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THE RISK PREMIUM IN THE MARKET FOR FORWARD FOREIGN EXCHANGE

by

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1. Introduction

This paper attempts to provide a theoretical link between future spot prices and current forward prices in the market for foreign exchange. A general equilibrium model is developed where economies experience country specific real and nominal shocks. Risk averse agents use forward foreign exchange markets as insurance against these shocks. The implications of this behavior for the existence and nature of a risk premium are discussed.

The question of the existence and nature of a risk premium in the market for foreign exchange is of enormous practical importance. One reason is that the efficacy of open market operations may depend upon the answer. If relative purchasing power parity holds ex ante, and bonds, which differ only in the currency in which they are denominated, are regarded as perfect substitutes then expected real interest rates will be equated across countries. In a small country open market operations will have no effect on expected real interest rates. If, however, bonds are not regarded as perfect substitutes, and a risk premium exists, then government policy which influences variables affecting risk premia will influence expected real interest rates. Thus, if risk premia depend upon stocks of debt, open market operations may be effective.

In recent years a plethora of empirical studies have indicated that the current forward rate is not an unbiased estimator of the future spot rate.[1/] If markets are efficient this is evidence of a time-varying risk premium. To explain the nature of a risk premium and to derive a testable equation, Hansen and Hodrick [forthcoming] and Viswanath [1982] rely on the real intertemporal asset pricing models of Brock [1980] and Lucas [1978], which give the result that the risk in a forward contract is caused by the covariance of the real profit of the contract and the intertemporal marginal rate of substitution.

A challenge for monetary theorists is to produce models of nominal assets which are analgous to the intertemporal pricing models for real assets. Such models would integrate monetary theory with modern financial theory.

In order to model forward markets in this paper a Samuelsonian overlapping generations model is employed. Overlapping generations models have been previously used to model foreign exchange markets by Karaken and Wallace [1977] and Nickelsburg [1980].

In both Karaken and Wallace's deterministic model and Nickelsburg's stochastic model, country-specific fiat monies are regarded as perfect substitutes. Hence, under laisez-faire the exchange rate is not determined. The assumption of perfect substitutability does not, however, seem to agree with reality. Americans prefer to hold dollars and the French prefer to hold francs. Lucas [1981] claims that the reason for this ". . . must have something to do with the local nature of information people have, but it is difficult to think of models that even make a beginning on understanding this issue."[2/]

Part of the allure of overlapping generations models is that valued fiat money arises endogenously in them. It would be desirable if the lack of perfect substitutability between currencies arose endogenously in a model of exchange rate determination. Such a phenomenon, however, will not occur here. It will instead be postulated that, while agents enter forward markets as a means of risk-sharing, they prefer to save their own currency. In fact, the extreme case, that they will save only their own currency, will be assumed.

Section two contains a general discussion of forward foreign exchange markets. In section three the model is set up, the forward market is described and the equilibrium is defined. In section four the roles of risk and hedging in the determination of a time-varying risk premium are discussed. In section five the existence of an equilibrium is proved.

2. The Forward Foreign Exchange Market

A foreign exchange contract is an agreement where each party agrees to pay the other an agreed upon amount of a specified currency on an agreed upon date. In a spot transaction the payment or "value" date coincides with the contract date.[3/] In a forward contract the value date occurs after the contract date.

Suppose an investor enters into a forward contract to purchase 1000 pounds with dollars at an exchange rate of 1.7 $/pound in three months. The investor, who is said to be "long" in sterling, pays nothing at the time of purchase. At the value date he pays $1700 and the seller, who is said to be "short" in sterling, delivers 1000 pounds.

Forward rates are tied to known, current spot prices and the distributions of unknown, future spot prices through two different types of behavior in the forward exchange market: covered interest arbitrage and speculation. Covered interest arbitrage, which entails no foreign exchange risk, works as follows. An agent purchases riskless, interest-bearing financial assets denominated in foreign currency and at the same time sells the foreign currency forward for delivery at the time of the sale of the interest-bearing assets. The relevant opportunity cost is the foregoing of purchasing interest-bearing assets denominated in the home currency. Under the assumption that there are no transactions costs, the rates of return on the alternative investment strategies must be equal. Hence the interest parity theorem, that the ratio of the forward price to the current spot price equals the ratio of the returns on the two countries' risk free financial assets, is obtained. Thus a relationship between the forward price and the current spot price is given in terms of known interest rates.

Speculative purchasing does entail foreign exchange risk. By convention the value of the contract at its signing date is zero; hence in the absence of margin requirements, the amount invested may be unlimited and there is no opportunity cost. The expected nominal return on this alternative depends on the forward rate and the spot rate expected to prevail in the delivery period. Thus the forward rate depends on, among other things, the distribution of the future spot rate.

The foreign exchange contract entails the trade of one currency for another. If one currency is, in some relevant sense, "riskier" than the other, and the risk is not completely diversifiable, a premium must be paid for the purchase of risk. Even in the absence of a risk premium, the forward price cannot be an unbiased estimator of the future price for both countries because the exchange rate for one country is the reciprocal of the exchange rate for the other country. By Jensen's inequality it will not generally be true that it is possible for the expected value of the future spot rate to equal the forward rate and for the expected value of the reciprocal of the future spot rate to equal the reciprocal of the forward rate. This phenomenon is known as Siegal's paradox.[4/] Thus, standard intuition has it that the forward rate is a combination of the expected future spot rate, a risk premium and a convexity term arising from Siegal's paradox.

3. The Model

3.1 The Enviorment

There are two countries, the home country, H, and the foreign country, F. In every period a single representative agent is born in each country. The agents live for two periods; thus there is a constant population of two agents in each country. At the start of the first period of life an agent is endowed with his country's output of the single, non-storable, costlessly-transportable consumption good. This output is exogenous and stochastic.

There also exist two country-specific fiat monies. The original stocks of the monies are held by the agents of generation zero at the start of period one. Additional money is injected into the economy by means of a stochastic, proportional transfer to the agents of each generation. Each agent receives an amount proportional to his savings at the beginning of his second period of life.

During their first period of life the agents of generation t trade with agents of generation t-1. The young agents allocate their endownment between consumption and savings. They trade part of their output to the old agents for fiat money. They may also enter into forward contracts.[5/] It is assumed that their are no margin requirements. By convention, forward contracts have zero value at the time of signing; hence there are no opportunity costs associated with entering into a forward contract.

In their second period of life the agents of generation t receive their proportional monetary transfers from the governments and settle their forward contracts. They then trade with the agents of generation t+1 to obtain the consumption good in exchange for fiat money.

There are no riskless interest-bearing assets. Interest arbitrage in this case would require that current spot and forward rates be identical, and thus agents would be indifferent between saving their own country's currency or entering into a spot-forward swap for the other country's currency.

In this case the home and foreign currency would be perfect substitutes, as in Karaken and Wallace [1977], and the exchange rate would not be determined. As the focus is on speculative behavior, the restriction that agents save only their own money will be imposed. It is this restriction which allows exchange rates to be determined.

