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ABSTRACT

This paper offers a way of efficiently estimating the parameters in demand functions for mark and dollar assets. The technique imposes the constraint that the parameters, rather than being determined arbitrarily, are based on investors' optimizing behavior regarding expected returns and variances. It dominates some previous empirical applications of finance theory in the respect that the expected returns are allowed to vary from period to period, which is a necessary feature of any macro model. The constraint that the parameters are based on mean-variance optimization is also tested, and not rejected.

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1. Introduction

One of the most rapidly-progressing sub-areas of international finance theory recently has been the application of the principles of expected utility maximization to the problem of international asset demand functions. Investors balance their portfolios between domestic and foreign assets as a function of the expected relative rate of return, that is, the interest differential in excess of expected exchange rate depreciation. The function itself is shown to depend on parameters such as the variance of the exchange rate and the degree of risk-aversion.

Substantive theoretical results include the following. (1) Only the supply of "outside" assets matters. For example, if residents of different countries consume a common basket of goods, then a current account imbalance that redistributes wealth among countries has no effect on aggregate asset demand or supply, and thus no effect on the relative price of domestic and foreign assets. (2) The portfolio share that is optimally allocated to a given country's assets can be expressed, with suitable assumptions, in a simple linear form: as the sum of a "minimum-variance" portfolio share that depends on the share of the consumption basket allocated to that country's goods and a "speculative" portfolio share that depends on the expected rates of return and the degree of risk aversion. (3) A necessary condition for domestic residents to have a greater propensity to hold domestic assets than the foreign residents, and thus for a current account surplus to increase the relative price of domestic assets, is that domestic residents have a greater propensity to consume domestic goods. However an additional necessary condition is that the coefficient of relative risk-aversion be greater than unity. Some specific references on the three results are (1) Frankel (1979), (2) Dornbusch (1980), and (3) Krugman (1980). However these points were made earlier, implicitly or explicitly, by Kouri (1976, 1977) and Kouri and de Macedo (1978).[1]

The empirical literature in this area has lagged behind the theoretical. Several people have taken the tests of the Capital Asset Pricing Model (CAPM) that have been developed for other financial markets and have extended them to foreign currencies.[2] These finance studies vary according to whether they take the relevant "market basket" to include real assets (equities) or nominal assets (bonds). They also vary according to what numeraire the investor is assumed to consider as riskless--domestic currency, domestic goods, or a basket of domestic and foreign goods--and according to whether investors who live in different countries are assumed to consider different numeraires as riskless.

But the finance studies all make the assumption that the expected currency returns perceived by investors are constant over time, and that the variances and covariances are constant as well. This assumption is made (usually implicitly) in order to be able to estimate the parameters from ex post sample data. In the case of the variances and covariances,

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[1] Other contributors to the literature include Adler and Dumas (1978), Grauer,Litzenberger and Stehle (1976), Fama and Farber (1979), Solnik (1973), Garman and Kohlhagen (1980), Stulz (1980) and Hodrick (1981).

[2] Examples are Roll and Solnik (1977), Cornell and Dietrich (1978), Kouri and Macedo (1978), Macedo (1980), and Dornbusch (1980b).

the stationarity assumption is perfectly appropriate. It is necessary if the parameters of the asset-demand functions are to be considered unchanging over time. However in the case of the expected returns, the stationarity assumption, while it may be appropriate from a micro CAPM perspective, is not appropriate for a macro model. It would imply that the arguments and values of the asset-demand functions, as opposed to the parameters of the functions themselves, are constant over time. It is an essential element of most macro models that expected returns, and thus asset demands, be allowed to vary over time.

This point is given extra practical relevance by the striking reversals in trend which the most important exchange rates have undergone in recent years. In 1977 and 1978 the dollar was depreciating steadily against the mark and other currencies. This fact partly explains why previous estimates of the optimal portfolios have given a larger-than-expected weight to the mark and a smaller-than-expected weight to the dollar.[3] But, as of 1981, the mark is down sharply and the

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[3] Dornbusch (1980b, p. 165) estimates that an optimal portfolio of these two currencies would be 56% in marks, of which 50% represents a minimum-variance portfolio and 6% represents speculation to exploit the higher return on the mark over the sample period of 1976 1 to 1979 2. Kouri and Macedo (1978, p. 129) find that an optimal five-currency portfolio would include a 37% share in marks, of which 33% represents a minimum-variance portfolio and 4% represents speculation. In their study the dollar also benefits from speculation. The big loser is the pound, which depreciated sharply over their earlier sample period of 1973-77. (The negative share calculated in the optimal portfolio for a currency like the pound does not jibe with the positive supply of pounds known to exist in the world market. Thus we know that either the actual portfolio held by investors is not in fact equal to the optimal one, or else the method of calculation of the optimal portfolio is incorrect.)

dollar up. If the estimates were redone on more recent data, they would certainly give a larger weight to the dollar and a smaller weight to the mark. The important point is that expected returns do vary over time, and any estimates that neglect this are suspect.[4]

Thus we require a measure of expected returns that can vary over time, rather than relying on the sample mean, and we require a measure of variances and covariances that computes squared deviations around this varying expected value, rather than computing deviations around the sample mean. At first this might appear to be asking the impossible. But this paper offers a strategy for constructing exactly such measures. The strategy involves imposing the constraint that actual asset-demand functions are in fact based on mean-variance optimization on the part of investors. It thus produces more efficient estimates of the parameters than earlier studies that do not impose any constraints regarding from where the asset-demand functions are derived.[5] The strategy also allows us to test formally the proposition that asset-demand functions are based on mean-variance optimization. We simply compare the likelihood with the constraint imposed, to the likelihood unconstrained.

