International Finance Discussion Papers

Number 199

January 1982

STRUCTURAL LAGS AND STABILITY IN INTERNATIONAL MACROMODELS

by

Richard Freeman

One of the features of many popular international macromodels that is troublesome is their apparent sensitivity to very slight changes in assumptions and initial conditions. This is especially the case for models with an asset sector in which holdings depend on the instantaneous expected rate of change of a variable, such as the exchange rate, and in which asset holders are taken to exhibit "myopic perfect foresight" in making their forecasts. Typically these models are only conditionally stable -- i.e., if they are disturbed from equilibrium, some free variable (often the current exchange rate) must accomplish a discrete, exactly correct jump so as to put the system on its unique stable trajectory to the new equilibrium. In the strict version of these models, even the slightest deviation from this path or the smallest error in this jump will lead -- in the absence of intervention or other bounds on the system -- to its collapse through an explosive movement of the exchange rate and other related variables.[1/]

The question of how best to interpret this theoretical tendency toward instability presents a dilemma. On one hand, to accept exactly correct jumps in variables as an empirical proposition seems to place an unduly heavy burden on the market's power to achieve intertemporal equilibrium on its own, particularly when forecasting skills in actual practice surely fall short of the perfect-foresight ideal. This would suggest that the actual foreign exchange market has a very high degree of endogenous instability and, therefore, must in practice be limited by some other constraints. On the other hand, if we regard such continuous models as no more than convenient approximations --

---

1/ Among the many models of this type now in print, two -- those by Dornbusch (1976) and Kouri (1976) -- have been especially influential. A fairly comprehensive treatment of conditional stability, or the saddlepath property as it is often referred to, is found in Gray and Turnovsky (1979a). The problem is not confined to international models; it is a potential issue in almost any model with an asset sector.

limiting cases of a more general specification with stability conditions that are more forgiving -- then the burden is shifted to the modeler to spell out more exactly the character of this more general case.

In this connection, it is known that models of the type described above sometimes can be transformed into models that are globally stable by inserting an adaptive-expectations rule in place of strict myopic perfect foresight. This modification has the unappealing feature, however, that it replaces an assumption that forecasters are always right with a mechanical rule that says, in effect, that forecasters are almost always wrong. Not only does this run counter to the spirit of rational expectations that pervades most of modern macroeconomic theory, but also in such a modified model global stability occurs only if certain limiting restrictions are met. When they are not, the original dilemma remains.[1/]

In the exercises below, we take a somewhat different approach. Instead, we look at the effect of introducing various discrete lags and leads into a basic international model to see which, if any, can give rise to global stability in a system which otherwise would be only conditionally stable. Although this modeling procedure is subject to some of the same shortcomings as is the use of an adaptive-expectations rule, the cases considered below have the attraction that their leads and lags can be related to plausible structural features of a real system. The findings here are aimed primarily at assessing a particular technical issue, but they are not without some policy implications as well. "Throwing some sand into the wheels" of the foreign exchange market

---

1/ Slowly adjusting adaptive expectations stabilize the simple Cagan model (Cagan, 1956), for example, and the same device has been used in other somewhat more elaborate frameworks. (See, for instance, Burmeister and Dobell, (1970), p. 186-189). The use of this approach is also discussed in footnote 1 , page 8 below. Gray and Turnovsky (1979a) consider an alternative approach, the slowing down of the exchange-rate adjustment itself.

has been suggested by more than a few analysts as a means of reducing the market's instability, and, as we shall see, not all types of "sand" will do for this task.[1/] Some may even give rise to greater variability in the foreign exchanges.

From a technical point of view, the introduction of a finite lead or lag into an otherwise continuous dynamic model gives rise to a system of mixed differential-difference equations. Since the formal analysis of these mixed systems is considerably more involved than it is for their simpler, non-mixed counterparts, in the next section we present the essentials for determining stability. These conditions are then used to evaluate the long-run properties of variations on a basic model. Following the classical format of most well-constructed fairy tales and many humorous anecdotes, a trio of cases are described in detail -- the first two of which are false leads that are shown to fail the stability tests. The point of including these cases is not only the conventional one of building the requisite suspense, but also to illustrate by examples that fail the nature of the conditions that must be met. The impetuous reader is welcome to jump ahead to the case that works, but should be forewarned that the findings there are limited at this point to a particular, somewhat special case. How far they can be generalized is not clear yet, but the preceding cases at least suggest the character of those limits.

