ldots,1} \Psi^{t-1}$. For all sequences of transition probabilities that yield Γ^T, the value of an option (put or call) is smallest for the following sequence: $\{I_{m.}, I, \ldots, I, 1_M \cdot \Gamma^T\}$, where 1_M is an Mx1 vector of ones. For this sequence, the value of the call option is given by $C^T[F_m] =$ $\text{Max}[F_m - x, \bar{\rho}^T \Gamma^T \cdot \text{Max}[F_m - x, 0]]$ where $\bar{\rho}^T = \prod_{t=T,\ldots,1} \rho^t$.

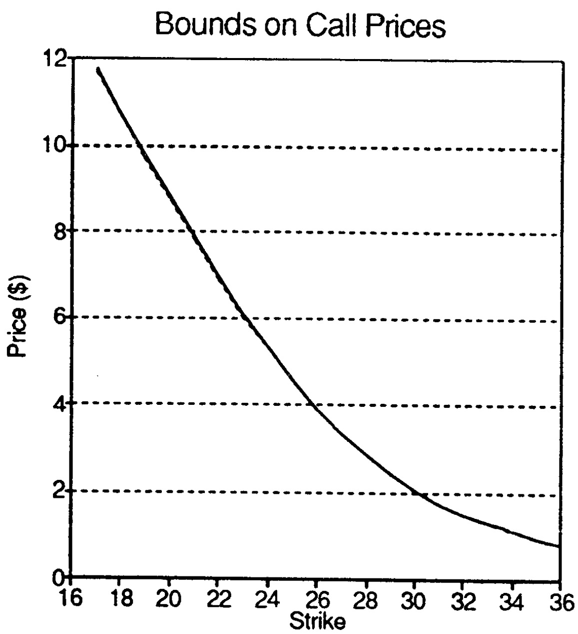
<u>Proof:</u>

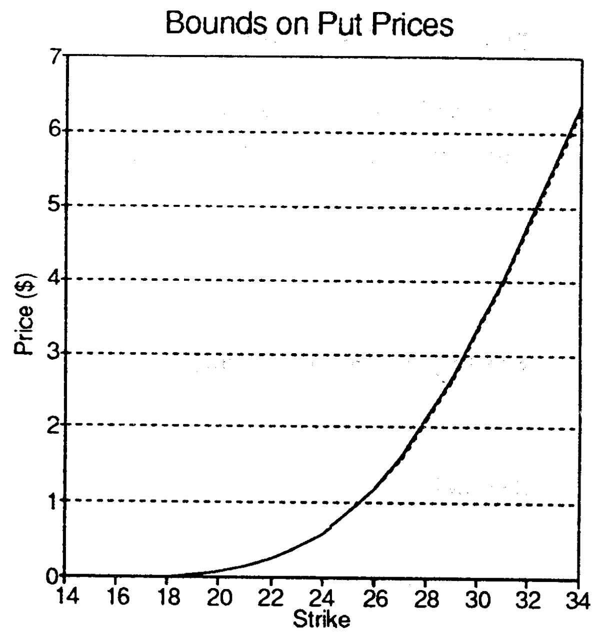
The proof is analogous to that of Theorem One and we only sketch it here. Replace $\psi_{m.}^T$ with $I_{m.}$ and replace Ψ^{T-1} with $\bar{\Psi}^{T-1} = 1_M \cdot \psi_{m.}^T \cdot \Psi^{T-1}$. The value of the option will be less than or equal to the option's value under the original sequence. Under this new sequence the option will not be exercised in period T-1 and its value in T-1 can be expressed in terms of F_m^T, $\bar{\Psi}^{T-1}$, and $C^{T-2}[F^{T-2}]$ where $C^{T-2}[F^{T-2}]$ is a function of the remaining original transition probabilities. No matter what the remaining original transition probabilities are, we can lower the value of the option in T-1 (while preserving Γ^T) by replacing $\bar{\Psi}^{T-1}$ with I and replacing Ψ^{T-2} with $\bar{\Psi}^{T-2} = \bar{\Psi}^{T-1} \cdot \Psi^{T-1}$. Under this new sequence the option will not be exercised in periods T-1 or T-2 and its value in period T-2 can be expressed in terms of F_m^T, $\bar{\Psi}^{T-2}$, and $C^{T-3}[F^{T-3}]$ where $C^{T-3}[F^{T-3}]$ is a function of the remaining original transition probabilities. The process continues until we are left with a sequence of the form $\{I_{m.}, I, \ldots, I, 1_M \cdot \Gamma^T\}$.