The next section of this paper shows how asset-demand functions can be estimated, in a world in which investors have different preferences depending on their country of residence, _without_ imposing the constraint

[4] One study of the optimal portfolio, by von Furstenberg (1981), does allow investors' expected returns to change each period; they are computed from the data observed up until the period in question.

[5] This refers to macro studies in which asset supplies are present as explanatory variables. In most portfolio-balance studies, the dependent variable is the exchange rate; examples include Branson, Halttunen and Masson [1977, 1979]. But the dependent variable is the relative rate of return (i.e. exchange rate depreciation in excess of the interest differencial) in Dooley and Isard [1979] and Frankel [1981], as it is in the present paper.

of mean-variance optimization. Section 3 derives theoretically the optimizing form of the functions. Section 4 estimates the asset-demand functions subject to the constraint that they are indeed of this form. In a comparison of the unconstrained likelihood from Section 2 with the constrained likelihood from Section 4, one is statistically unable to reject the constraint. This evidence would tend to support the hypothesis that actual asset-demand functions are indeed optimizing. However, the power of the test is probably very low. Thus the contribution of the paper may lie primarily in the estimation framework, which is of general macroeconomic applicability. Section 5 briefly discusses extensions of the framework, in particular relaxations of two simplifying assumptions made in the paper: (a) the limitation of the portfolio to two assets - dollar bonds and mark bonds, and (b) the assumption that exchange rate variability is the only source of uncertainty, for example because prices are "sticky" in the short run.

2. Estimation of Unconstrained Asset-Demand Functions

In this paper we assume that investors allocate their portfolio between mark bonds and dollar bonds only. Let x_{it} be the share allocated to marks, by residents of country i at time t. The asset-demand function β_i gives us the demand as a function of the interest rate on marks i_t^{DM}, the interest rate on dollars $i_t^{\$}$, and the expected depreciation of the mark (from time t to time t+1) Δs_t^e:

$$x_{i_t} = \beta_i \, (i_t^{DM}, \, i_t^{\$}, \, \Delta s_t^e) \, .$$

More specifically, assume that β_i is linear in the expected relative return z_t^e:

$$x_{i_t} = a_i + b(z_t^e) \; , \quad a_i > 0 \; , \quad b > 0 \tag{1}$$

where $z_t^e \equiv i_t^{DM} - i_t^{\$} - \Delta s_t^e$. This functional form is assumed for two reasons: (1) <u>some</u> form must be assumed for estimation, and (2) mean-variance optimization implies such a linear form, as will be seen in the next section. But the important point is that we are not constraining the parameters a and b to be anything in particular. They could be based on investors' arbitrary "tastes" for assets as easily as on mean-variance optimization. Of course we have already restricted the function somewhat; for example many macroeconomic models include real income levels, representing a transactions demand for the assets.[6]

If all investors in the market had the same preferences, then we could

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[6] I tested various other possibilities such as including income in Frankel [1981], on which this section is based.

estimate equation (1) by itself.[7] But, in general, asset-demand functions certainly vary, residents of each country having a relatively greater preference for their own currency. For the purposes of this paper we distinguish among residents of Germany ($i = G$), the United States ($i = US$) and the rest of the world ($i = R$). The mark shares of the three countries' portfolios are given by

$$M_{G_t}/W_{G_t} = a_G + b(z_t^e)$$

$$M_{US_t}/W_{US_t} = a_{US} + b(a_t^e)$$

$$M_{R_t}/W_{R_t} = a_R + b(z_t^e),$$

where $M_{i_t} \equiv$ holdings of marks by residents of country i, and $W_{i_t} \equiv$ total wealth (marks and dollars) held by residents of country i, expressed in marks. Presumably $a_G > a_R > a_{US}$.[8]

If we had data on mark assets broken down by country of holder, we could estimate each of these equations separately. But given adequate data on only the aggregate supply of marks M_t, we must aggregate the three equations. We do this by defining the countries' shares in aggregate world wealth W_t: $w_{G_t} = W_{G_t}/W_t$, $w_{US_t} = W_{US_t}/W_t$ and $w_{R_t} = W_{R_t}/W_t$. We multiply each equation by that country's w_{i_t} and add them up. We

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[7] Two common kinds of models satisfy this description. Some, like Branson, Haltunnen and Masson [1977], assume that domestic residents are the only ones who hold domestic assets, and thus are the only investors who count in the market. Others, like Frankel [1979] and Dornbusch [1980a], assume that all investors, domestic and foreign alike, share the same preferences. In Frankel [1980] I call the former "small-country" portfolio-balance models and the latter "uniform-preference" portfolio-balance models, and give further references.