3.2 The Consumers' Problems

Time t variables:

$p(t)$:= the good price in country H currency.

$e(t)$:= the exchange rate, or the price of country F's currency in terms of country H's currency.

$f(t)$:= the forward price of country F's currency at time t for delivery at time t+1, in terms of country H's currency.

$z^i(t)$:= the number of units of country F's country purchased forward by the generation t agent in country i; i = H, F.

$m^i(t)$:= the number of units of country i's currency saved by the generation t agent in country i; i; i = H, F.

$x^i(t)$:= output in country i; i = H, F.

$[\tau^i(t) - 1]\, m^i(t)$:= monetary transfer from the government of country i to the generation t agent in country i.

$c_j^i(t)$:= time t-1+j consumption by the generation t agent in of country i; i = H, F, j = 1, 2.

All individuals of generations $t \geq 1$ have the identical utility function

$$(1) \quad U[c_1^i(t)] + E_t \{V[c_2^i(t)]\} ,$$

where E_t is the expectations operator conditioned on current and past values of all variables and

$U: R \rightarrow]-\infty, \infty[$ and

$V: \underset{\sim}{R} \rightarrow]-\infty, \infty[$

are strictly increasing, twice differentiable, strictly concave functions. The Inada conditions are assumed to hold, thus

$$(2) \quad \begin{aligned} &U' \rightarrow \infty \text{ as } c_1 \rightarrow 0,\ V' \rightarrow \infty \text{ as } c_2 \rightarrow 0 \\ &U' \rightarrow 0 \text{ as } c_1 \rightarrow \infty,\ V' \rightarrow 0 \text{ as } c_2 \rightarrow \infty. \end{aligned}$$

The function $V'(c_2)c_2$ is assumed to be strictly increasing. This implies that the substitution effect of a price change will dominate the income effect.

In period $t \geq 1$ the generation t agent of country i, i = H, F, maximizes (1) with respect to $\{c_j^i\}$, m^i and z^i subject to

$$(3)\quad p(x^i - c_1^i) \geq \begin{cases} m^H \text{ if i if i} \\ em^F \text{ if } i = F \end{cases}$$

and

$$(4)\quad p'c_2^i - (e' - f)z^i \leq \begin{cases} \tau^{H'}m^H \text{ if } i = H \\ \tau^{F'}e'm^F \text{ if } i = F, \end{cases}$$

where an unprimed variable is a time t variable and a primed variable is a time t+1 variable.

Equation (3) says that the nominal value (in terms of country H's currency) of savings must be less than or equal to the nominal value of endownment minus first period consumption. Equation (4) says that the nominal value (in terms of country H's currency) of second period consumption of the agent minus his profit on his forward contract must be less than or equal to the nominal value of his savings.

3.3 The Underlying Stochastic Process

$\{S(t)\} \equiv \{\{x^i\}, \{\tau^i\}\}_{t \geq 1}$ assumed to be the realization of a known, stationary, first-order Markov process with

$$(5)\ \Pr[x^H(t+1) \leq x^{H'}, x^F(T+1) \leq x^{F'}, \tau^H(t+1) \leq \tau^{H'},$$

$$\tau^F(t+1) \leq \tau^{F'} \big| x^H(t) = x^H, x^F(t) = x^F, \tau^F(t) = \tau^H, \tau^F(t) = \tau^F] =$$

$$G(x^{H'}\ \ x^{F'}, \tau^{H'}, \tau^{F'} \big| x^H, x^F, \tau^H, \tau^F).$$

The stochastic process takes its values on the set

$$\tilde{S} := [\underline{x}^H, \bar{x}^H] \times [\underline{x}^F, \bar{x}^F] \times [\underline{\tau}^H, \bar{\tau}^H] \times [\underline{\tau}^F \times \bar{\tau}^F]$$

where $\underset{\sim}{0} < (\underline{x}^H, \underline{x}^F, \underline{\tau}^H, \underline{\tau}^F) < (\bar{x}^H, \bar{x}^F, \bar{\tau}^H, \bar{\tau}^F) < \underset{\sim}{\infty}$.

$G(\cdot|\cdot)$ is a cumulative distribution function with the continuous, strictly positive density function $g(\cdot|\cdot)$, which has a continuous derivative. In addition

(6) $\mu(A) > 0$ implies $\Pr[S(t+1)\varepsilon A \,|S(t) = S] > 0$,

$\forall \; S(t),\; S(t+1)\varepsilon \; \tilde{S}$,

$\forall \; A \subseteq S$,

where $\mu(\cdot)$ denotes Lebesque measure.

3.4 Solutions to the Consumers' Problems

U^i is strictly concave in m^i and $E_t(V^i)$ is a Lebesque integral of strictly concave functions in m^i and z^i, where U^i and V^i are U and V evaluated at country i consumption levels. Hence, the objective function is strictly concave, and the young agents' problems have unique solutions which are given by

$$(7)\quad E_t\left[\frac{\tau^{H'}V'(c_2^H)}{p'}\right] = \frac{U'(c_1^H)}{p}$$

$$(8)\quad E_t\left[\frac{(e'-f)V'(c_2^H)}{p'}\right] = 0$$

$$(9)\quad E_t\left[\frac{\tau^{F'}e'V'(c_2^F)}{p'}\right] = \frac{U'(c_2^F)}{p}$$

$$(10)\quad E_t\left[\frac{e'-f)V'(c_2^F)}{p'}\right] = 0.$$

Equations (8) and (10) are similar to the equation derived in the Richard and Sunderasen [1981] model of commodity forward markets, and have the same interpretation. The expected marginal utility of consumption of the home good at time t+1, from entering into a forward contract at time t, is zero.

3.5 The Government

Under floating exchange rates the governments control the money supply by means of the exogenous, stochastic policy variables $\{\tau^i\}$. Money supplies grow according to

$$(11)\quad M^{i'} = M^i\tau^i;\ i = H, F.$$

In a more elaborate model, governments might also intervene in the market for foreign exchange by buying and selling foreign currency. Here it is assumed that there is no intervention.

3.6 Equilibrium

3.6.1 Market Clearing

Market clearing requires

$$(12)\quad c_1^{H'} + c_1^{F'} + c_2^{H} + c_2^{F} = x^{H'} + x^{F'}$$

$$(13)\quad m^i = M^i \tau^i;\ i = H, F.$$

$$(14)\quad z^H + z^F = 0.$$

Equations (12) - (14) are a system of four equations, but by Walras' Law only three of them are independent.

3.6.2 Definitions

Substituting equations (12) - (14) into equations (7) - (10) gives the following definition.