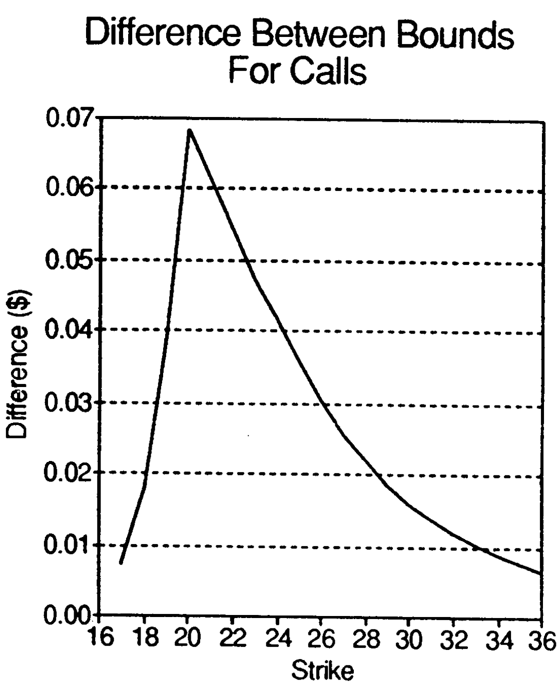
Under this sequence the value of the option in period T can be expressed as $C^T[F_m] = \text{Max}[F_m - x, \bar{\rho}^T \Gamma^T \cdot \text{Max}[\underline{F} - x, 0]$. This completes the proof.