Stability in Mixed Systems

In this section, we shall spell out briefly and without formal proof conditions that are necessary and sufficient for global stability of a mixed

---

1/ For example, Tobin has recently suggested that this might be done through a tax on foreign-exchange conversions (Tobin, 1978), a view which is discussed (and opposed) by Dornbusch (1980).

system of linear difference-differential equations. Attention is limited here to systems of order no larger than two in both differences and derivatives, but similar conditions apply to higher-order systems.1/

First, it is convenient as a preliminary step to rearrange the system's equations to form a single second-order, mixed equation in one of the variables of the following form:

(i) $L(x) + K = 0,$

$$\text{where } L(x) = q_2 \ddot{x}(t + 2w) + q_1 \dot{x}(t + 2w) + q_0 x(t + 2w) + m_2 \ddot{x}(t + w) + m_1 \dot{x}(t + w) + m_0 x(t + w) + n_2 \ddot{x}(t) + n_1 \dot{x}(t) + n_0 x(t),$$

K is a constant, and w is an arbitrary finite lead. Substitution of a solution of the form $x = e^{\lambda t}$ into $L(x)$ allows us to express (i) as

(ii) $L(e^{\lambda t}) + K = J(\lambda) e^{\lambda t} + K,$

where $J(\lambda)$, the characteristic function corresponding to L, is

(iii) $$J(\lambda) = (q_2 \lambda^2 + q_1\lambda + q_0)e^{2\lambda w} + (m_2 \lambda^2 + m_1\lambda + m_0)e^{\lambda w} + (n_2 \lambda^2 + n_1\lambda + n_0).$$

As with simpler, non-mixed dynamic systems, the signs of the roots to the characteristic equation,

(iv) $J(\lambda) = 0,$

1/ A useful summary of techniques for evaluating stability in mixed systems is found in Gray and Turnovsky (1979b). For a more detailed and comprehensive treatment, see Bellman and Cooke (1963).

are crucial to determining stability.[1/]

Necessary and sufficient conditions that equation (iv) and the system from which it is derived be globally stable are

A. (1) All characteristic roots (i.e., the roots of (iv)) have non-positive real parts, and

(2) All characteristic roots with a zero real part be non-complex.

A useful concept for assessing whether or not condition A. is met is that of a principal term. In this context the latter is a term, $c_{rs}\,\lambda^r e^{s\lambda w}$, of the polynomial

$$M(\lambda, e^{\lambda w}) = \sum_{i=o}^{u} \sum_{j=o}^{v} c_{ij}\,\lambda^i e^{j\lambda w},$$

such that $c_{r,s} \neq 0$ and for any other term $c_{ij}\lambda^i e^{j\lambda w}$ with $c_{ij} \neq 0$, either $r > i$, $s > j$ or $r > i$, $s = j$ or $r = i$, $s > j$. It has been shown that condition A. is met only if

B. Polynomial $J(\lambda)$ in (iii) has a principal term.

Otherwise, $J(\lambda) = 0$ has an infinity of roots with a positive real part. Hence, the presence of a principal term is a necessary condition for stability; its absence is indicative of endogenous instability in the original system.

1/ Note that the characteristic equation is typically a transcendental equation which has an infinite number of roots -- a fact which often compounds considerably the difficulties encountered in solution and analysis of stability.

Next, let us write $J(i\lambda)$ as

$$J(i\lambda) = F(\lambda) + i\, G(\lambda).$$

Then, if $J(\lambda)$ has a principal term (of order (r,s)), necessary and sufficient conditions for A. to hold are that 1/

C. (1) All the roots of $G(\lambda) = 0$ are real, and

(2) $G'(\lambda) \cdot F(\lambda) > 0$ for all these roots.

Finally, a useful procedure for establishing C.(1) above involves application of the following property:

For the function $G(\lambda)$ above to have only real roots, it is necessary and sufficient that

D. In the interval $\left[-(2\pi k - \delta), (2\pi k + \delta)\right]$ where δ is an arbitrary constant, and k is a positive integer,

$G(\lambda)$ must have exactly $4sk+r$ real roots, starting with an interval defined by a sufficiently large k.

Various combinations of these conditions are used below to diagnose the stability properties of several variations on a basic perfect-foresight model. This basic model is presented next.

The Basic Open-Economy Model

A fairly broad class of models of an open economy with a continuously adjusting exchange rate can be represented in linear form by the following two semi-reduced-form, dynamic equations:

1/ This statement of necessary and sufficient conditions, originally due to Pontryagin, is but one of several alternative versions. For more detail, see Bellman and Cooke (1963), Chapter 13. Depending on the problem, other versions may be more convenient to apply.

(1) $\dot{V}(t) = a_1 E(t) + a_2 V(t) + a_0$,

(2) $E(t) = k_1 \varepsilon(t) + k_2 V(t) + k_0$,

where V is the level of home-country wealth,

E is the exchange rate (in units of home/foreign currency),

ε is the expected instantaneous rate of change of the exchange rate,

and a_1, a_2, a_0, k_1, k_2, and k_0 are constants. The following sign conventions also apply:

$$a_1 > 0,\ a_2 < 0,\ k_1 > 0,\ k_2 < 0.$$

Equation (1), which expresses goods-market equilibrium, reflects the fact that the current account (measured by $\dot{V}$) will tend to be raised by a depreciation of the exchange rate (higher E) or by reduced domestic wealth (lower V). Equilibrium in the asset market, as shown in (2) implies that a higher current exchange rate (i.e., a lower valued home currency) is required when there is an expectation of more rapid depreciation (ε higher) or when domestic wealth (V) is smaller.

It is conventional to introduce myopic perfect foresight into this model by substituting $\dot{E}(t)$ for $\varepsilon(t)$ in (2) so that

(3) $E(t) = k_1 \dot{E}(t) + k_2 V(t) + k_0$.