Definition 1. An equilibrium is a bounded sequence, $\{p(t), e(t), f(t), z^H(t), z^F(t)\}_{t \geq 1}$ such that $f(t) > 0$, $\{p(t)\}$ and $\{e(t)\}$ are bounded away from zero and

$$(15)\quad E_t\left[\frac{\tau^{H'}}{p'} V'\left(\frac{\tau^{H'}}{p'} M^{H'} + \frac{e' - f}{p'} z^H\right)\right] = \frac{1}{p} U'\left(x^H - \frac{M^{H'}}{p}\right)$$

$$(16)\quad E_t\left[\frac{e'\tau^{F'}}{p'} V'\left(\frac{e'\tau^{F'}M^{F'}}{p'} + \frac{e'-f}{p'} z^F\right)\right] = \frac{e}{p} U'\left(x^F - \frac{eM^{F'}}{p}\right)$$

$$(17)\quad E_t\left[\frac{e'-f}{p'} V'\left(\frac{\tau^{H'}}{p'} M^{H'} + \frac{e'-f}{p'} z^H\right)\right] = 0$$

$$(18)\quad E_t\left[\frac{e'-f}{p'} V'\left(\frac{e'\tau^{F'}}{p'} M^{F'} + \frac{e'-f}{p'} z^F\right)\right] = 0$$

$$(19)\quad z^H + z^F = 0,$$

for every possible realization of $\{S(t)\}$.

It is well known that models involving rational expectations give rise to a multiplicity of equilibria.[6/] This is because there may be equilibria where extraneous variables, such as variables unrelated to the model or lagged values of state variables, are important only because agents believe they are important. In order to render a rational expectations model with multiple equilibria useful it is necessary to single out a particular equilibrium. McCallum [1981] suggests that an appropriate candidate is the equilibrium depending on the information set of smallest dimesionality.

In this model the economy at time t is described by the vector of time t state variables, $\{M^H, M^F, \tau^H, \tau^F, x^H, x^F\}$. Since the economy at time t is completely summarized by this vector, the path to the current state should not matter in determining prices. Thus, the equilibria focused upon are equilibria where prices are solely a function of the current state.

Given knowledge of the joint distribution function of the random variables consumers should be able to determine the pricing function, and hence their optimal decision rules. Rational expectations will ensure that the prices that agents believe will prevail if a given state is realized, are in fact the prices which will prevail.

At this point it is conjectured that goods prices must vary proportionately with the money stocks and the spot and forward exchange rates must vary proportionately with the ratio of the money stocks. Then the price of the good may take the form

$$(20)\quad p = M^H \phi_1(S),$$

the exchange rate may take the form

$$(21)\quad e = \frac{M^H}{M^F}\ \phi_2(S)$$

and the forward price may take the form

$$(22)\quad f = \frac{M^{H'}}{M^{F'}}\ \hat{f}(S),$$

where it is recalled that $S := \{\tau^H, \tau^F, x^H, x^F\}$.

<u>Definition 2.</u> An <u>equilibrium pricing function</u> is a continuous function

$$\phi := (\phi_1, \phi_2) : \tilde{S} \rightarrow \underset{\sim}{R}_{++},$$

where $\phi_1(S)$ is the price of the good (divided by M) in state S, $\phi_2(S)$ is the exchange rate (weighted by M^F/M^H) in state S and

$$(23)\quad E\{ \frac{\tau^{H'}}{\phi_1(S')} V'[\frac{\tau^{H'}}{\phi_1(S')} + \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} \frac{z}{M^{*'}}]|S\} = \frac{\tau^H}{\phi_1(S)} U'[x^H - \frac{\tau^H}{\phi_1(S)}]$$

$$(24)\quad E\{ \frac{\tau^{F'}\phi_2(S')}{\phi_1(S')} V'[\frac{\tau^{H'}\phi_2(S')}{\phi_1(S')} + \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} \frac{z^F}{M^{*'}}]|S\}$$

$$= \frac{\tau^F\phi_2(S)}{\phi_1(S)} U'[x^F - \frac{\phi_2(S)\tau^F}{\phi_1(S)}]$$

$$(25)\quad E\{ \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} V'[\frac{\tau^{H'}}{\phi_1(S')} + \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} \frac{z^H}{M^{*'}}]|S\} = 0$$

$$(26)\quad E\{ \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} V'[\frac{\phi_2(S')\tau^{F'}}{\phi_1(S')} + \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} \frac{z^F}{M^{*'}}]|S\}=0$$

$$(27)\quad z^H + z^F = 0,$$

for every possible realization of $\{S(t)\}$.

Notice that expectations in a stationary equilibrium are conditioned only upon S. Prices are a function of S, and hence add no additional information.

<u>Conjecture.</u> There exists an equilibrium pricing function.

In the next section the market for forward contracts is discussed. It will be shown that bankruptcy cannot occur and that equations (25) - (26) can be solved for a unique triplet $\{\hat{f}, z^H, z^F\}$, which is a function of S and $\phi(\cdot)$. In section 5 the existence of an equilibrium is proved.

3.8. The Market for Forward Contracts

Given the demand for savings, equations (25) - (26) give the home and foreign country demand for forward purchases of the currency of country F. Suppose that $\phi(\cdot)$ is an element of an arbitrary set $\tilde{P}(\tilde{S})$ of continuous functions on S such that $\phi(\tilde{S}) > \underset{\sim}{0}$ for every $S \varepsilon \tilde{S}$, $\phi(\cdot) \varepsilon \tilde{P}(\tilde{S})$. Then the left-hand sides of equations (25) and (26) are strictly decreasing in z^H and z^F respectively. Thus, there exist functions

$$(28) \quad C^H: \tilde{S} \times \tilde{P}(\tilde{S}) \times [-\infty, \infty] \to [-\infty, \infty]$$

$$C^F: \tilde{S} \times \tilde{P}(\tilde{S}) \times [-\infty, \infty] \to [-\infty, \infty],$$

such that

$$(29) \quad z^H = C^H(S, \hat{f})$$
$$z^F = C^F(S, \hat{f})$$

solve equations (25) and (26).

By the assumption that $V'(c_2)c_2$ is an increasing function, both C^H and C^F are strictly decreasing in f. Thus, as the price of forward contracts increases, the demand for forward contracts by both countries decreases.

$$\text{As } \hat{f} \to \inf_{S \in \tilde{S}} \phi_2(S), \quad z^H \to \infty \text{ and } -z^F \to -\infty,$$

$$\text{as } \hat{f} \to \sup_{S \in \tilde{S}} \phi_2(S), \quad z^H \to -\infty \text{ and } -z^F \to \infty.$$

There is no opportunity cost to entering a forward contract, thus as profit on a forward contract becomes positive with certainty, demand by the home country becomes infinitely positive and supply by the home country becomes infinitely negative. As profit on a forward contract becomes negative with certainty, demand by the home country becomes infinitely negative and supply by the foreign country becomes infinitely positive. Thus in equilibrium

$$\hat{f} \in \left] \inf_{S \in \tilde{S}} \phi_2(S), \sup_{S \in \tilde{S}} \phi_2(S) \right[.$$

Equations (27) and (29) imply that if the market for forward contracts clears

$$(30) \quad c^H(S, \hat{f}) + c^F(S, \hat{f}) = 0.$$

The left-hand side of equation (30) is monotonically decreasing in $\hat{f}$ with

$$C^H + C^F \rightarrow \infty \text{ as } \hat{f} \rightarrow \inf_{S \in \tilde{S}} \phi_2(S),$$

$$C^H + C^F \rightarrow -\infty \text{ as } \hat{f} \rightarrow \sup_{S \in \tilde{S}} \phi_2(S).$$

Thus, there exists a continuous function

$$\hat{f}: \tilde{S} \rightarrow [\ \inf_{S \in \tilde{S}} \phi_2(S).\ \sup_{S \in \tilde{S}} \phi_2(S)[$$

such that

$$(31) \quad \hat{f} = \hat{f}(S) > 0$$

solves equation (30). Substituting $\hat{f}(S)$ into equation (29) gives a continuous function

$z: \tilde{S} \times \tilde{M}^F \rightarrow \underset{\sim}{R}$, where $\tilde{M}^F$ is the set of all possible values of M^F, such that

$$(32) \quad \frac{z^H}{M^F} = -\frac{z^F}{M^F} = z(S).$$

The shapes of $\hat{f}(\cdot)$ and $z(\cdot)$ depend upon the shape of $\phi(\cdot)$.