Chart 1

For a Typical Day's Estimated Distribution







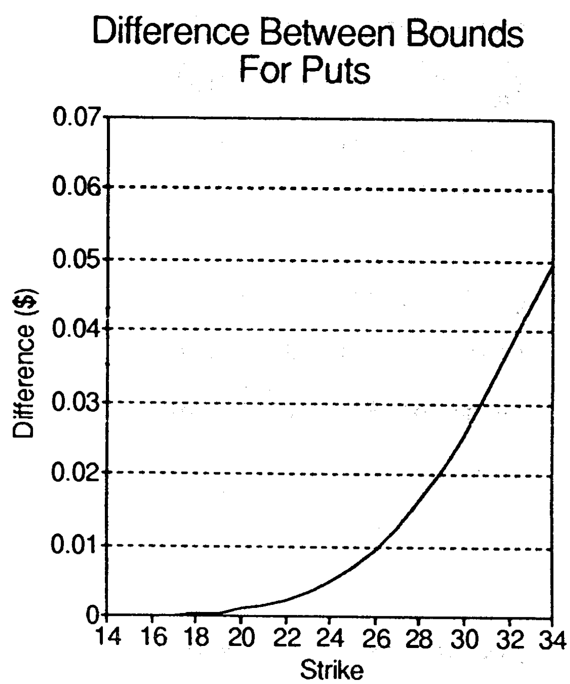


Chart 2

Observationally Equivalent Density Functions