Equations (1) and (3) then constitute a well defined system in two variables which displays the conditional stability property referred to earlier. Under the sign conventions indicated, one of the roots of the corresponding characteristic equation for this system is assured to be positive. Accordingly, if the system is dislodged from equilibrium, unless there is some mechanism (such as a jump in a variable) to bring the system to its unique, stable trajectory, it will diverge from equilibrium.1/

Model Variations

I. Information lags in expectations formation

Consider now the effect on the basic model if current expectations of the rate of change of the exchange rate are always formed on the basis of obsolete information -- either because the market provides information to participants with a lag or because there is some systematic delay in market participants' processing of information. In either case, if τ is a measure of this lag, we must rewrite equation (3) as

$$(3.a) \quad E(t) = k_1 \dot{E}(t - \tau) + k_2 V(t) + k_0.$$

Equations (1) and (3.a) now constitute a mixed differential-difference equation system. To determine how stability properties are influenced by this lag, we make use of the procedures and conditions discussed earlier.

First, by differentiation of (3.a) and suitable algebraic manipulations, we can obtain the following second-order equation in E:

$$(4.a) \quad m_1 \dot{E}(t + \tau) + m_0 E(t + \tau) + n_2 \ddot{E}(t) + n_1 \dot{E}(t) + \bar{n} = 0,$$

where

1/ The system in (1) and (2) can be stabilized by introducing an adaptive rule of the following type for the rate of change of the expected exchange rate:

$$\dot{\varepsilon}(t) = \alpha \left[\dot{E}(t) - \varepsilon(t)\right], \alpha > 0$$

It can be shown that, when α is sufficiently small, all characteristic roots of the system formed by (1), (2), and the adaptive rule above are negative, making the system globally stable.

$m_1 = 1/\tau > 0,$

$m_o = -(k_2 a_1 + a_2) > 0,$

$n_2 = -k_1/\tau^2 < 0,$

$n_1 = a_2 k_1/\tau < 0,$

$\bar{n} = (a_2 k_o - k_2 a_o)$.

The homogeneous part of the corresponding characteristic function is, therefore,

$$(5.a)\quad J(\lambda) = (m_1\lambda + m_1)\, e^{\lambda\tau} + (n_2\lambda^2 + n_1\lambda).$$

It is evident by inspection that this polynomial has no principal term; hence, on the basis of condition B., we can determine immediately that an information lag in the formation of exchange-rate expectations -- of any duration -- cannot serve to stabilize the original instantaneous system.1/ In fact, when there is no principal term, the characteristic equation has an infinity of roots with a positive real part. This profusion of unstable roots means that even the usual device of an initial jump in the exchange rate will be inadequate to bring the system to a stable trajectory.2/

II. Intermittant participation in the asset market

As an alternative approach, let us suppose instead that individual transactors in the asset market do not or cannot participate on a continuous basis. Instead, they enter the market at intervals no shorter than θ periods in duration, say, either because of a minimum holding-period requirement on assets or because, as a practical matter, the bidding and performance on contracts can only be done at discrete intervals. In fact, let us assume that the

---

1/ In some other versions of this case (not formally reported here) in which additional leads and lags to other variables were added, this characteristic indicating instability in the system was found to persist. For example, a version in which the level of wealth in equation (3) was lagged as well -- producing an effect not unlike a transmission lag in the foreign-exchange market -- had no principal term.

2/ One possible approach to resolving this dilemma is to suspend equation (3.a) for a period of length τ following a shock and assume that in this interval the system finds a segment of a continuous (possibly unique), stable trajectory by, in effect, making shifts in $E(t + \gamma)$ for all real γ in $[0,\tau]$, an infinite number of momentary shifts.

intervals between entry are exactly of size Θ. Also, to finesse problems that might arise in aggregation across individuals, we assume that the timing of transactors' participation is staggered so that active transactors are distributed evenly over any interval of length Θ, and that each transactor holds a proportionate share of total wealth.[1/] Accordingly, if we maintain a strict perfect-foresight rule (though no longer a myopic perfect-foresight rule), the appropriate modification of (3) is

$$(3.b)\quad E(t) = \frac{k_1}{\Theta}\,(E(t+\Theta) - E(t)) + k_2\, V(t) + k_o.$$

Thus, the average per-period rate of change of the exchange rate $(E(t + \Theta) - E(t))/\Theta$, replaces $\varepsilon(t)$ in equation (3). In effect, in (3.b) it is assumed that any market participant at a given time, t, forecasts E for the time of his next entry into the market, $t + \Theta$, and that this forecast is exactly equal to the actual outturn, $E(t + \Theta)$.

Together with (1), this equation can be used to obtain the following first-order equation in E:

$$(4.b)\quad m_1\, \dot{E}(t + \Theta) + m_o\, E(t + \Theta) + n_1\, \dot{E}(t) + n_o\, E(t) + \bar{n} = 0,$$

where

$$m_1 = \frac{k_1/\Theta^2}{1 + k_1/\Theta} > 0 ,$$

$$m_o = \frac{-a_2}{1 + k_1/\Theta} > 0 ,$$

$$n_1 = 1/\Theta > 0 ,$$

$$n_o = \frac{k_2 a_1}{1 + k_1/\Theta} + a_2 < 0 ,$$

$$\bar{n} = \frac{k_2 a_o - a_2 k_o}{1 + k_1/\Theta} .$$

1/ A model which deals with this issue more elegantly is found in Gray and Turnovsky (1979b). The extra detail related to this point appears to make little difference to the qualitative results, however.