By the Inada conditions, it must be the case that agents choose z^H and z^F to be small enough in absolute value so that with probability one

$$c_2^i > 0;\ i = H, F.$$

Thus in equilibrium z will always be small enough so that

$$c_2^i > 0; \; i = H, F,$$

and bankruptcy cannot occur.

4. The Nature of the Risk Premium

Adding and subtracting from equation (17) or equation (18) gives

$$(33)\quad f = E(e') + \left[\frac{E_t\left(\frac{e'V^{i'}}{p'}\right)}{E_t\left(\frac{V^{i'}}{p'}\right)} - \frac{E\left(\frac{e'}{p'}\right)}{E_t\left(\frac{1}{p'}\right)}\right]$$

$$+ \left[\frac{E_t\left(\frac{e'}{p'}\right)}{E_t\left(\frac{1}{p'}\right)} - E_t(e')\right];\ i = H,\ F.$$

This says that the forward rate is equal to the expected future spot rate plus a term which can loosely be thought of as a risk premium, because it is only for risk neutral utility functions that it equals zero, and a convexity term arising from Siegal's paradox.

This decomposition is similar to the one derived in Stockman [1978]. A difficulty with equation (33) is that it suggests that the variable of interest is the expected difference between the future spot and current forward rates, or the expected nominal profit of a forward contract. Actually agents are interested in the expected difference between the real future spot rate, e'/p', and the real forward rate, f/p'. $E_t[(e' - f)/p]$, rather than $E_t(e' - f)$, is the appropropriate measure of the risk premium. Thus, a more interesting way to formulate the equations describing equilibrium in the market for forward contracts is

$$(34) \quad E_t \frac{(e' - f)}{p'} = - \frac{Cov_t(\frac{e' - f}{p'}, V^{i'})}{E_t (V^{i'})} ; \quad i = H, F$$

This is similar to the equilibrium condition in Richard and Sundarsasen [1981].

By equation (34), a positive premium or a positive expected profit on a forwrd exchange contracts exists when

$$(35) \quad Cov_t(\frac{e' - f}{p'}, V^{i'}) < 0; \; i = H, F.$$

This occurs when higher than average are profits on long positions and lower than average profits on short positions are associated with lower than average marginal utilities, and hence, higher than average consumption levels. In this case purchasing currency forward does not provide protection from risk, but selling currency forward does. The forward buyer of foreign currency is insuring the forward seller of foreign currency. Hence, the buyer must make a positive expected profit and the seller will make a positive expected loss.

5. Existence of Equilibrium

Defining an equilibrium to be a pricing function has been done previously by Bental [1979] and Lucas [1972] among others. Proving such an equilibrium exists amounts to finding a fixed point in a function space. This is done in Lucas by applying the contraction mapping theorem and in Bental by applying Schauder's generalization of the Brower fixed point theorem to Banach spaces. Here I follow Bental in applying Schauder's theorem. The proof is similar to Bental's in structure.

The first step in the proof is to restrict the collection of functions over which the search must be conducted to a compact, convex set. Then the equilibrium conditions are used to construct a continuous mapping from the set into itself. Schauder's theorem ensures that such a map has a fixed point, which will be the desired equilibrium pricing function.

$\tilde{S} \subset \underset{\sim}{R}^4$ is the sample space of the random state variable over which the pricing function is to be defined. $\tilde{S}$ is closed and bounded; hence $\tilde{S}$ is compact in $\underset{\sim}{R}^4$.

For the moment, assume bounds on the pricing functions exist. Let

$$\Phi_j \equiv \{\phi \in C^2 \big| \phi: \tilde{S} \to \underset{\sim}{R};\ \underline{\phi}_j(\cdot) < \phi(\cdot) < \bar{\phi}_j(\cdot);$$

$$D_k\phi(\cdot) < \overline{M}\ ;\ k = 1, \ldots, 4\};\ j = 1, 2,$$

where $(\underline{\phi}_j(\cdot), \bar{\phi}_j(\cdot))$, $j = 1, 2$ are the appropriate bounds, D_k denotes the kth partial derivative and

$$||\phi_i|| = \sup_{S\varepsilon\tilde{S}} |\phi_i(S)|.$$

The set of pricing functions to which attention is restricted is $cl(\Phi_1) \times cl(\Phi_2)$.[7/] $cl(\Phi_1)$ is the set of admissable candidates for the good pricing function and $cl(\Phi_2)$ is the set of admissable candidates for the exchange rate pricing functions. A generic element of Φ_j will be denoted ϕ_j, j = 1,2.

It will first be shown that the appropriate bounds for the pricing functions exist. Then it will be shown that $cl(\Phi_1) \times cl(\Phi_2)$ is convex and compact. Finally, equations (23) and (24) will be used to construct a continuous map from $cl(\Phi_1) \times cl(\Phi_2)$ into itself. By Shauder's theorem such a map has a fixed point which will be the equilibrium pricing function.

To establish the existence of the appropriate bounds on the pricing functions, consider first the good price. For every $S \varepsilon \tilde{S}$ and every continous $\phi_2(\cdot)$, equation (23) gives the relationship between $\phi_1(\cdot)$, the time t+1 pricing function, and $\phi_1(S)$, the time t price. The right-hand side of equation (23) is strictly increasing in $\phi_1(S)$.

$$\text{As} \quad \phi_1(S) \to \infty, \quad \frac{\tau^H}{\phi_1(S)} U'[x^H - \frac{\tau^H}{\phi_1(S)}] \to 0$$

$$\text{As} \quad \phi_1(S) \to \frac{\tau^H}{x^H}, \quad \frac{\tau^H}{\phi_1(S)} U'[x^H - \frac{\tau^H}{\phi_1(S)}] \to \infty$$

Let

$$\tilde{\pi}(\tilde{S}) \equiv \{\phi_1 \varepsilon\, C^2 | \frac{\tau^H}{X} < \phi_1(\cdot) < \infty\}.$$

Then there exists a function P: $\tilde{S} \times \tilde{\pi}(\tilde{S}) \rightarrow]\frac{\tau^H}{x^H}, \infty]$ where $P[S, \phi_1(\cdot)] = \phi_1(S)$, which solves equation (23).