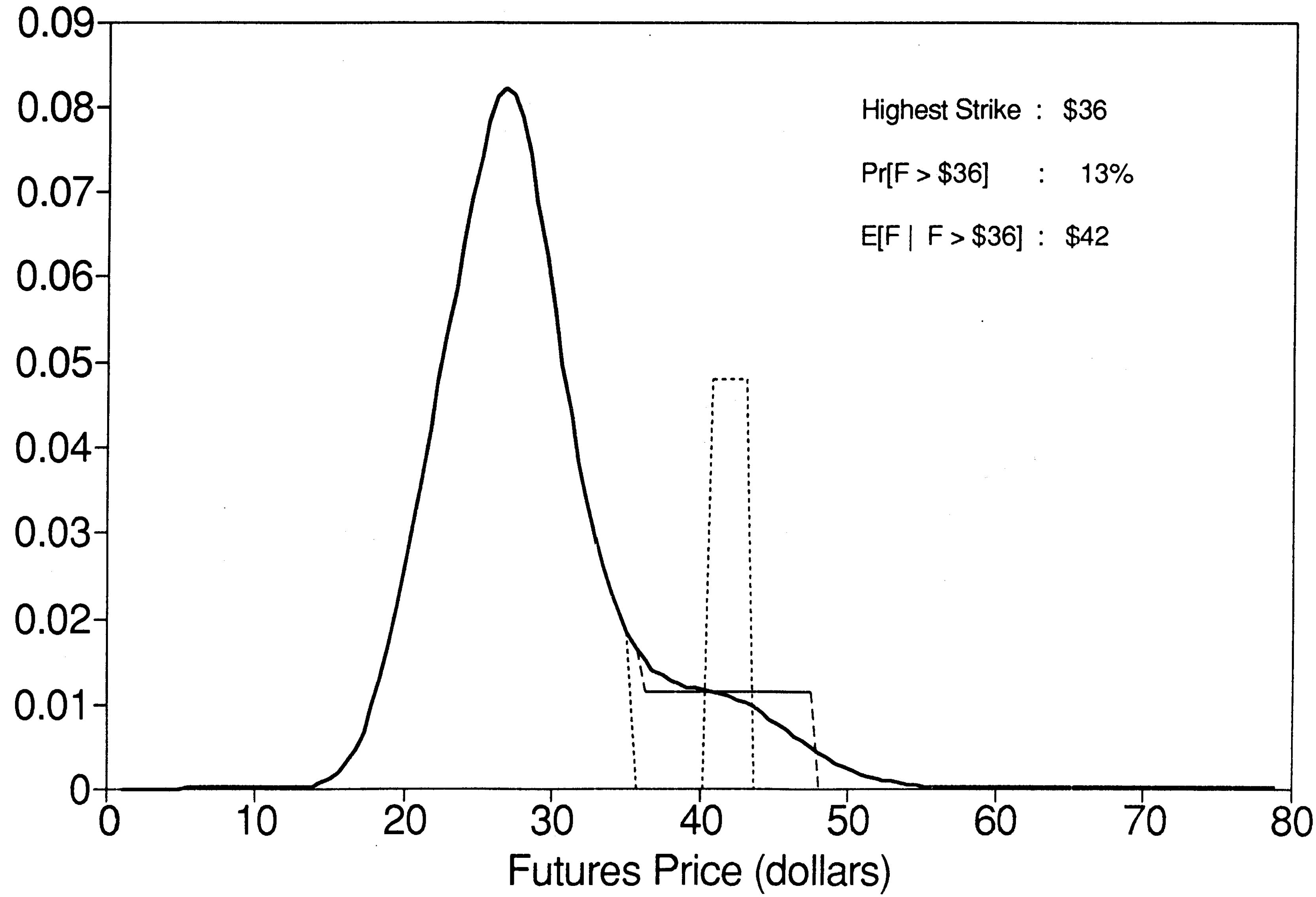
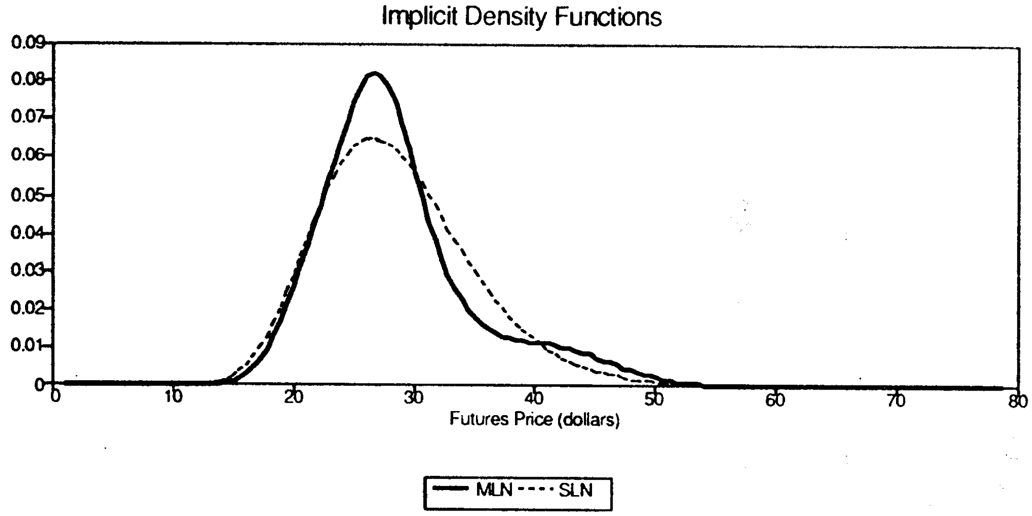
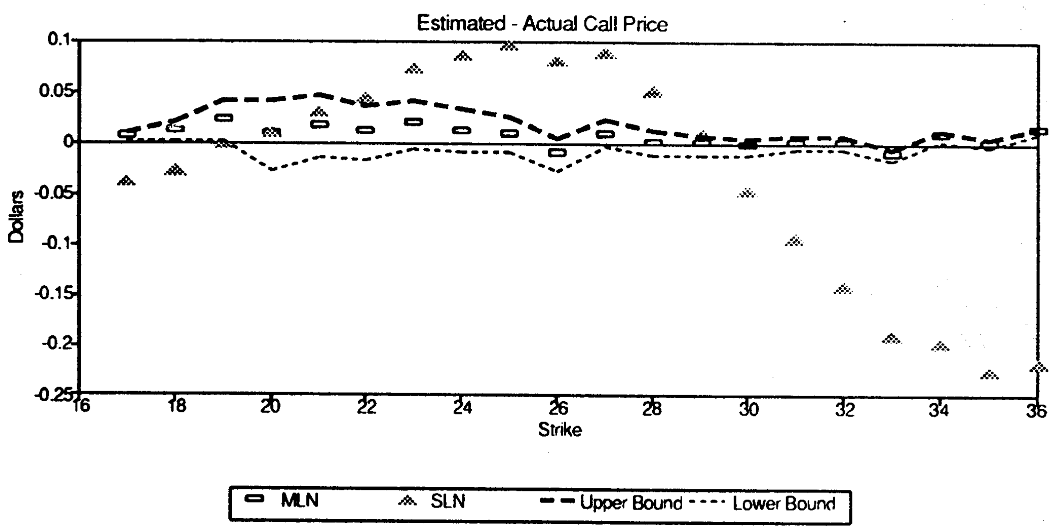