The homogeneous part of the corresponding characteristic function is, therefore,

$$(5.b) \quad J(\lambda) = (m_1\lambda + m_0)e^{\lambda\theta} + (n_1\lambda + n_0),$$

which clearly has a principal term. Proceeding as before, when we write $J(i\lambda)$ as $F(\lambda) + i\,G(\lambda)$, we obtain

$$(6.b.1) \quad F(\lambda) = m_0 \cos\lambda - m_1 \lambda \sin\lambda + n_0,$$

$$(6.b.2) \quad G(\lambda) = m_1 \lambda \cos\lambda + m_0 \sin\lambda + n_1 \lambda.$$

Condition C. above includes the requirement that all the zeros of $G(\lambda)$ be real, which may be established in turn via condition D. It is evident by inspection that $\lambda = 0$ is one solution to $G(\lambda) = 0$. To help in evaluating the other zeros of $G(\lambda)$, we divide (6.b.2) by λ and write it as

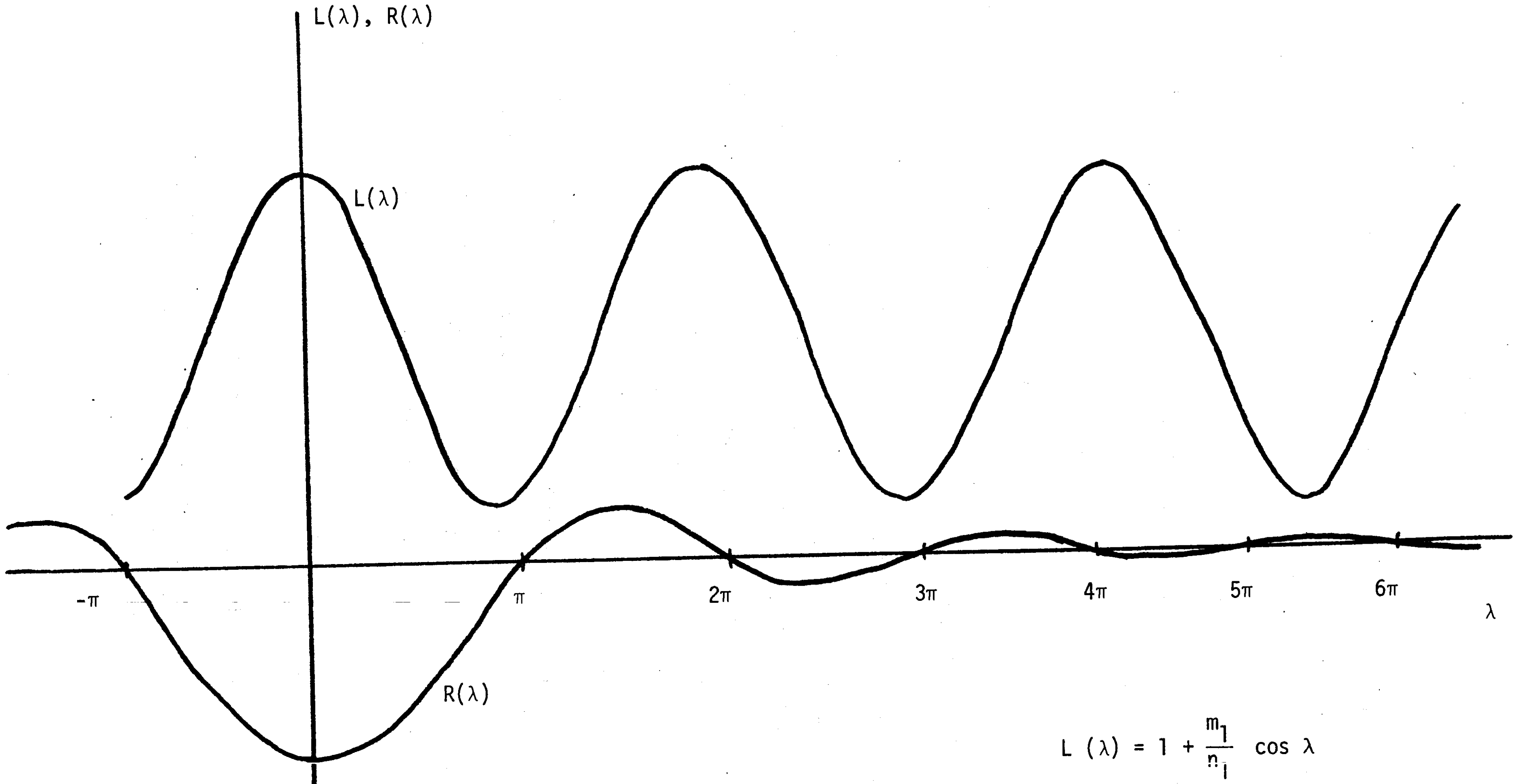
$$(7.b) \quad 1 + \frac{m_1}{n_1} \cos\lambda = -\frac{m_0}{n_1}\,\frac{\sin\lambda}{\lambda}.$$

The left and right-hand sides of (7.b) are depicted in Figure 1 as $L(\lambda)$ and $R(\lambda)$, respectively. Since $\sin\lambda/\lambda$ diminishes toward zero as λ gets large, $L(\lambda)$ will continue to intersect $R(\lambda)$ for large values of λ only if $|m_1/n_1| > 1$. But, referring back to (4.b), in this case

$$|m_1/n_1| = \left| \frac{\theta/k_1}{1 + \theta/k_1} \right| < 1.$$

Accordingly, condition D. can never be satisfied by a positive value of θ of any size, and therefore, this system cannot be made globally stable by the introduction of a lead in this form.