Proposition 1. There exists $\bar{\phi}_1(\cdot) \in \{\phi_1 | \frac{\tau^H}{x^H} \leq \phi_1(\cdot) < \infty\}$ such that for every $\phi_1(\cdot) \in \{\phi_1 \in C^2 | \frac{\tau^H}{x^H} \leq \phi_1(\cdot) \leq \bar{\phi}_1(\cdot)\}$, $P[S, \phi_1(\cdot)] \leq \bar{\phi}_1(S)$, for every $S \in \tilde{S}$.

If proposition 1 is true, then $\bar{\phi}_1(\cdot)$ constitutes an upperbound for $\phi_1(\cdot)$.

Proof. See the appendix. The proof of this proposition is based on a proof in Peled [1980].

Proposition 2. There exists $\underline{\phi}_1(\cdot) \in \{\phi_1 | \frac{\tau^H}{x^H} < \phi_1(\cdot) \leq \bar{\phi}_1(\cdot)\}$, such that for every $\phi_1(\cdot) \in \{\phi_1 \in C^2 | \underline{\phi}_1(\cdot) \leq \phi_1(\cdot) \leq \bar{\phi}_1(\cdot)\}$, $P[S, \phi_1(\cdot)] \geq \underline{\phi}_1(\cdot)$, for every $S \in \tilde{S}$.

Proof. See the appendix.

$\underline{\phi}_1(\cdot)$ constitutes a lower bound for $\phi_1(\cdot)$.

Consider now the exchange rate. For every $S \in \tilde{S}$, $\phi_1(\cdot) \in [\underline{\phi}_1(\cdot), \bar{\phi}(\cdot)] \cap C^2$, equation (24) gives the relationship between $\phi_2(\cdot)$, the time t+1 exchange rate function, and $\phi_2(S)$, the time t exchange rate. The right-hand side of equation (24) is strictly increasing in $\phi_2(S)$.

$$\text{As} \quad \phi_2(S) \rightarrow 0, \quad \frac{\tau^F \phi_2(S)}{\phi_1(S)} \; U'[x^F - \frac{\phi_2(S)\tau^F}{\phi_1(S)}] \rightarrow 0$$

$$\text{As} \quad \phi_2(S) \rightarrow \frac{x^F \phi_1(S)}{\tau^F}, \quad \frac{\tau^F \phi_2(S)}{\phi_1(S)} \; U'[x^F - \frac{\phi_2(S)\tau^F}{\phi_1(S)}] \rightarrow \infty.$$

Thus, there exists a function R: $\tilde{S} \times \{\phi_2 \in C^2 \big| 0 < \phi_2(\cdot) < x^{H'}\phi_1(\cdot)/\tau^{F'}\} \to [0, x^H\phi_1(\cdot)/\tau^F[$, where $R[S, \phi_2(\cdot)] = \phi_2(S)$ which solves equation (24).

Proposition 3. For every $\phi_1 \in \{\phi_1 \in C^2 \mid \underline{\phi}_1(\cdot) < \phi_1(\cdot) < \bar{\phi}_1(\cdot)\}$, there exists $\underline{\phi}_2(\cdot)$ where $0 < \underline{\phi}_2(\cdot) < x^F\phi_1(\cdot)/\tau^F$ such that for every $\phi_2 \in \{\phi_2 \in C^2 | \underline{\phi}_2(\cdot) < \phi_2(\cdot) < x^F\phi_1(\cdot)/\tau^F\}$, $R[S, \phi_2(\cdot)] > \underline{\phi}_2(S)$ for every $S \in \tilde{S}$.

Proof. Similar to the proof of proposition 2.

Proposition 4. For every $\phi_1 \in \{\phi_1 \in C^2 | \underline{\phi}_1(\cdot) < \phi_1(\cdot) < \bar{\phi}_1(\cdot)\}$, there exists $\bar{\phi}_2(\cdot)$ where $\underline{\phi}_2(\cdot) < \bar{\phi}_2(\cdot) < x^F\phi_1(\cdot)/\tau^F$ such that for every $\phi_2(\cdot) \in \{\phi_2 \in C^2 | \underline{\phi}_2(\cdot) < \phi_2(\cdot) < \bar{\phi}_2(\cdot)\}$, $R[S, \phi_2(\cdot)] < \bar{\phi}_2(S)$ for every $S \in \tilde{S}$. Moreover $\sup \phi_2(S) < x^F\phi_1(S)/\tau^F$.

Proof. Similar to the proof of proposition 1.

Propositions 1-4 have established the existence of the bounds on the pricing functions.

Proposition 5. $cl(\Phi_1) \times cl(\Phi_2)$ is convex and compact.

Proof. See the appendix.

$\tilde{S}$ is compact and Φ_i is bounded; hence by Arzela's theorem, Φ_i is relatively compact.[8/] By Tychonoff's theorem $cl(\Phi_1) \times cl(\Phi_2)$ is compact.

Theorem 1. There exists $\phi \in cl(\Phi_1) \times cl(\Phi_2)$ such that ϕ is an equilibrium pricing function.

The strategy will be to show that for every admissable time t+1 price vector, there exists a time t price vector and the mapping from time t+1 prices to time t prices has a fixed point.

Proposition 6. For every $S \varepsilon \tilde{S}$, $\phi \varepsilon \Phi$, there exists a unique $\theta \varepsilon \Phi$ such that

$$(36)\quad E\{\frac{\tau^{H'}}{\underline{\phi}_1(S')} V'[\frac{\tau^{H'}}{\phi_1(S')} + \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} z(S)]|S\} = \frac{\tau^H}{\theta_1(S)} U'[x^H - \frac{\tau^H}{\theta_1(S)}]$$

$$(37)\quad E\{\frac{\tau^{F'}\phi_2(S')}{\phi_1(S')} V'[\frac{\tau^{F'}\phi_2(S')}{\phi_1(S')} - \frac{\phi_2(S') - \hat{f}(S)}{\phi_1(S')} z(S)]|S\} =$$

$$\frac{\tau^F\theta_2(S)}{\theta_1(S)} U'[x^F - \frac{\tau^F\theta_2(S)}{\theta_1(S)}].$$

If this proposition is true then $\theta \equiv \phi$ constitutes an equilibrium. Thus it will be sufficient to show that $\mathcal{M}: \Phi \to \Phi$, such $\mathcal{M}(\phi) = \theta$, has a fixed point. To prove proposition 6, propositions 7-9 are employed.

Proposition 7. For every $S \varepsilon \tilde{S}$, $\phi \varepsilon \Phi$, there exists a unique $\theta_1(S) \varepsilon [\underline{\phi}_1(S), \bar{\phi}_1(S)]$, such that (36) holds.

Proof. $\theta_1(S) = P[S, \phi_1(\cdot)]$. $\theta_1(S) \varepsilon [\underline{\phi}_1(S), \bar{\phi}_1(S)]$ by propositions 1 and 2.

Proposition 8. Let $\theta_1(S)$ be given by proposition 7. For every $S \varepsilon \tilde{S}$, $\phi \varepsilon \Phi$, there exists a unique $\theta_2(S) \varepsilon [\underline{\phi}_2(S), \bar{\phi}_2(S)]$, such that (37) holds.