Chart 3

For a Typical Day





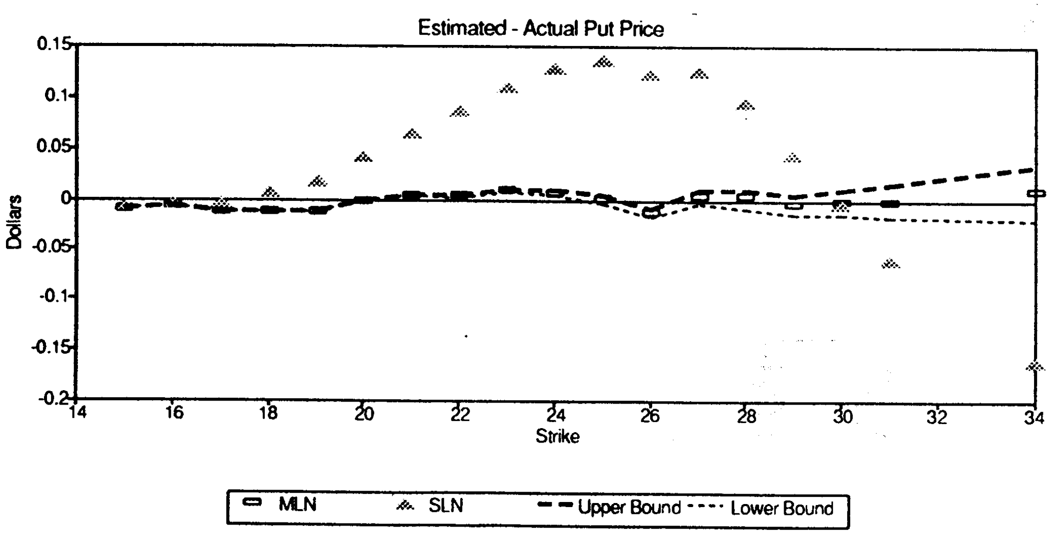


Chart 4

All Contracts
Futures Price - Expected Price
vs. Change in Futures Price

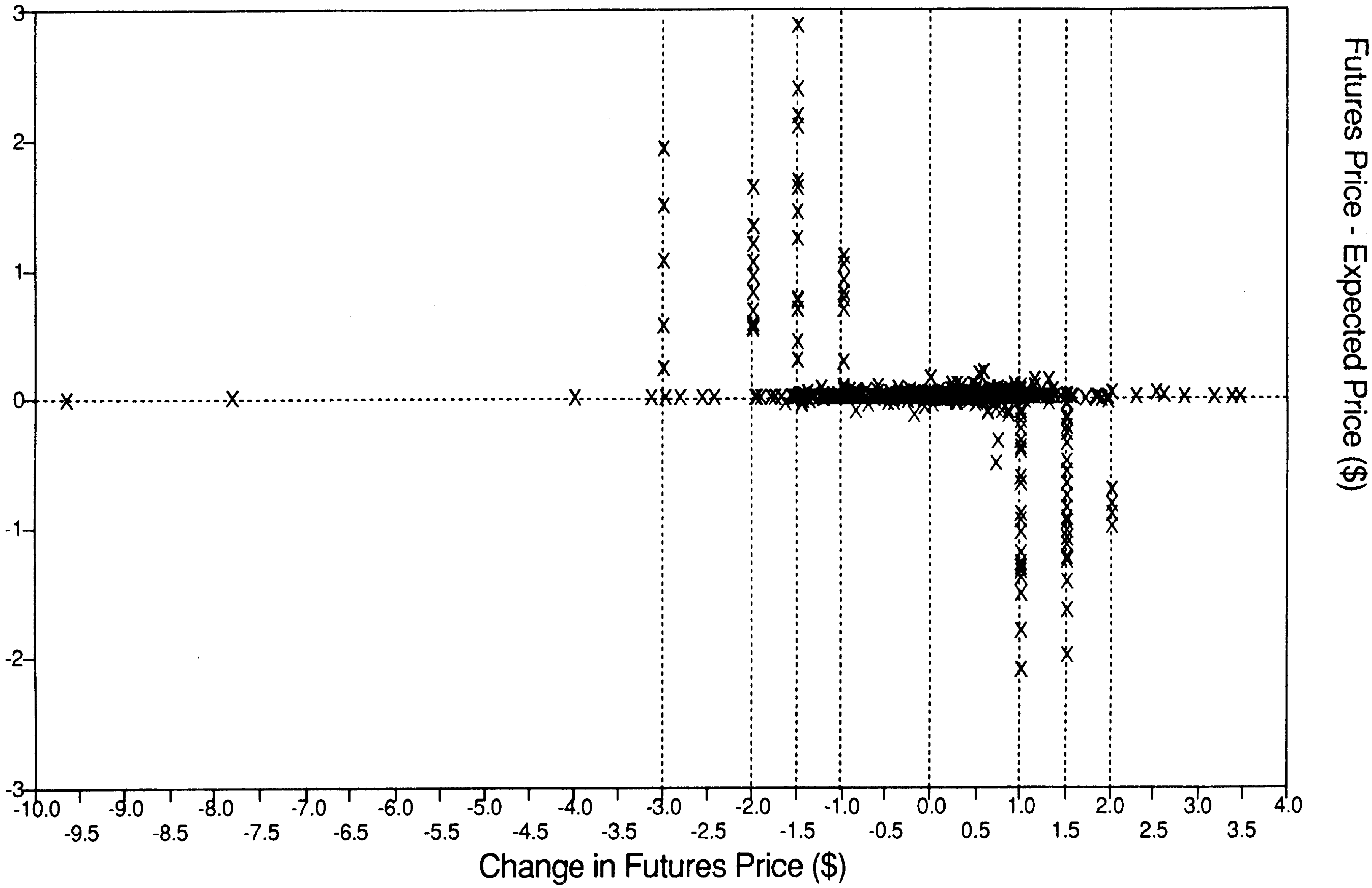
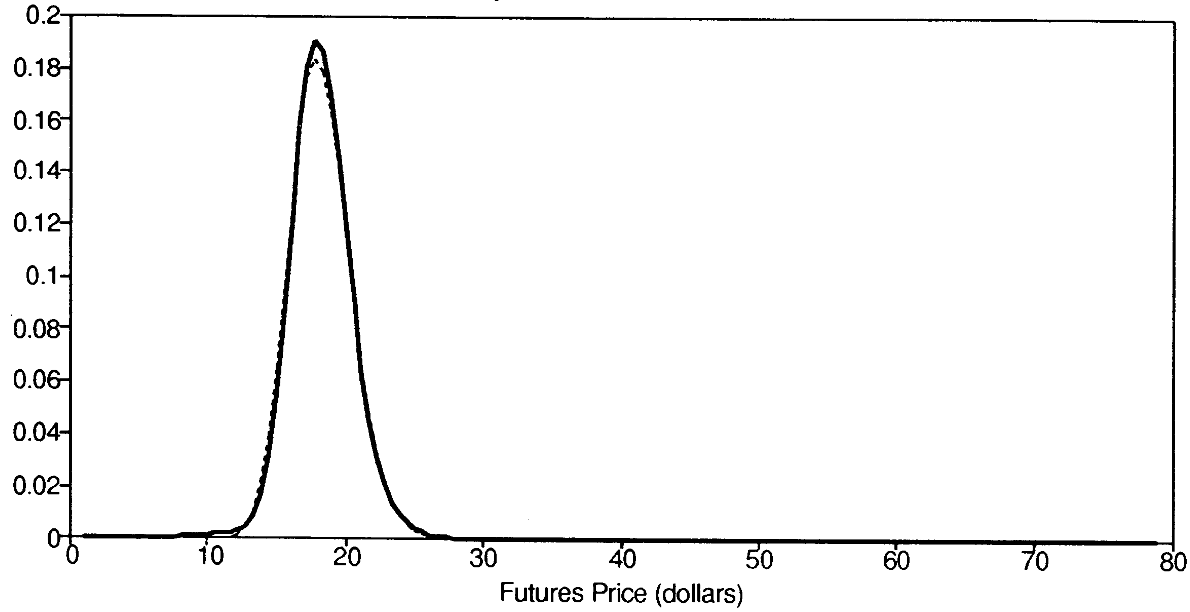