Figure 1



$$L(\lambda) = 1 + \frac{m_1}{n_1} \cos \lambda$$

$$R(\lambda) = - \frac{m_o \sin \lambda}{n_1 \lambda}$$

III. Intermittant participation with expectations based on data from previous entry.

As a third alternative, consider next the following specification, which combines features of the previous two: We continue to assume that transactors in the foreign-exchange market enter only at intervals (defined by Θ , as before), but now we assume that their projection of the rate of change of the exchange rate for the next interval is the same as its actual rate of change over the most recent interval. Accordingly, equation (3) is revised to

$$(3.c) \quad E(t) = \frac{k_1}{\Theta}(E(t) - E(t-\Theta)) + k_2 V(t) + k_o$$

Proceeding as before, (3.c) can be joined with (1) to obtain

$$(4.c) \quad m_1 \dot{E}(t+\Theta) + m_o E(t+\Theta) + n_1 \dot{E}(t) + n_o E(t) + \bar{n} = 0$$

where

$$m_1 = 1/\Theta > 0 ,$$

$$m_o = -\frac{k_2 a_1}{1 - k_1/\Theta} - a_2 \gtrless 0 \text{ ?},$$

$$n_1 = \frac{k_1/\Theta^2}{1 - k_1/\Theta} \gtrless 0 \text{ ?},$$

$$n_o = -\frac{a_2 k_1/\Theta}{1 - k_1/\Theta} \gtrless 0 \text{ ?},$$

$$\bar{n} = \frac{a_2 k_o - k_2 a_o}{1 - k_1/\Theta} .$$

Since the homogeneous part of the corresponding characteristic function has exactly the same form as (5.b), corresponding equations for this case that are identical in form to (6.b.1), (6.b.2), and (7.b) can be derived, with the only difference lying in the interpretation of coefficients. In particular, in this case the ratio m_1/n_1 is given by

$$m_1/n_1 = \frac{\Theta}{k_1} - 1 \ .$$

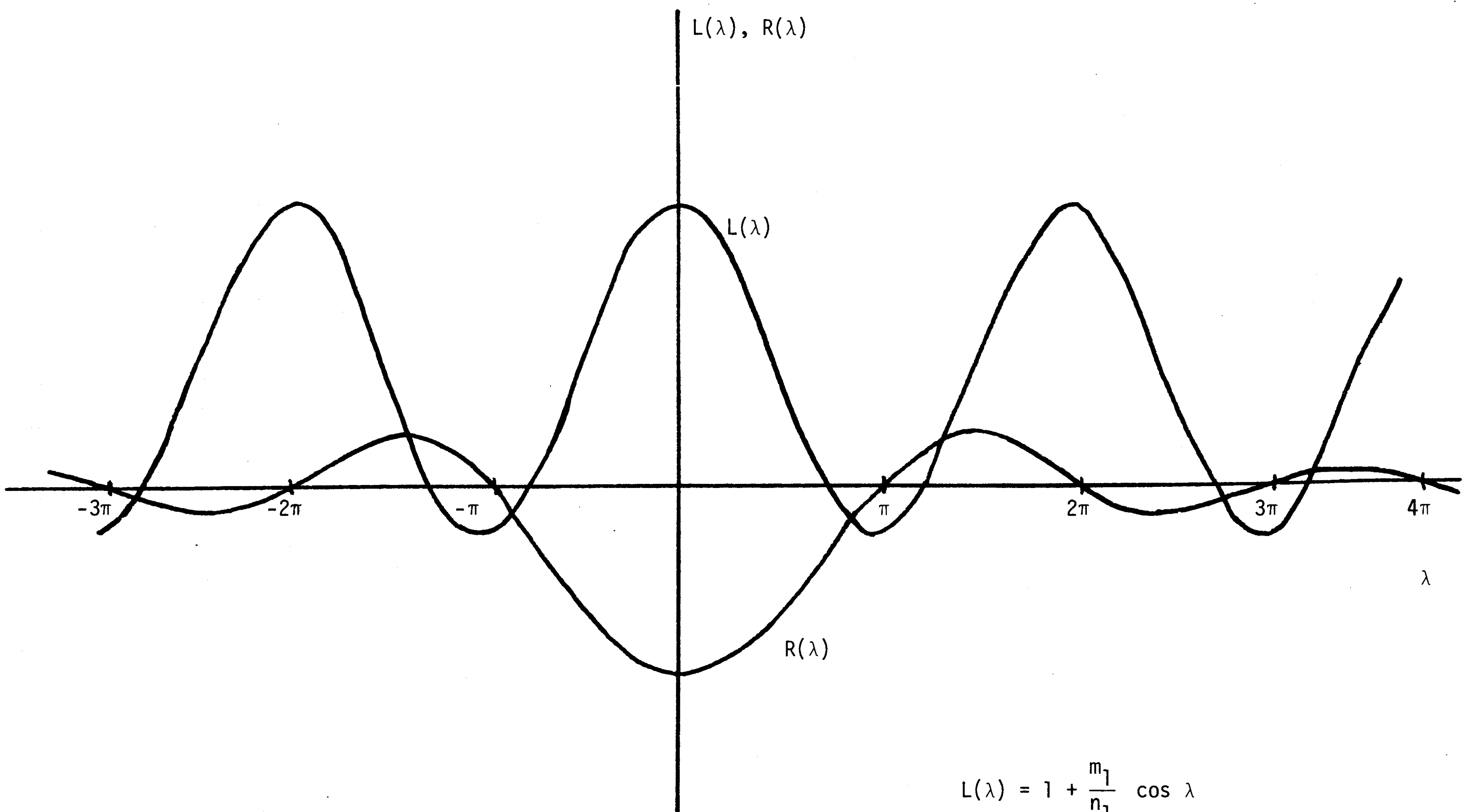
Evidently, for relatively small values of Θ, i.e., when $\frac{\Theta}{k_1} \leq 2$, the ratio m_1/n_1 will be no greater than unity in absolute value, and condition D. cannot be met for the same reason as in the previous case. However, when the lag measured by Θ is sufficiently large, i.e., when $\frac{\Theta}{k_1} > 2$, then $L(\lambda)$ will continue to intersect $R(\lambda)$ at indefinitely large positive (and negative) values for λ . An illustration is provided in Figure 2.[1/]