Proof. $\theta_2(S) = R[S, \phi_2(\cdot)]$. $\theta_2(S) \varepsilon [\underline{\phi}_2(S), \bar{\phi}_2(S)]$ by propositions 3 and 4.

Proposition 9. $\theta \varepsilon \Phi$.

Proof See the appendix.

Proposition 10. $\mathcal{M}: \Phi \to \Phi$ is continuous.

Proof. See the appendix.

Proof of Theorem 1. is continous; hence

$$\mathcal{M}(\mathrm{cl}(\phi_1),\ \mathrm{cl}(x_2)) = (\mathcal{M}_1(\mathrm{cl}(\phi_1)), \mathcal{M}_2(\mathrm{cl}(\phi_2)))$$

$$\subseteq (\mathrm{cl}(\mathcal{M}_1(\phi_1)),\ \mathrm{cl}(\mathcal{M}_2(\phi_2))$$

$$\subseteq (\mathrm{cl}(\phi_1),\ \mathrm{cl}(\phi_2)) \text{ by the argument in the proof of proposition 10.}$$

Hence, $\mathcal{M}(\mathrm{cl}(\phi_1),\ \mathrm{cl}(\phi_2)) \subseteq (\mathrm{cl}(\phi_1),\ \mathrm{cl}(\phi_2))$. $(\mathrm{cl}(\phi_1) \times \mathrm{cl}(\phi_2))$ is convex and compact. Hence there exists a fixed point.

Thus, it is established that equilibrium pricing functions

$$\phi_1(S) = p/M$$

and

$$\phi_2(S) = eM^*/M$$

exist.

6. Conclusion

This work has attempted to model risk premia in foreign exchange markets in a choice-theoretic, general equilibrium framework. In this framework forward contracts are used as a method of sharing risk. Time-varying risk premia arise and are seen to be related to the efficacy of such contracts as hedges.

Allowing trade in forward contracts, but requiring that agents save only their own country's currency, allows exchange rate determination in an overlapping generations model. It is readily observable that currencies are not actually regarded as perfect substitutes. Agents in a given country tend to hold portfolios consisting predominately of that country's currency. The interesting question is why this non-substitutability should exist. This question is still to be answered.

7. Appendix

Proof of Proposition 1

Claim 1. For every $S \in \tilde{S}$ there exists $\gamma(S) \in]\tau^H/x^H, \infty[$ such that $P[S, \phi_1(\cdot)] < \phi_1(S)$ for every $\phi_1 \in \{\phi_1 \in C^2 | \gamma(S) < \phi_1(\cdot) < \infty\}$.

Proof of Claim 1. Suppose that the claim is not true for some $S \in \tilde{S}$. Then for every sequence $\{\gamma^n\}$ on $]\tau^H/x^H, \infty[$ such tht $\gamma^n \to \infty$, there exists a sequence of functions $\{\phi_1^n(\cdot)\}$ such that $\phi_1^n \in \{\phi_1 \in C^2 | \gamma^n < \phi_1^n(\cdot) < \infty\}$ and $P[S, \phi_1^n(\cdot)] > \phi_1^n(S) > \gamma^n$ for every n. Then there exists a neighborhood, $\mathcal{N}(S)$, of S such that $P[S, \phi_1^n(\cdot)] > \phi_1^n(S)$ for every $S' \in \mathcal{N}(S)$. By assumption, the probability measure of $\mathcal{N}(S)$ is strictly greater than zero. As $n \to \infty$, $\gamma^n \to \infty$; hence $\phi_1^n(\cdot) \to \infty$ and $P[S, \phi_1^n(\cdot)] \to \infty$. $U'(x^H - \tau^H/P[S, \phi_1^n(\cdot)]) \to U'(x^H)$, a strictly positive, finite constant. $V'(\tau^{H'}/\phi_1^n(S') + [\phi_2(S') - f(S)]/\phi_1^n(S')) \to V'(0) = \infty$. Thus,

$$E_t\left[\frac{\tau^H P[S, \phi_1^n(\cdot)]V'}{\phi_1^n(\cdot)U'} \right] \to \infty$$

at S. This is a contradiction; hence Claim 1 is proved.

Choose such a $\{\gamma(S)\}_{S\in\tilde{S}}$.

Claim 2. For every $S \in \tilde{S}$, $P[S, \phi_1(\cdot)]$ is bounded from above by some $\varepsilon(S) \in [\tau^H/x^H, \infty[$ for every $\phi_1 \in \{ \phi_1 \in C^2 | \tau^H/x^H < \phi_1(\cdot) < \gamma(S)\}$.

Proof of Claim 2. Suppose not for some $S \in \tilde{S}$. Then there exists a sequence $\{\phi_1^n(\cdot), P[S, \phi_1^n(\cdot)]\}$ where $\phi_1^n \in \{\phi_1 \in C^2 | \tau^H/ x^H < \phi_1(\cdot) < \gamma(S)\}$ such that $P[S, \phi_1^n(\cdot)] \to \infty$. $V'/\phi_1(\cdot)$ evaluated at a strictly positive, bounded price is strictly positive; hence this is a contradiction and claim 2 is proved.

Choose such a $\{\varepsilon(S)\}_{S\in\tilde{S}}$.

Let $\bar{\phi}_1(S) = \max\{\gamma(S), \varepsilon(S)\}$ for every $S \varepsilon \tilde{S}$.

Claim 3. For every $S \varepsilon \tilde{S}$, for every $\phi_1 \varepsilon \{\phi_1 \varepsilon C^2 | \tau^H/x^H < \phi_1(\cdot) < \bar{\phi}_1(\cdot)\}$, $\bar{\phi}_1(S)$ bounds $P[S, \phi_1(\cdot)]$ from above.

Proof of Claim 3.

$P[S, \phi_1(\cdot)] < \varepsilon(S) < \bar{\phi}_1(S)$

$$\text{for every } \phi_1(\cdot) \varepsilon \{\phi_1 \varepsilon C^2 | \frac{\tau^H}{x^H} < \phi_1(\cdot) < \gamma(S)\}$$

$P[S, \phi_1(\cdot)] < \phi_1(S) < \bar{\phi}_1(S)$

$$\text{for every } \phi_1(\cdot) \varepsilon \{\phi_1 \varepsilon C^2 | \gamma(S) < \phi_1(\cdot) < \infty\}$$

This proves claim 3.

Proof of Proposition 2.

Suppose not for some $S \varepsilon \tilde{S}$. Then there exists a sequence $\{\phi_1^n(\cdot), P[S, \phi_1^n(\cdot)]\}$ where $\phi_1^n \varepsilon \{\phi_1 \varepsilon C^2 | \tau^H/x^H <_1\phi(\cdot) < \bar{\phi}_1(\cdot)\}$ and $\phi_1^n \rightarrow \phi_1^0 \varepsilon \{\phi_1 \varepsilon C^2 | \tau^H/x^H < \phi_1(\cdot) < \bar{\phi}_1(\cdot)\}$ such that $P[S, \phi_1{}^n(\cdot)] \rightarrow \tau^H/x^H$. $V'/\phi_1(\cdot)$ evaluated at a strictly positive finite function is finite; hence this is a contradiction.