[8] We are assuming that the responsiveness with respect to expected returns, b , is the same for all investors. In the next section this will be seen to hold if all have the same degree of risk-aversion. (It is also necessary if we are to have $x_G > x_R > x_{US}$ for all z .)

thus obtain an expression for the share of the world portfolio occupied by marks:

$$x_t \equiv M_t/W_t = a_G w_{G_t} + a_{US} w_{US_t} + a_R(1 - w_{G_t} - w_{US_t}) + b(z_t^e), \tag{2}$$

where we have used the fact that $w_{G_t} + w_{US_t} + w_{R_t} = 1$.

Equation (2) is simply a weighted average of equation (1) over the three countries, where the weights are their shares in world wealth.[9]

So far we have not said anything about measuring z_t^e, the expected relative rate of returns on mark assets. This is not a trivial task, in light of the nonobservability of expected depreciation, and our unwillingness to assume it constant as in the previous finance studies.

The solution adopted here is to invert equation (2), so that the expected relative return is expressed as a function of the asset supplies and the distribution of wealth:

$$z_t^e = -\left(\frac{a_R}{b}\right) - \left(\frac{a_G - a_R}{b}\right) w_{G_t} + \left(\frac{a_R - a_{US}}{b}\right) w_{US_t} + \frac{1}{b} x_t \tag{3}$$

Let us check the economic relationships in (3). If the relative supply of marks x_t is high, they must pay a high expected relative return z_t^e in order to be willingly held. Also if world wealth is redistributed toward U.S. residents (e.g. by a current account surplus with the rest of the world), then the relative return on marks z_t^e must rise, in order for them to be willingly held, because U.S. residents have a lower preference for them $(a_R - a_{US} > 0)$. On the other hand if world wealth is redistributed toward German residents (e.g. by a current account surplus with the rest of the world), then z_t^e must fall, because German residents have a higher

[9] This trick is borrowed from Dornbusch [1980a, appendix]. Notice that in the case $a_G = a_R = a_{US}$, equation (2) reduces to the "uniform-preference" model mentioned in footnote 7.

preference for marks ($a_G - a_R > 0$).

To deal with the unobservability of expectations, we make the assumption that investors form them rationally. The ex post relative return ($z_{t+1} \equiv i_t^{DM} - i_t^{\$} - \Delta s_t$), which is observable, is assumed equal to the expected return z_t^e plus a random error ε_{t+1}. By "random", we mean uncorrelated with all information available at the beginning of the period over which the return is measured:

$$z_{t+1} = z_t^e + \varepsilon_{t+1}, \quad E(\varepsilon_{t+1} \mid I_t) = 0 .$$

Substituting into (3),

$$z_{t+1} = -(\frac{a_R}{b}) - (\frac{a_G - a_R}{b})w_{G_t} + (\frac{a_R - a_{US}}{b})w_{US_t} + \frac{1}{b}x_t + \varepsilon_{t+1} . \qquad (4)$$

The parameters of equation (4) can now be estimated by regression. All the variables (z_{t+1}, w_{G_t}, w_{US_t}, and x_t) are observable. And the regression error is simply the expectational error ε_{t+1}, which we know to be uncorrelated with the right-hand-side variables by the assumption of rational expectations. Indeed, the reason we inverted equation (2) to begin with was so that ε_{t+1} would enter the left-hand-side variable rather than one of the right-hand-side variables, allowing us to use regression estimation.[10]

Table 1 reports regressions of equation (4) for the period January 1974 to October 1978. Data are discussed in appendix 2. Unfortunately,

[10]Note the importance of the strong assumption that the asset-demand function (1) is correctly specified, so that the only source of regression error is the expectational error. If the asset stocks are measured with error, or if any other determinants of asset demands have been omitted, then a regression of equation (4) will produce estimates that are biased and inconsistent. But the defense of this procedure is that, as a way of estimating asset-demand functions, it is a step forward from the typical finance studies, which would require not only an absence of error terms other than the expectational error, but also the assumptions that (a) the actual functions are based on mean-variance optimization, and (b) the expected returns are constant. In this paper we rule out the latter and consider the former open to testing.

even the few implications of our hypothesis so far - a negative constant term and coefficient on German wealth, and positive coefficients on U.S. wealth and the supply of marks - are not borne out. The coefficients are generally wrong in sign, and always statistically insignificant. In light of the high standard errors, there seems no purpose in unscrambling the point estimates to obtain estimates of the original parameters a_G, a_{US}, a_R and b, even though they are fully identified.[11]

However, one assumption that we have already made is borne out. The absence of serial correlation in the error term supports the hypothesis of rational expectations.

Perhaps the main lesson to be drawn from Table 1 is the very low degree of precision that plagues estimation of general portfolio-balance equations, and the need to bring additional information to bear. This provides the motivation for considering the constraints placed on the parameters by the hypothesis, developed in the following section, that they are derived from mean-variance optimization by investors. If one believes this hypothesis, then the resulting estimates will be more precise.