To see whether or not condition D. can be satisfied when $\frac{\Theta}{k_1} > 2$, let us determine first if the system's characteristic equation has the correct number of roots. Since the principal term of (5.b) is of order (1,1), condition D. requires there to be $4k + 1$ real roots over intervals of size $\left[(-2k\pi-\delta),\ (2k\pi+\delta)\right]$ for a sufficiently large value of k. We have already taken note of the fact that zero is one of those real roots. Also, since $L(\lambda)$ and $R(\lambda)$ are symmetric about $\lambda = 0$, there are the same number of negative and positive roots, and we can restrict our attention to counting the latter. If we label the positive roots successively starting from $\lambda_0 = 0$ as $\lambda_0, \lambda_1, \lambda_2$ etc., it may be seen that the odd-labeled roots $\left[\lambda_j; j=1,3,5,7\ldots\right]$ are found, one each, in the segments $\left[j\pi,\ (j+\tfrac{1}{2})\pi\right]$ -- i.e., one root in each occurance of the "second quadrant", when λ is interpreted as a radial measure. Similarly, even-labeled roots ($\lambda_j; j=2,4,6,8\ldots$) are located, one each, in the segments $\left[(j-1)\pi, j\pi\right]$ -- i.e, in either the "third" or "fourth quadrants" in each "cycle" of λ. Furthermore, the occurances in the "fourth quadrant", if relevant at all, are limited to relatively small λ's in the sequence of even-labeled roots. Hence, if we take the arbitrary constant δ to be zero, it is clear that $G(\lambda)$ will

---

1/ When the latter condition is met, coefficients m_0, m_1, n_0, and n_1 in (3.c) are all positive.

Figure 2



$$L(\lambda) = 1 + \frac{m_1}{n_1} \cos \lambda$$

$$R(\lambda) = - \frac{m_o}{n_1} \frac{\sin \lambda}{\lambda}$$

have exactly 4k+1 zeros for any integer k on an interval of size $\left[-2k\pi,\ 2k\pi\right]$. Condition D. is met, and this establishes in turn condition C.(1).

Next, we need to determine the conditions under which C.(2) is met, i.e., conditions such that $G'(\lambda)\cdot F(\lambda) > 1$ for all zeros of $G(\lambda)$. The function $G'(\lambda)$ is found by differentiating (6.b.2) to get

$$(8.c)\quad G'(\lambda) = n_1 + (m_1 + m_o)\cos\lambda - m_1\lambda\sin\lambda\,,$$

while $F(\lambda)$, given earlier, is

$$(6.b.1)\quad F(\lambda) = n_o + m_o\cos\lambda - m_1\lambda\sin\lambda\ .$$

First, for λ_o, we have

$$G'(\lambda) = 1 + m_o + m_1 > 0,$$
$$F(\lambda) = m_o + n_o > 0,$$

and condition C.(2) is clearly met for this root. Next, consider the odd-labeled roots.[1/] Let us use $G(\lambda) = 0$ to rewrite (6.b.1) and (8.c) as

$$(9.c.1)\quad F(\lambda) \doteq m_1 - \frac{n_1 m_o}{m_1} - \left(\frac{\sin\lambda}{\lambda}\right)\left[\frac{m_o^2}{m_1} + m_1\lambda^2\right]\ ,$$

$$(9.c.2)\quad G'(\lambda) = -\frac{n_1 m_o}{m_1} - \left(\frac{\sin\lambda}{\lambda}\right)\left[\frac{m_o^2}{m_1} + m_o + m_1\lambda^2\right]\ .$$

For odd-labeled roots (all of which lie in the "second quadrant"), $\sin\lambda/\lambda$ is positive. Also, it is readily shown that $m_1 - \frac{n_1 m_o}{m_1}$ is negative. Hence, both $F(\lambda)$ and $G'(\lambda)$ must be negative, and C.(2) is always met for odd-labeled roots.