Suppose $\inf \phi_1(S) = \tau^H/x^H$. then there exists a sequence $S^n \rightarrow S^0$ such that for some $\phi_1(\cdot) \varepsilon \{\phi_1 \varepsilon C^2 | \underline{\phi}_1(\cdot) < \phi_1(\cdot) < \bar{\phi}_1(\cdot)\}$, $P[S^n, \phi_1^n(\cdot)] \rightarrow P[S^0, \phi_1^n(\cdot)] \rightarrow \tau^H/x^H$. $\tilde{S}$ is closed; hence $S^0 \varepsilon \tilde{S}$ and this is a contradiction.

Proof of Proposition 5.

Convexity is obvious.

I now show Φ_i; $i = 1, 2$, is equicontinuous. Applying the mean value theorem to each component of S, and using the bounds on the partial derivatives of ϕ_i; gives

$$|\phi_i(S^0) - \phi_i(S^1)| < \overline{M}\, \Sigma_{j=1}^4 \, |S_j^0 - S_j{}^1|$$

$< 4\overline{M} \sup_j \mid S_j^0 - S_j^1 \mid$ for every $S^0, S^1 \in \tilde{S}$, $\phi_j \in \Phi_j$. $\| S^0 - S^j \| < \delta$ implies $\sup_j \mid S_j^0 - S_j^1 \mid < \delta$; hence choose $\delta = \varepsilon/(4M)$. Then for every $\varepsilon > 0$, $\| S^0 - S^j \| < \delta$ implies $|\phi_i(S^0) - \phi_i(S^1)| < \varepsilon$, for every $S^0, S^1 \in \tilde{S}$, $\phi_i \in \Phi_i$. Thus ϕ_i is equicontinuous and the proof is complete.

Proof of Proposition 9.

The continuity of both sides of equations (23) and (24) in S guarantees that θ is a continuous function in S.

It is now shown that the partial derivatives of θ are uniformly bounded. Let S_j be the jth component of S, j = 1,..., 4.

$$\frac{\partial\theta_1}{\partial S_j} = \frac{\partial}{\partial S_j} \{E [\frac{\tau^{H'} V^{H'}}{\phi_1(S')} |S] - \frac{\tau^H U^{H'}}{\theta_1} \} / \frac{\partial}{\partial\theta_1} (\frac{\tau^H U^{H'}}{\theta_1})$$

$$\frac{\partial\theta_2}{\partial S_j} = \frac{\partial}{\partial S_j} \{E [\frac{\tau^{F'}\phi_2(s') V^{F'}}{\phi_1(S')} |S]' - \frac{\tau^F \theta_2 U^{F'}}{\theta_1} \} / \frac{\partial}{\partial\theta_2} (\frac{\phi^F \theta_2 U^{F'}}{\theta_1}) + \frac{\partial}{\partial\theta_1}$$

$$(\frac{\tau^F \theta_2 U^{F'}}{\theta_1}) \frac{\partial\theta_1}{\theta S_j} ,$$

where V^H and U^H are V and U evaluated home levels of consumption and V^F and U^F are V and U evaluated at foreign levels of consumption.

$$\left| \frac{\partial}{\partial\theta_1} (\frac{\tau^H U^{H'}}{\theta_1}) \right| = \frac{\tau^H}{\theta_1{}^3} (U^{H''} + V^{H'}\theta_1) \text{ and } \left| \frac{\partial}{\partial\theta_2} (\frac{\tau^F \theta_2 U^{F'}}{\theta_1}) \right|$$

$= \frac{\tau^F}{\theta_1} \left(U^{F'} + \theta_2 U^{F''} \frac{\tau^F}{\theta_1} \right)$ (are continuous functions) on $\tilde{S}$; hence they reach their minima on $\tilde{S}$. Prices are strictly positive and finite; hence consumption is strictly positive and finite and the minima must be greater than zero. Denote the smallest of the minima $\overline{M}_1$.

$$\text{(A.1)} \qquad \left| \frac{\partial}{\partial S}\left(\frac{\tau^H U^{H'}}{\theta_1}\right) \right| = \left(\frac{\theta_1 U^{H'} - \tau^H U^{H''}}{\theta_1^2}, 0, -\frac{U^{H''}\tau^H}{\theta_1}, 0 \right)$$

$$\text{(A.2)} \qquad \left| \frac{\partial}{\partial S}\left(\frac{\tau^F \theta_2 U^{F'}}{}\right) \right| = \left(0, \frac{\theta_1 U^{F'} - \tau^F \theta_2 U^{F''}}{\Theta_1^2}, 0, \frac{U^{F''}\tau^F}{\theta_1} \right).$$

All of the components of the vectors (A.1) and (A.2) are continuous functions of S on the compact set $\tilde{S}$; hence they reach their maxima on $\tilde{S}$. Choose $\overline{M}_2$ to be the largest such bound

$$\frac{\partial}{\partial S_j} E\left[\frac{\tau^{H'} V^{H'}}{\phi_1(S')} \middle| S \right] = E\left\{ \frac{\tau^{H'} [\phi_2(S') - \hat{f}(S)]}{\phi_1(S')} V^{H''} \middle| S \right\}$$

$$\left[\frac{\partial Z}{\partial f} - Z(S) \right] \frac{\partial \hat{f}(S)}{\partial S_j} + \frac{1}{g(S)} \int_{\tilde{S}} \frac{\tau^{G'} V^{H'}}{\phi_1(S')} \frac{\partial g(S,S')}{\partial S} dS'$$

$$- \left[\frac{\partial g(S)}{\partial S} / g(S) \right] E \left[\frac{\tau^H V^{H'}}{\phi_1(S')} \middle| S \right].$$

All terms are continuous functions of S. g(S) reaches its minima on $\tilde{S}$ and is strictly positive at the minima, and $\frac{\partial g(S)}{\partial S}$ and $\frac{\partial g(S.S')}{\partial S}$ are continuous, and hence, bounded.

$$\frac{\partial f}{\partial S_j} = \frac{E\,(\Delta^2 U^{H''})\ \frac{\partial}{\partial S_j}\ E(\Delta U^{F'}) + E(\Delta^2 U^{F''})\ \frac{\partial}{\partial S_j}\ E(\Delta^{H'})}{E\,(\Delta^2 U^{H''})\ E\,[\frac{U^{F'}}{\phi_1(S')} - z\,\frac{U^{F''}}{\phi_1(S')}] + E(\Delta^2 U^{F''})E[\frac{U^{F'}}{\phi_1(S')} + z\,\frac{U^{H''}}{\phi_1(S')}]},$$

where $\Delta \equiv \dfrac{\phi_2(S') - f(S)}{\phi_1(S')}$, which is also uniformly bounded.

hence

$$\left| \frac{\partial}{\partial S_j} E\left[\frac{\tau^{H'} V^{H'}}{\phi_1(S')} \middle| S\right] \right|$$

is bounded from above. Similarly

$$\left|\frac{\partial}{\partial S_j} E\left[\frac{\tau^{F'} \phi_2(S') V^{F'}}{\phi_1(S')} \middle| S\right] \right|$$

is bounded from above. Let $\overline{M}_3$ be the largest of the two bounds.

$$\left|\frac{\partial}{\partial \theta_1} \left(\frac{\tau^F \theta_2 U^{F'}}{\phi_1}\right) \right| = \frac{\tau^F \theta_2}{\theta_1^{\,3}} \left(U^{F''} \tau^F \theta_2 + U^{F'} \theta_1\right),$$

which is bounded from above by some finite $\overline{M}_4$.