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[11]We should not be concerned with the very high sums of squared residuals-and consequent very low R^2s - in Table 1 (and later, in Table 2). The empirical literature on the forward exchange market has shown that deviations of the forward discount (or, equivalently, the interest differential) from ex post spot depreciation are enormous, regardless whether they are random. Thus we would expect a high SSR and low R^2 even if the explained sum of squares were significantly greater than zero.

Table 1: Unconstrained Asset-Demand Functions

OLS

Dependent Variable: z_{t+1} , relative return on marks

z_{t+1} measured as: German-U.S. interest differential minus depreciation of mark

Sample: Jan. 1974 - Oct. 1978 (58 obs.)

Asset supplies measured as:	Coefficients: Constant	w_{G_t}	w_{US_t}	x_t	D.W.	R^2	SSR	$V(\hat{\varepsilon})$*	log likelihood*
Total assets	.246 (.256)	.138 (.731)	-.133 (.188)	-.654 (1.122)	1.82	.04	.05617	.0009684 [.00104]	118.96 [116.88]
Bonds only	.152 (.149)	.152 (.346)	-.064 (.098)	-.774 (.909)	1.82	.05	.05598	.0009651 [.00104]	119.05 [116.98]
Monetary base only	-.083 (.191)	-.144 (.249)	.028 (.033)	.354 (.948)	1.87	.03	.05703	.0009832 [.00106]	118.52 [116.45]

z_{t+1} measured as: forward discount minus depreciation of mark

Sample: March 1974 - October 1978 (56 obs.)

Asset Supplies measured as:	Coefficients: Constant	w_{G_t}	w_{US_t}	x_t	D.W.	R^2	SSR	$V(\hat{\varepsilon})$*	log likelihood*
Total assets	.313 (.985)	.581 (.758)	-.210 (.194)	-1.207 (1.156)	1.92	.04	.05014	.0008953 [.0009643]	117.05 [114.97]

(Standard errors are reported in parentheses.)

* Maximum likelihood estimates of the variance and log likelihood are reported.

[Unbiased estimates are reported in brackets.]

3. Derivation of Asset-Demand Functions from Mean-Variance Optimization

In this section we derive the correct form for the asset-demand of an investor who maximizes a function of the mean and variance of his end-of-period real wealth.[12] The analysis is lifted wholesale from Dornbusch [1980a]. The reader familiar with that paper or with the general approach, which is standard in the CAPM literature, is urged to skip to the next section, and thus to conserve his patience for the rest of the journey ahead.

Let $\tilde{W}_{i_t}$ be the real wealth of investors of country i. Their decision variable is x_{i_t}, the fraction of the portfolio that they put into mark assets. The real rate of return on the mark assets is $i_t^{DM} - \pi_{i_{t+1}}^{\$} - \Delta s_{t+1}$ and the real rate of return on dollar assets is $i_t^{\$} - \pi_{i_{t+1}}^{\$}$, where i_t^{DM} is the current one-period interest rate on mark assets, $i_t^{\$}$ is the current one-period interest rate on dollar assets, $\pi_{i_{t+1}}^{\$}$ is the dollar-denominated inflation rate in the goods consumed by residents of country i, and Δs_{t+1} is the depreciation rate of the mark between times t and t + 1. Thus end-of-period wealth depends on x_{i_t} and the realized rates of return:

$$\tilde{W}_{i_{t+1}} = \tilde{W}_{i_t}\{x_{i_t}(i_t^{DM} - \pi_{i_{t+1}}^{\$} - \Delta s_{t+1}) + (1 - x_{i_t})(i_t^{\$} - \pi_{t+1}^{\$}) + 1\}$$

$$= \tilde{W}_{i_t}\{x_{i_t}(i_t^{DM} - i_t^{\$} - \Delta s_{t+1}) + i_t^{\$} - \pi_{i_{t+1}}^{\$} + 1\}. \tag{5}$$

If the relative return on marks is positive, end-of-period wealth is an increasing function of x_{i_t}. Notice that the inflation rate drops out of the relative return.

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[12] The assumption that returns are normally distributed is sufficient to imply that investors look only at the mean and variance. The normality assumption might be justified by an appeal to Brownian motion observed at discrete intervals, and is necessary for the maximum likelihood estimation in any case.