Chart 5

Implicit Density Functions

On July 10 for October Contract



On October 18 for January Contract

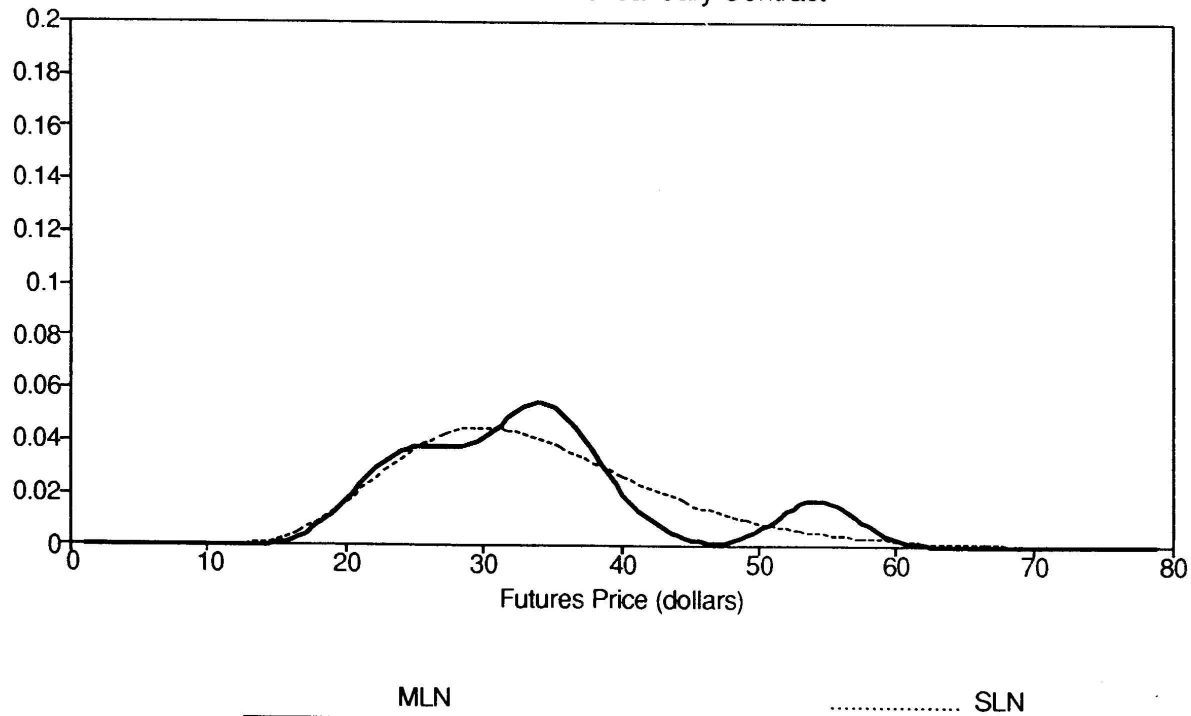
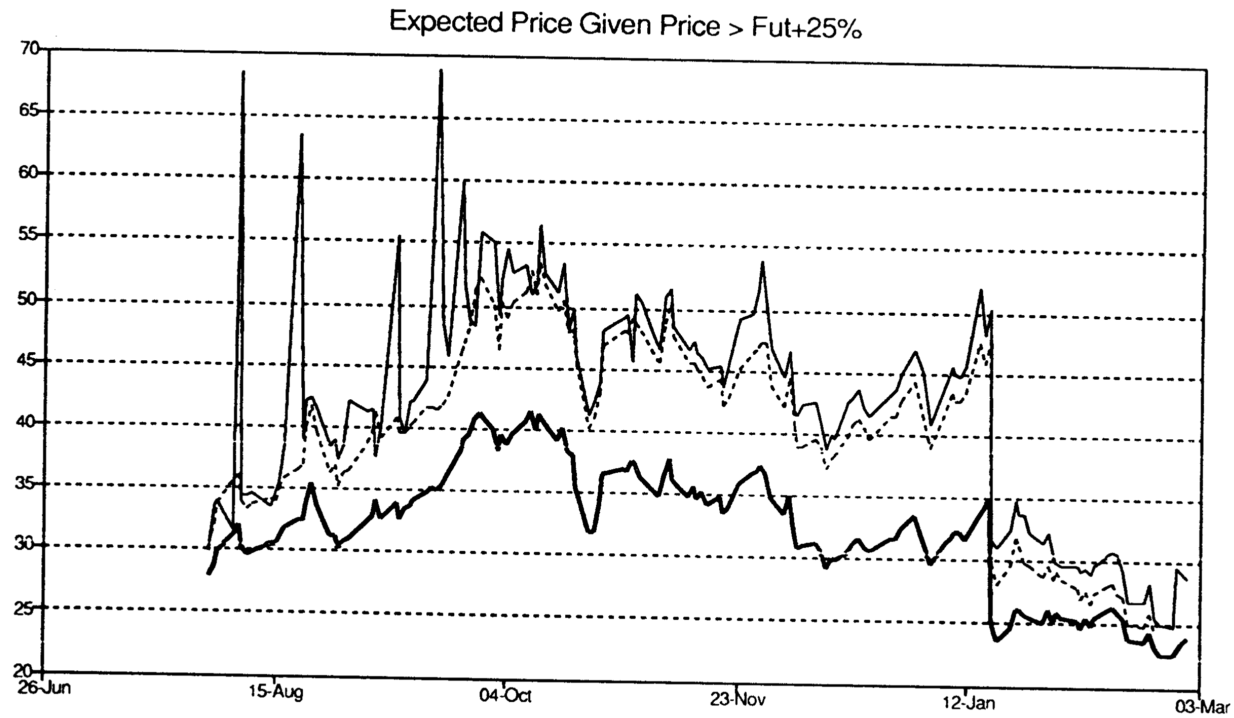
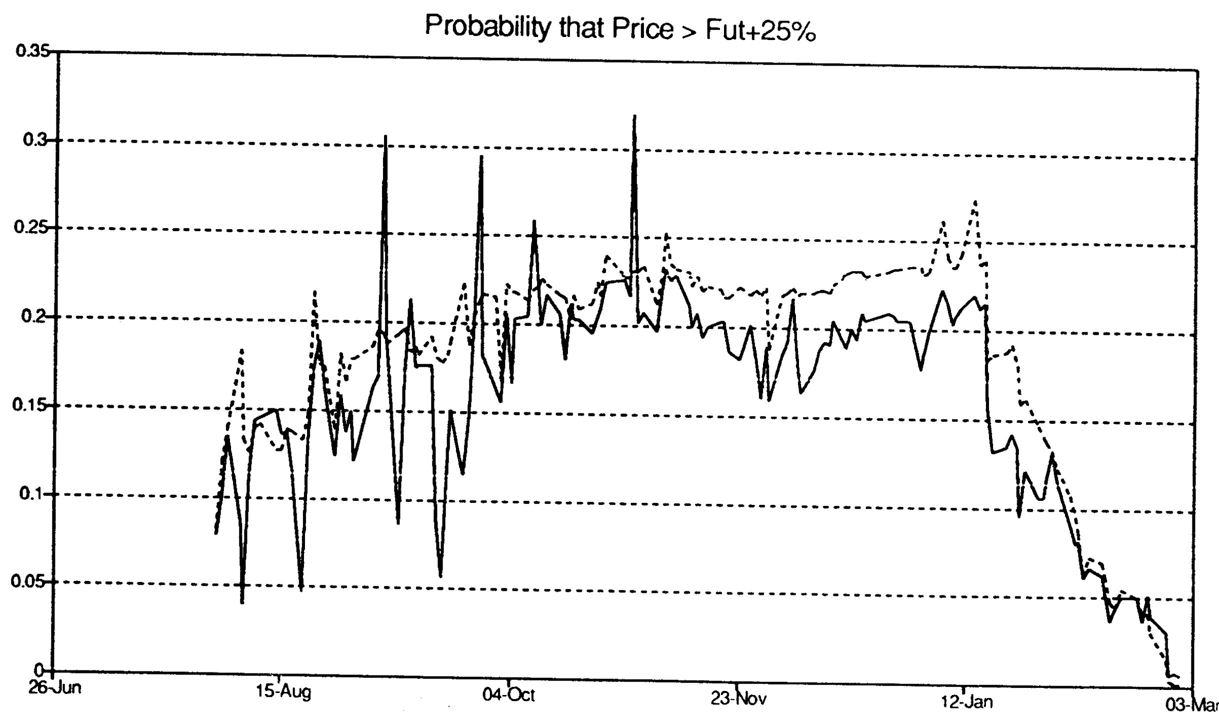