Then

$$\left| \frac{\partial \theta_1}{\partial S_j} \right| < \frac{\overline{M}_2 + \overline{M}_3}{\overline{M}_1} \equiv \overline{M}^1$$

$$\left| \frac{\partial \theta_2}{\partial S_j} \right| < \frac{(1 + \overline{M}_4)\ (\overline{M}_2 + \overline{M}_3)}{\overline{M}_1} \equiv \overline{M}^2.$$

Let $\overline{M} = \max\ \{\overline{M}^1, \overline{M}^2\}$.

Proof of Proposition 10.

Let $\{\phi^n\}\ \Phi$ where $\phi^n \to \phi^0$ uniformly. Note that Φ is not closed; hence ϕ^0 may not be in Φ. I show $(\phi^n) \to (\phi^0)$ uniformly.

For every $S \in \tilde{S}$, $p \in [\phi_1(S), \bar{\phi}_1(S)]$, let

$$(\psi(p) \equiv \frac{\tau^H U^{H'}}{p} - E[\frac{\tau^{H'} V^{H'}}{\phi_1(S'')} \big| S]),$$

where $U^{H'}$ is evaluated at p and $V^{H'}$ is evaluated at $\phi^0(S')$.

$$\psi^n(p) \equiv \frac{\tau^H U^{H'}}{p} - E[\frac{\tau^{H'} V^{H'}}{\phi_1^n\ (S')} \big| S],$$

where $U^{H'}$ is evaluated at p and $V^{H'}$ is evaluated at $\phi^n(S')$.

Let p^n be the unique p such that $\psi^n(p^n) = 0$, $p^n \in [\underline{\phi}_1(S), \bar{\phi}_1(S)]$ for every n; hence by the Bolzano-Weierstrass theorem, $\{p^n\}$ has a convergent subsequence. W.l.o.g., $p^n \to p^0$.

Claim 1. $|\psi(p^0)| = 0.$

Proof of Claim 1.

$$|\psi(p^o)| < \left| \frac{\tau^H U'(p^0)}{p^0} - \frac{\tau^H U'(p^n)}{p^n} \right| +$$

$$\left| \frac{\tau^H V'(p^n)}{p^n} - E[\frac{\tau^{H'} V'(\phi^n)}{\phi_1{}^n(S')} |S] \right| +$$

$$\left| E[\frac{\tau^{H'} V'(\phi^n)}{\phi_1^n \ (S')} |S] - E[\frac{\tau^{H'} V'(\phi^0)}{\phi_1^0 \ (S')} |S] \right|$$

$$= \left| \frac{\tau^{H'} U'(p^0)}{p^0} - \frac{\tau\ U'^H(p^n)}{p^n} \right| + \left| E[\frac{\tau^{H'} V'(\phi^n)}{\phi\ {}_1^n(s')} |S] - E[\frac{\tau^{H'} V'(\phi^0)}{\phi_1^0 \ (S')} |S] \right|.$$

$\frac{\tau^H U'}{p}$ is continuous in p on $[\underline{\phi}_1(S), \bar{\phi}_1(S)]$; hence for every $\varepsilon/2 > 0$, there exists $\delta > 0$, such that

$$\left| \frac{\tau^H U'(p^0)}{p^0} - \frac{\tau^H U'(p^n)}{p^n} \right| < \frac{\varepsilon}{2}$$

for every $|p^o - p^n| < \delta$. $p^n \rightarrow p^0$; hence for every $\delta > 0$, there exists N_1 such that $|p^0 - p^n| < \delta$ for every $n > N_1$.

$E[\frac{V'(\phi^n)\tau^H}{\phi_1^n \ (S')} |S]$ is continuous in ϕ_1 on $\{\phi_1 \ \varepsilon\ C^2 | \underline{\phi}_1(\cdot) < \phi_1(\cdot) < \bar{\phi}_1(\cdot)\}$;

hence for every $\varepsilon/2 > 0$, there exists $\delta > 0$, such that

$$\left| E[\frac{\tau^{H'} U^{H'}(\phi^n)}{\phi_1^n \ (S')} |S] - E[\frac{\tau^{H'} U^{H'}(\phi^0)}{\phi_1^0 \ (S)}]|S \right| < \varepsilon/2$$

for every $|\ \phi^n(\cdot) - \phi^0(\cdot)| < \delta$. $\phi^n \to \phi^0$ uniformly; hence for every $\delta > 0$, there exists N_2 such that

$$||\phi^n(\cdot) - \phi^0(\cdot)|| < \delta$$

for every $n > N_2$. Choose $N = \max\{N_1, N_2\}$.

Thus there exist functions

$$p^n = \theta_1^n\ (S)$$

$$p^0 = \theta_1^0\ (S)$$

where $\theta_1^n\ (\cdot) \to \theta_1^0\ (\cdot)$ pointwise. $D_j\theta_j^n < \overline{M}$, $j = 1, \ldots, 4$; hence $\{\theta_1^n\}$ is equicontinuous. Thus $\theta_1^n \to \theta_1^0$ uniformly. This completes the proof of claim 1.

The argument for θ_2 is simiar. Thus is continuous. The preceding argument also showed $(\theta_1{}^0, \theta_2{}^0) = \mathcal{M}(\phi^0)$ is well-defined.

FOOTNOTES

1/ See Bilson[1981], Cumby and Obstfeld [1983], Dooley and Shafer [1981], Geweke and Feige [1979], Hakkio [1981], Hansen and Hodrick [1980, forth.], Hsieh [1981] and Viswanath [1982].

2/ Lucas [1981], p. 41.

3/ Actually the value date is usually one or two days after the contract date. This technicality will be ignored.

4/ See Siegal [1972]. McCulloch [1975] claims this is emprically trivial. The magnitude, however, obviously depends on the shape of the distribution function.

5/ In reality there are both forward and futures markets for foreign exchange. The fine differences between the two are discussed in Richard and Sundarasen [1981]. All contracts here are one period contracts and thus forward and futures contracts are equivalent.

6/ See, for example, McCallum [1981] or Wallace [1980].

7/ Actually elements of Φ_1 are pricing functions of the good weighted by $1/M^*$ and elements of Φ_2 are pricing functions for the exchange rate weighted by M^F/M^H. The term "pricing function" will be used for convenience.

8/ See Kolmogorov and Fomin [1975], p. 102.

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