We define the dollar-denominated inflation index to be a weighted average of dollar-denominated inflation in German-produced goods and dollar-denominated inflation in U.S.-produced goods, with the weights α_i and $(1 - \alpha_i)$ equal to shares in consumption:

$$\pi^{\$}_{i_t} = \alpha_i(\pi^{DM}_{G_t} - \Delta s_t) + (1 - \alpha_i)(\pi^{\$}_{US_t}) .$$

Now we make a major simplifying assumption: goods prices are non-stochastic when denominated in the currency of the producing country. Only the exchange rate is uncertain.[13] Then the mean and variance of end-of-period wealth (5) are as follows:

$$E(\tilde{W}_{i_{t+1}}) = \tilde{W}_{i_t}\{x_{i_t}(i^{DM}_t - i^{\$}_t - E(\Delta s_{t+1})) + i^{\$}_t - [\alpha_i(\pi^{DM}_{G_{t+1}} - E(\Delta s_{t+1})) + (1 - \alpha_i)(\pi^{\$}_{US_{t+1}})] + 1\}$$

$$V(\tilde{W}_{i_{t+1}}) = \tilde{W}_{i_t}{}^2\{[- x_{i_t} + \alpha_i]^2 V(\Delta s_{t+1})\} .$$

The hypothesis is that investors maximize a function of the mean and variance

$$U[E(\tilde{W}_{i_{t+1}}), V(\tilde{W}_{i_{t+1}})] .$$

We differentiate with respect to x_{i_t} :

$$\frac{dU}{dx_{i_t}} = U_1\frac{dE(\tilde{W}_{i_{t+1}})}{dx_{i_t}} + U_2\frac{dV(\tilde{W}_{i_{t+1}})}{dx_{i_t}} = 0$$

$$U_1\tilde{W}_{i_t}(i^{DM}_t - i^{\$}_t - E(\Delta s_{t+1})) + U_2\tilde{W}_{i_t}{}^2 2[x_{i_t} - \alpha_i]V(\Delta s_{t+1}) = 0$$

$$i^{DM}_t - i^{\$}_t - E(\Delta s_{t+1}) = (- \frac{U_2}{U_1}\tilde{W}_{i_t} 2)V(\Delta s_{t+1})[x_{i_t} - \alpha_i] .$$

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[13] This assumption is made by Krugman, but is considered only one special case by Kouri [1976] and Dornbusch [1980a]. Assuming that prices are sticky, at least in the short run - and in this case we are talking about one month - is of course standard in Keynesian macroeconomics.

The expectation and variance of the exchange rate change are conditional on all information available at time t. We use the notation of section 2, in which z_{t+1} was defined to be the relative return on marks and ε was defined to be the expectational error $(\Delta s_{t+1} - E\Delta s_{t+1})$. Also we define ρ to be the measure of relative risk-aversion: $\rho \equiv 2U_2\tilde{W}_{i_t}/U_1$.[14] Thus our expression for the expected relative return on marks is

$$E(z_{t+1}) = \rho V(\varepsilon)[x_{i_t} - \alpha_i] . \tag{6}$$

This expression for the expected relative return is analogous to the equation that was estimated in the previous section, except that we have not yet aggregated over the countries. But let us invert to get the form analogous to the earlier asset-demand function:

$$x_{i_t} = \alpha_i + \frac{1}{\rho V(\varepsilon)} E(z_{t+1}) \tag{7}$$

Compare this equation to our general unconstrained asset-demand function (1). The two are the same, with $a_i \equiv \alpha_i$ and $b \equiv \frac{1}{\rho V(\varepsilon)}$. Following Kouri and Macedo, Dornbusch [1980a] calls the first term the minimum-variance portfolio. If the investor is highly risk-averse $(\rho = \infty)$ or

[14] The Arrow-Pratt measure of risk aversion is defined as $\rho \equiv -u''W/u'$, where u(W) is the utility function, the expectation of which is to be maximized. One can take a Taylor-series approximation to Eu(W) and differentiate it with respect to E(W) and V(W) to show that the two definitions of ρ are equivalent.

The utility function will have a constant coefficient of relative risk-aversion if it is exponential in form:
$u(\tilde{W}) = \frac{1}{\gamma} \tilde{W}^{\gamma}$, where $\rho = 1 - \gamma$.
(The solution to the one-period maximization problem considered here will be the correct solution to the general intertemporal maximization problem, if the utility function is further restricted to the logarithmic form, the limiting case as γ goes to zero, which implies $\rho = 1$, or if events occurring during the period are independent of the expected returns that prevail in the following period.)

the returns are very uncertain ($V(\varepsilon) = \infty$), he will hold marks and dollars in the same proportions as he consumes German and U.S. goods. Of course, he would have to give up some expected return to hold the minimum-variance portfolio. They call the second term the speculative portfolio. A higher expected relative return on marks induces investors to hold more of them than the minimum-variance portfolio, to an extent limited only by the variance of the exchange rate and the degree of risk-aversion. For example, under risk-neutrality ($\rho = 0$), the two assets become perfect substitutes and arbitrage insures that $E(z_{t+1}) = 0$.

Before we proceed to the estimation of the parameters of equation (6) or (7), which is the main point of this paper, we must add a footnote to the foregoing derivation, pointed out by Krugman: equations (5) to (7) involve some sleight-of-hand. Is the expected rate of mark depreciation, which enters Ez_{t+1}, defined as the percentage increase in the mark cost of dollars $E(S_{t+1}/S_t)-1$? Or is it the percentage decrease in the dollar cost of marks $-[E(S_t/S_{t+1}) - 1]$? The two are not equivalent, by Jensen's inequality. The latter definition is correct only if $\alpha = 0$. For example, under risk-neutrality ($\rho = 0$), dollar assets and mark assets will have the same expected purchasing powers in this case, i.e. in terms of U.S. goods; then $E\, z_{t+1}$ so defined is zero. But <u>to the extent that investors consume German goods</u> ($\alpha > 0$), <u>the variance of the mark/dollar exchange rate will have a positive effect on the expected purchasing power of dollar assets</u>, due to Jensen's inequality. Thus the variance should enter $E\tilde{z}_{t+1}$ even under risk-neutrality.