Conditions that relate to the even-labeled roots are more involved, largely because, for roots located the "third quadrant", $\sin\lambda$ and $\cos\lambda$ carry the same algebraic sign. Inasmuch as the even-labeled roots increase without bound,

1/ Since $F(\lambda)$, $G(\lambda)$ and $G'(\lambda)$ are all symmetric around $\lambda=0$, we need only examine properties of the positive roots.

however, it is evident that for these roots we seek to establish conditions (i.e., conditions on underlying parameters of the system) such that both $F(\lambda) > 0$ and $G'(\lambda) > 0$ for all even-labeled roots.[1/] By using (9.c.1) and (9.c.2) together with $G(\lambda) = 0$, one can show that this task is equivalent to establishing conditions under which

$$\text{(10.c)} \quad \text{Min} \left\{ \begin{array}{l} \left[\frac{-m_1^2}{n_1 m_o} \lambda \sin \lambda \right] - \left[\frac{m_o}{n_1} \frac{\sin \lambda}{\lambda} \right] + \left[\frac{n_o m_1}{n_1 m_o} \right] \\ \left[\frac{-m_1^2}{n_1 m_o} \lambda \sin \lambda \right] - \left[\frac{m_o}{n_1} \frac{\sin \lambda}{\lambda} \right] - \left[\frac{m_1}{n_1} \frac{\sin \lambda}{\lambda} \right] \end{array} \right\} > 1 ,$$

for all even-labeled roots. Taking note of the fact that $\sin \lambda < 0$ for all even-labeled roots, it is apparent that all terms on the left-hand side of (10.c) are positive. It also can be confirmed that increases in Θ will cause each of the three major terms in each line on the left-hand side of (10.c) to increase, with the exception of the third term in the lower line (which declines but always remains positive).

Although condition (10.c) is exact, because it involves endogeneously determined values of λ (and an infinite set of them, as well) it fails to shed much light on the circumstances under which (10.c) holds and, thereby, under which C.(2) can be met for all roots. Our main interest here, however, is on the effect of introducing the lag Θ on stability. Focusing on Θ and holding other parameters constant, it is readily shown that for sufficiently small values of Θ (i.e., values of Θ that approach $2k_1$ from above) inequality (10.c) cannot be satisfied since $\sin \lambda$ and, therefore, the left-hand side of the second line in (10.c) can be made arbitrarily close to zero.

1/ The distinction being made here is that we can reject the other possible way to satisfy $G'F > 0$ for all λ -- i.e., both $G' < 0$ and $F < 0$.

Thus, as might be expected on the basis of the original basic model, this system is not globally stable for very short lags.

Furthermore, it is relatively easy to establish that the system can be made globally stable by a sufficiently large value of Θ . To confirm this important property, we offer the following argument: Let us define $\tilde{\lambda}$ as the smallest even-labeled root (for a given Θ) such that

$$-\frac{m_1^2}{n_1 m_0} \lambda \sin \lambda > 1$$

is satisfied for all $\lambda > \tilde{\lambda}$. (The existence of such a $\tilde{\lambda}$ is guaranteed by a non-zero lower bound on $|\sin \lambda|$.) Since all these roots satisfy (10.c) for this <u>or</u> any larger value of Θ , we now concentrate on the finite set of even-labeled roots for which $\lambda < \tilde{\lambda}$. It is easily shown that increases in Θ will expand the amplitude of both $L(\lambda)$ and $R(\lambda)$ in Figure 2. As can be seen, this will tend to increase the value of all even-labeled roots and, more importantly, raise $R(\lambda)$ for these roots. Since

$$R(\lambda) = -\frac{m_0}{n_1} \frac{\sin \lambda}{\lambda} ,$$

it is always possible to find a sufficiently large Θ (the least of which we designate as $\tilde{\Theta}$) such that $R(\lambda) > 1$ for all $\lambda < \tilde{\lambda}$, thereby satisfying (10.c). Accordingly, for $\Theta \geq \tilde{\Theta}$, condition <u>C.(2)</u> is satisfied for <u>all</u> roots, condition <u>A.</u> is met as well, and the system must be globally stable.[1/]

<u>Additional Comments</u>

Several points that bear on stability in international macromodels are illustrated by the three examples above. First, Case III shows that it is

---

1/ In practice, global stability may be achieved by values of Θ much below $\tilde{\Theta}$. The value of $\tilde{\Theta}$ is of interest only in that it guarantees that, for a sufficiently large Θ , the system can be made stable.

possible for plausible systemic "frictions" to invalidate the instability often associated with the use of myopic perfect foresight in these models. The contrast between findings in this case and the two previous, however, suggests that some compromise with <u>both</u> the "myopic" and the "perfect" aspects of this assumption are needed. It is noteworthy that neither making exchange-rate forecasts non-instantaneous (Case II) nor systematically obsolete (Case I) by itself is sufficient to bring on stability. In fact, these two cases are associated with a higher order of instability than the original model -- in the sense that a larger number of free variables is needed to place the system, once disturbed, on its stable dynamic path.