Chart 6

April Contract

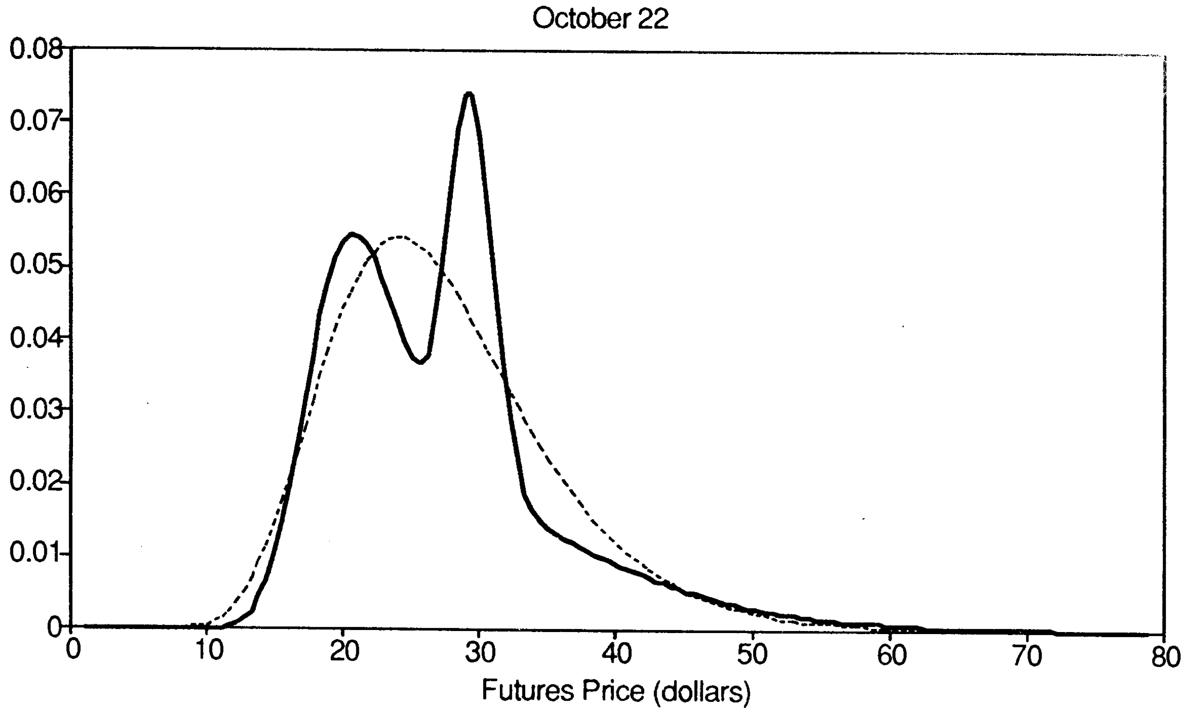




MLN .... SLN Future + 25%

Chart 7

Mining of Wells: Density Functions from April Contract