Similarly, the former definition is correct only if $\alpha = 1$. For example, under risk-neutrality dollar and mark assets will have the same expected purchasing powers, in this case in terms of German goods; then $E\, z_{t+1}$ so defined is zero. But to the extent that investors consume U.S. goods $(\alpha < 1)$, the variance of the dollar/mark exchange rate will have a positive effect on the expected purchasing power of mark assets, due to Jensen's inequality. Thus, again, the variance should enter $E\, z_{t+1}$ even under risk-neutrality.

This rather counter-intuitive, but important, point of Krugman's is discussed, and the estimation technique is modified accordingly, in Appendix 1. We stick with the Dornbusch formulation in the text, as it is more intuitive. The ultimate finding of this paper, a statistical inability to reject the hypothesis that asset-demand functions are based on mean-variance optimization, is the same in either formulation.

4. Estimation of Asset-Demand Functions Constrained to be Optimizing

In the last section we found that the linear form assumed in equation (1) is mean-variance optimizing, provided

$a_i = \alpha_i$, the share of consumption occupied by German goods,

and $b = \frac{1}{\rho V(\varepsilon)}$, where ρ is the constant of relative risk-aversion and $V(\varepsilon)$ is the variance of exchange rate prediction errors. To aggregate across residents of the three countries, Germany, the United States, and the rest of the world, we can substitute directly into equation (4), the pre-constrained aggregate form:

$$z_{t+1} = -\rho V(\varepsilon)\alpha_R - \rho V(\varepsilon)(\alpha_G - \alpha_R)w_{G_t} + \rho V(\varepsilon)(\alpha_R - \alpha_{US})w_{US_t} + \rho V(\varepsilon)x_t + \varepsilon_{t+1} \tag{8}$$

We could simply estimate equation (8) by OLS. If we use actual import and consumption averages for α_R , α_G and α_{US} , then the coefficients are overidentified (by three). This is what we want; overidentifying restrictions are necessary for hypothesis-testing. Table 2 presents regressions of equation (8) using the same data sample as Table 1. The constraints are imposed by constructing the variable

$$y_t = -\alpha_R - (\alpha_G - \alpha_R)\, w_{G_t} + (\alpha_R - \alpha_{US})\, w_{US_t} + x_t$$

and then running the regression

$$z_{t+1} = \frac{1}{b}\, y_t + \varepsilon_{t+1} .$$

The shares of consumption allocated to German, as opposed to U.S., goods, were taken to be $\alpha_G = 0.986$, $\alpha_R = 0.469$ and $\alpha_{US} = 0.005$, as explained in appendix 2.

Table 2: Constrained Asset - Demand Functions (Dornbusch version)

OLS

Dependent Variable: z_{t+1} , relative return on marks

Independent Variable: $y_t \equiv -\alpha_R - (\alpha_G - \alpha_R)w_{G_t} + (\alpha_R - \alpha_{US})w_{US_t} + x_t$

z_{t+1} measured as: German - U.S. interest differential minus depreciation of mark

Sample: Jan. 1974 - Oct. 1978 (58 obs.)

Asset supplies measured as:	Coefficient $\rho V(\hat{\varepsilon})$	D.W.	R^2	SSR	$V(\hat{\varepsilon})$ *	log likelihood *
Total assets	.035 (.033)	1.88	-.00	.05877	.001013 [.001031]	117.64 [117.14]
Bonds only	.015 (.014)	1.88	-.00	.05873	.001013 [.001030]	117.66 [117.16]

z_{t+1} measured as: forward discount minus depreciation of mark

Sample: Mar. 1974 - Oct. 1978 (56 obs.)

Asset supplies measured as:	Coefficient $\rho V(\hat{\varepsilon})$	D.W.	R^2	SSR	$V(\hat{\varepsilon})$ *	log likelihood *
Total assets	.019 (.033)	1.98	-.00	.05216	.0009314 [.0009480]	115.94 [115.44]

(Standard errors are reported in parentheses.)

* Maximum likelihood estimates of the variance and log likelihood are reported. [Unbiased estimates are reported in brackets.]

First consider a formal test of the constraint. To illustrate, we take the last regression, in which z_t is calculated using the forward exchange rate rather than the interest differential. The constrained log likelihood is 115.94 . The unconstrained log likelihood in Table 1 was 117.05 . Twice the difference is distributed χ^2 with three degrees of freedom. We easily fail to reject the hypothesis that actual asset-demand functions are in fact based on mean-variance optimization. (Intuitively, the imposition of the three constraints worsened the fit, i.e. the sum of squared residuals, very little.)