It is also evident that, from a technical point of view, the manner by which the particular constellation of assumptions in Case III brings about stability is quite similar to that associated with an adaptive rule. Although it is imbedded in a macro structure, the source of instability in these models is in some fundamental sense in the overly rapid adjustment of the exchange-rate forecasting process. Hence, in order to stabilize the system it is natural to look first at mechanisms that either slow the forecasting process or the adjustment of the exchange rate itself. Both an adaptive expectations rule and the lag developed in Case III perform this function. The assumptions of Case III, in effect, allow the substitution of the left-hand derivative for the right-hand derivative of the current expected exchange rate for a given cohort of transactors in the foreign-exchange market.[1/] In episodes when the exchange rate would otherwise tend to diverge at an accelerating pace away from equilibrium, it

1/ Actually, for any cohort of transactors, their common forecasts of the rate of change of the exchange rate are taken only at separated points. These rates of change can be applied to line segments that can then be joined in the obvious way to form a continuous path for the expected exchange rate for this cohort, with kinks at each time of market entry. It is the derivatives on the left-hand side and right-hand sides of these kinked points that are referred to above. For a discussion of when such an expectations structure may be consistent with perfect foresight in a similar example, see Gray and Turnovsky (1979b) and Burmeister and Turnovsky (1976).

is this feature which exercises a degree of control on the system and, if strong enough, permits the return of the system to its equilibrium. As with an adaptive rule, however, there is a clear tension between this specification and strict rationality, since the potential for stability depends on agents sustaining forecasts that are incorrect over finite intervals.

It is somewhat reassuring to be able to add one more argument to what is still a fairly short list that indicates that the foreign exchange market may be more stable than some recent models would have us believe. At this point, it is a rather small comfort, however, since the conditions for stability are still fairly specific and difficult to quantify. Until the model's structural parameters are better known and the effects of some of its special assumptions are better explored, it is difficult to say whether the critical lag is more on the order of months or minutes. If it is the latter, however, it suggests that the lags and frictions associated with the normal course of business may help to keep the foreign exchange market under control, rather than contribute to its instability.

For similar reasons one has to be wary at this point of proposals which would deliberately introduce or enlarge a lag of this type in the system so as to achieve greater stability. Even putting the question of enforcement of the lag aside for the moment, to be certain that greater stability would be the result we would have to be assured that expectations are formed in the fashion described in the model. If they were not -- and instead closer to the strictly rational standard of Case II, for example -- then less rather than greater stability might be the consequence. Also, sight should not be lost of the fact that attempting to stabilize the system by this device might require the introduction of a very large lag with its own attendant economic distortions on the real side of the economy that we have not even considered here.

Despite these qualifications I feel that it is of more than theoretical interest that global stability can be induced through a structural lag, since in many contexts this approach is more easily supported than some other competing stabilizing devices. Furthermore, for reasons mentioned above, we have deliberately confined attention to lags related to equation (2), the asset-market equilibrium equation. It is interesting to speculate whether some form of lag in the goods-market relationship -- where lags have a long tradition and are, if anything, likely to be more pronounced -- might be an additional source of stability. This, however, is an open question that must be left for later research.[1/]

1/ Some of these possibilities have been explored in another paper (Freeman, 1982) in which it was found that the introduction of "J-curve" effects into the current account tended, if anything, to increase the instability of the system.

BIBLIOGRAPHY

R. Bellman and K. Cooke (1963), Differential-Difference Equations, Academic Press, New York, 1963.

E. Burmeister and A.R. Dobell (1970), Mathematical Theories of Economic Growth, Macmillan, London, 1970.

P. Cagan (1956), "The Monetary Dynamics of Hyperinflation," in M. Friedman, ed., Studies in the Quantity Theory of Money, Chicago University Press, Chicago, 1956.

R. Dornbusch (1976), "Exchange Rate Expectation and Monetary Policy," Journal of Political Economy, December 1976.

R. Dornbusch, "Exchange-Rate Economics: Where Do We Stand?", Brookings Papers on Economic Activity, 1980:1.

R. Freeman (1982), "J-Curves and Stability of the Foreign-Exchange Market," International Finance Discussion Paper No.198, Board of Governors of the Federal Reserve System, Washington, D.C., January 1982.

M. Gray and S. Turnovsky (1979a), "The Stability of Exchange Rate Dynamics Under Perfect Myopic Foresight," Journal of International Economics, October 1979.

M. Gray and S. Turnovsky (1979b), "Expectational Consistency, Information Lags, and the Formulation of Expectations in Continuous Time Models," Econometrica, November 1979.

P. Kouri (1976), "The Exchange Rate and the Balance of Payments in the Short Run and in the Long Run: A Monetary Approach," Scandinavian Journal of Economics, April, 1976.

J. Tobin (1978), "A Proposal for International Monetary Reform," Cowles Foundation Discussion Paper #506, Yale University, October 1978.