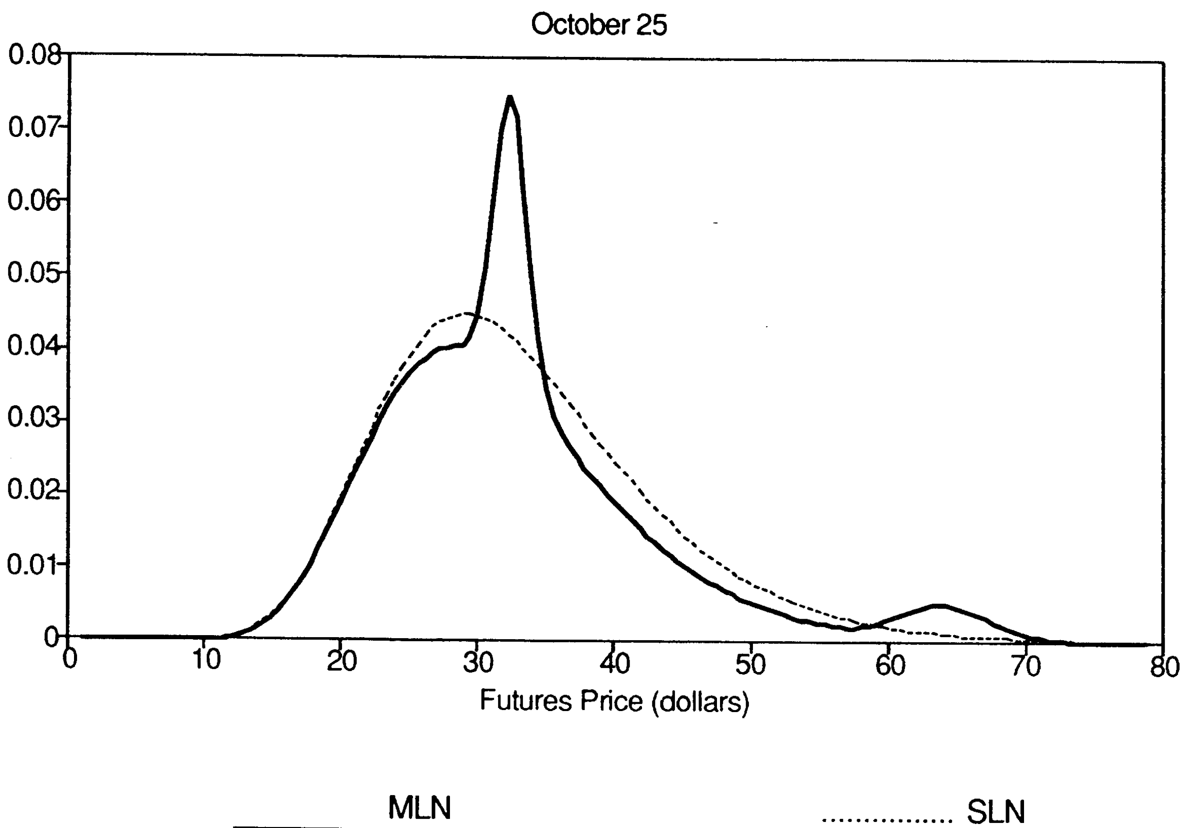
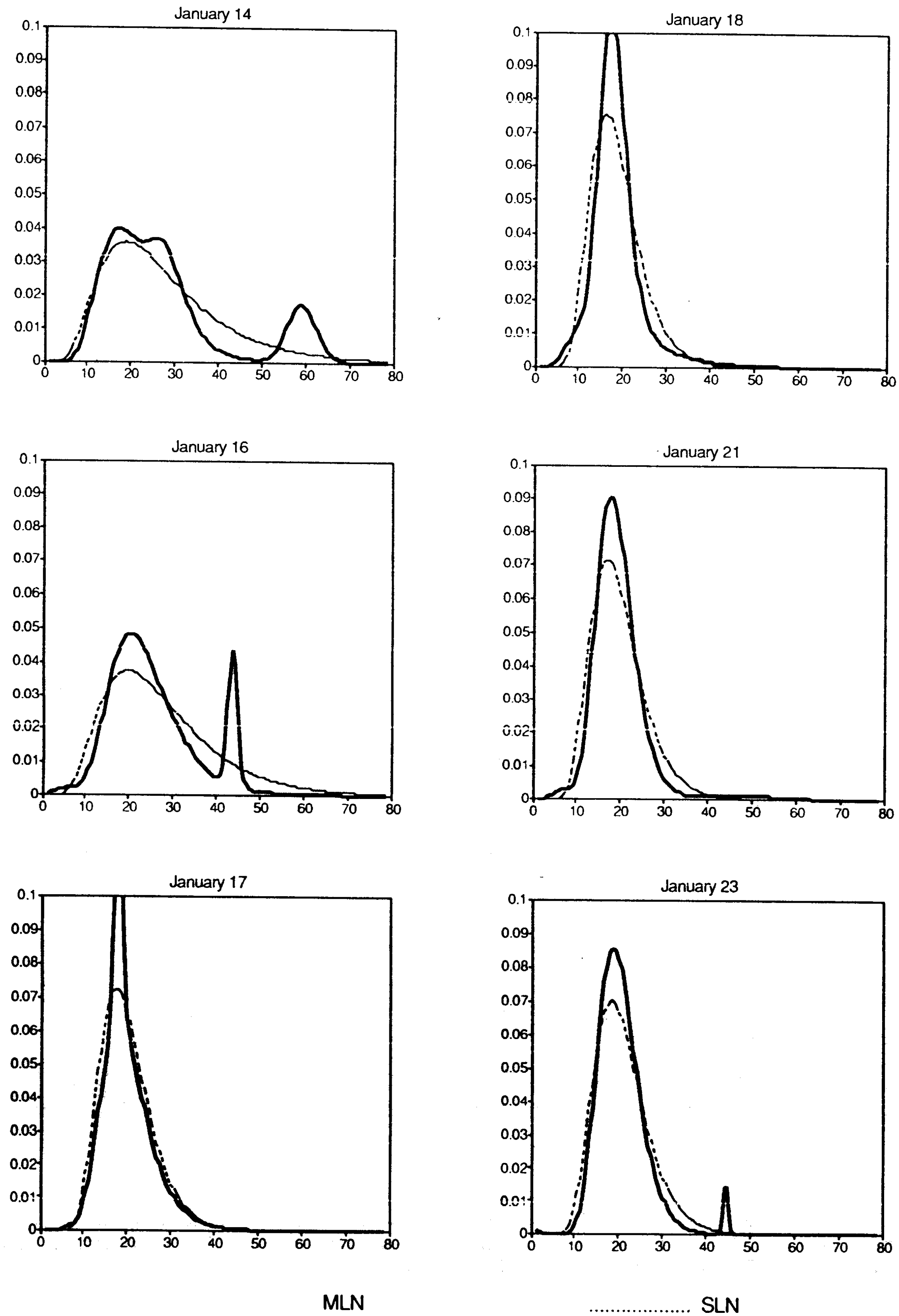


Chart 8

Air War: Density Functions from April Contract



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