If we accept the optimization hypothesis either on a priori grounds or on the basis of the likelihood ratio test, then we can use it to get efficient estimates of the parameters. The estimate of $\frac{1}{b}$, 0.019 , is presumably more efficient than its estimate in Table 1 (the coefficient of x_t). Indeed, unlike before, it is now of the correct positive sign. For example, an increase in the expected relative return on marks z of 100 basis points would raise the demand for marks by .019% of the aggregate portfolio. If asset supplies were unchanged, this point estimate would imply (as of May 1980, when the share of the portfolio in marks x happened to be .19038) an appreciation of the mark of .023%. Unfortunately the standard error is still too high for the estimates to be statistically significant.

Under the optimization hypothesis, $\frac{1}{b} = \rho V(\varepsilon)$. Let us say we are willing to ignore the high standard error and use the point estimate for $\widehat{\frac{1}{b}} = 0.019$; then how can we separate out ρ from $V(\varepsilon)$? The introduction promised a measure of the expected relative return that was allowed to change over time and an estimate of the variance around the changing expected value, rather than around the sample mean. Under the hypotheses we have adopted (rational expectations and optimization), an efficient estimate of the expected relative return z_t^e is simply the fitted value from the regression in Table 2, $\hat{z}_{t+1} = \frac{\hat{1}}{b} y_t$. An efficient estimate of the expectational error is simply the residual $\hat{\varepsilon}_{t+1}$. And an efficient estimate of the variance is simply the variance of the residual. In Table 2 the variance is 0.0009314 . We also get an estimate of the coefficient of relative risk aversion: $\hat{\beta} = \frac{0.0187}{0.0009314} = 20.08$.

The high standard errors attached to the coefficients are a consequence of (1) the small numbers of overidentifying restrictions imposed by the hypothesis, relative to the number of observations, and (2) the high sum of squared residuals (SSR). Given the high SSR in the unconstrained case, a high SSR in the constrained case was inevitable. To obtain more efficient estimates, we would need additional overidentifying restrictions, the prospect of which arises in the next and final section.

5. Future Research

In the foregoing analysis two assumptions have been made that it would be desirable to relax in the interest of greater realism: (a) the limitation of the portfolio to two assets, and (b) the non-stochastic nature of goods prices.

Relaxing the first of these assumptions, i.e. extending the number of assets, not only is more realistic, but turns out to have the additional advantage that the hypothesis of optimization then imposes additional constraints. This should allow a more powerful likelihood ratio test of the hypothesis than in the present paper. And, assuming we continue to accept the optimization hypothesis, imposing the constraints should earn us more precise estimates than in the present paper.

The outcome of the optimization problem in the multiasset case, analogous to equation (6), turns out to be simply

$$z_{t+1} = \rho\Omega[x_t - \alpha] + \varepsilon_{t+1}$$

where z_{t+1}, x_t, α and ε_{t+1} are now vectors of dimension $n-1$ = the number of assets (not counting the dollar). The testable constraint is that Ω (an $n-1 \times n-1$ matrix) is the variance-covariance matrix of ε, which is the vector of expectational errors pertaining to the relative returns on the various assets. If the constraints are to be imposed, the parameters cannot be estimated by OLS, but instead must be estimated by maximum likelihood. Nor is the programming problem as easy as it is in the M.L.E. of the Krugman version of the one-dimensional equation, discussed in Appendix 1. But results on the multicountry case should be ready soon.

To relax the second assumption, the nonstochastic nature of goods prices, is a matter of explicitly deflating rates of return by inflation rates as measured by price indices. Up until now even the standard efficiency tests, for example tests of serial correlation in our z_t, have not used price data. Some of them recognize the relevance of Jensen's inequality (or "the Siegal Paradox"). But nevertheless they all simply test that the forward rate equals the expected future spot rate. Doing it right would mean testing that the forward rate equals the ratio of the expected future purchasing power of foreign currency to the expected future purchasing power of domestic currency.[15] The culmination of this line of research would be to repeat all the estimation in the present paper allowing uncertainty in the price levels as well as the exchange rate.

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[15]See Engel [1981]. Frenkel and Razin (1980) offer the first test to recognize explicitly that the equation for the expected future spot rate should include not only the forward rate but also a term representing the covariance of the spot rate with the purchasing power of domestic currency, even under the null hypothesis of market efficiency and risk-neutrality. However they calculate the covariance term with ex post data, implicitly assuming that the expected purchasing powers of the currencies are constant, which is inconsistent with the intrinsically present fact that the expected exchange rate varies. In other words, their approach to the efficiency hypothesis is subject to the same limitation as the literature on the mean/variance-optimization hypothesis cited in footnote 2.

APPENDIX 1: The Krugman Version

As explained at the end of Section 3, Krugman points out that Jensen's inequality is not merely a mathematical annoyance that can be swept away by an appeal to approximation, but is substantive to the question of how the parameters of the asset-demand functions depend on ρ and $V(\Delta s_t)$. His equation (18), translated from his continuous-time model to our discrete-time notation, is

$$Ez_{t+1} = \rho V(\Delta s_t)\, x_{i_t} - (\rho - 1)\, V(\Delta s_t)\, \alpha_i \tag{6'}$$

where depreciation Δs_t is defined as the percentage change in S (as opposed to the percentage change in 1/S). It differs from our equation (6) by the addition of a $V(\Delta s_t)\alpha_i$ term. An increase in the variance raises the expected purchasing power of dollar assets over German goods, due to Jensen's inequality. and thus (even under risk-neutrality) raises the necessary expected relative nominal return that must be paid on mark assets. When (6') is aggregated across the three countries of residence, we get a discrete-time version of Krugman's equation (21):

$$Ez_{t+1} = \rho V(\Delta s_t) X_t + (\rho - 1)\, V(\Delta s_t)[-\alpha_R - (\alpha_G - \alpha_R)\, w_{G_t} + (\alpha_R - \alpha_{US})\, w_{US_t}] . \tag{8'}$$

Unlike the Dornbusch form, our equation (8), equation (8') is not homogeneous in $\rho V(\Delta s_t)$. Thus we must take advantage of the constraint between the coefficients and the variance of the regression error <u>as part of the estimation process</u>. Imposing a constraint between coefficients and the variance of the error is unusual in econometrics. It cannot be done by OLS, but requires maximum likelihood estimation. Fortunately in the simple two-asset case, it is very easy to write down the likelihood function and to compute the values of ρ and $V(\Delta s_t)$ that maximize it.

The results are given in Table 3. As in the Dornbusch formulation, the reduction in the likelihood that results from imposing the constraint is very small; we are again unable to reject the hypothesis that the parameters of the asset-demand functions are based on mean-variance optimization. The likelihoods under the Dornbusch and Krugman formulations are so extremely close as to permit no possible inference as to which fits the data better. The point estimate of the variance is essentially the same as it was in the Dornbusch formulation. The point estimate of the constant of relative risk-aversion is somewhat larger than before.

Table 3: Constrained Asset-Demand Functions (Krugman version)

Maximum Likelihood Estimation

$$z_{t+1} = \rho V(\Delta s)\ x_t + (\rho - 1)\ V(\Delta s)[-\alpha_R - (\alpha_G - \alpha_R)w_{G_t} + (\alpha_R - \alpha_{US})w_{US_t}] + \varepsilon_{t+1}$$

Z_{t+1} measured as: forward discount minus depreciation of mark

Asset supplies measured as: total assets

Sample: March 1974 - Oct. 1978 (56 obs.)

$\hat{\rho}$	$\widehat{V(\Delta s)}$	log likelihood
27.53	.000931	115.59
(1.06)	(.000926)	

(Standard errors reported in parentheses.)

APPENDIX 2: Data

The total net supply of assets denominated in the currency of a particular country (Germany or the United States) was calculated as the stock of federal debt outstanding (whether monetized by the Central Bank or not) plus the Central Bank's cumulative sales of domestic assets in foreign exchange intervention (measured as its international reserve holdings corrected for valuation changes) minus a measure of the holdings by foreign central banks of the country's assets in the form of foreign exchange reserves. The net supply of bonds to the private market was calculated as the total net supply of assets minus the monetary base.

For purposes of distinguishing German-held wealth and U.S.-held wealth, in each country the current account surplus and federal debt were cumulated to arrive at the private sector's total claims on outside assets.

The data sample was ended at October 1978 because the calculation of reserve holdings becomes especially difficult after that date due to the issuing of mark-denominated Carter notes by the U.S. Treasury, the holding of foreign exchange reserves valued at current exchange rates by the Federal Reserve, and the turning over of reserves to the European Monetary System by the Bundesbank.

Further details on the above data calculations, and sources, are given in the appendix to Frankel [1981].

The tests were run once using the interest differential and once using using the forward discount. The two are theoretically the same, and in practice very close, by covered interest parity. In the former case, one-month Eurocurrency interest rates were used: the mean of the bid-ask spread on the last day of each month, as reported by the <u>Financial Times</u>

of London the first day of the following month. The exchange rate was also end-of-month, as reported in the IMF's International Financial Statistics. In the latter case, the 30-day forward rate and spot rate were both taken from the Wall Street Journal.

Computing α_G, α_{US} and α_R, the consumption shares allocated by residents of the various countries to German rather than U.S. goods, involves a number of arbitrary assumptions. The numbers used in this paper were computed as follows:

$$\alpha_{US} \equiv \text{(U.S. imports from Germany)/(U.S. income)} = 0.005$$

$$\alpha_G \equiv 1 - \text{(German imports from U.S.)/(German income)} = 0.986$$

$$\alpha_R \equiv \text{(R.O.W. imports from Germany)/(R.O.W. imports from Germany and U.S.)} = 0.469 .$$

The arbitrary assumptions are that all goods are denominated in the currency of the producer, and that R.O.W. currency values are uncorrelated with the mark/dollar exchange rate. The data are for 1974, taken from the IMF's Direction of Trade Annual 1969-75.

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