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DYNAMIC INSTABILITY IN RATIONAL EXPECTATIONS MODELS: AN ATTEMPT TO CLARIFY

by

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I. Introduction and Summary

A prevalent feature of deterministic rational expectations macro-models is dynamic instability. Examples abound of both open and closed economy macro-models that "fly off the handle" unless precisely correct initial conditions on prices and/or exchange rates are satisfied.[1/] A common practice in calculating the equilibrium of these models is to arbitrarily select the stable arm of the model. This practice, in effect, rules out explosive and implosive solutions by assumption. It is sometimes justified by reference to an underlying framework involving intertemporal optimizing behavior.[2/] A frequently cited example of such a framework is provided by Brock (1975).[3/]

The primary purpose of the present paper is to explore the conditions under which the popular procedure of specifying a dynamically unstable macro-model and then arbitrarily selecting the stable arm of that model may be justified. Many of results of this paper can be found elsewhere. Most of them, however, come from a literature so technical as to

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render them inaccessible to the wide audience which needs to use them. It is hoped that this paper will help to rectify that situation.

The paper begins, in section II, with a brief review of two commonly cited papers. The first, by Sargent and Wallace (1973), provides an extremely simple example of a dynamically unstable rational expectations macro-model. The second, by Brock (1975), addresses the question of the non-uniqueness of equilibrium in this model by exploring the same question in a maximizing framework that generates similar dynamics. The results of the section do not provide unqualified support for the practice of arbitrarily selecting the stable arm of dynamically unstable models. In Brock's model, the steady-state is shown to be the unique equilibrium solution to the model only if certain restrictions on the household's utility function are met. If these restrictions are not met, there exist a multiplicity of equilibrium solutions to Brock's problem -- the steady-state and an infinite number of implosive real balance paths. The question of the economic relevance of these restrictions is postponed to a later section.

In section III, a transactions cost model of money demand is developed and analyzed. In this framework money is held because it reduces transactions costs; money does not enter the household's utility function directly, as it does in Brock's model. Among the results of the section are the following: First, the price and real balance dynamics of this transactions cost model are identical to those of Brock's model. It is generally not possible to rule out a multiplicity of equilibrium solutions to the model, all but one of which involves an ever decreasing level of real money balances. Further, the conditions under which uniqueness of equilibrium can be insured in the model are precisely analogous to those of the Brock model. Second, an intuitive development of the Euler equation associated with the model is undertaken. It is shown that the Euler equation may be regarded as a restriction on the path of prices that must hold if various finite horizon arbitrage schemes are to be unprofitable along the path.

A more careful study of explosive and implosive real balance paths is undertaken in section IV. It is verified that explosive real balance paths are not equilibrium solutions. Along such paths, the household can increase its lifetime utility by exploiting an open-ended arbitrage opportunity -- one <u>not</u> ruled out by the Euler equation, which all these paths necessarily satisfy. This arbitrage scheme is shown to produce a transversality condition which must be satisfied by all optimal real balance paths. It is violated by all the explosive real balance paths examined in this paper. Thus, we are able to provide an arbitrage interpretation of the use of transversality conditions in infinite horizon maximization problems. Implosive real balance paths are shown to satisfy the transversality condition appropriate for the two models discussed in this paper. They can, nevertheless, be ruled out if they lead to infeasible (negative) levels of prices or consumption. This can occur in either model if certain restrictions are satisfied. However, applying a result recently developed by Obstfeld and Rogoff (1981), these restrictions are shown to be economically irrelevant. The major conclusion of the section is that uniqueness of equilibrium can not be guaranteed in either model under any reasonable set of assumptions about the economic role of real money balances.

In the last section of the paper, more general versions of both Brock's model and the transactions cost model are presented. These more general models are not necessarily dynamically unstable. It is argued that saddle-point instability is a desirable property of models intended to explain the behavior of variables (such as exchange rates or interest rates) that are observed to "jump" in response to exogenous shocks. A condition is developed that guarantees saddle-point dynamics in the general versions of the two models of this paper. In Brock's model, there is no *a priori* presumption as to whether or not this condition should be met. In the transactions cost model of this paper, the condition is necessarily violated. However, in an alternative transactions cost framework the condition is necessarily satisfied. The conclusion of the section is that the answers to some questions may depend critically on the specific story one chooses to tell about *why* households hold money. Models in which money enters the utility function directly can not adequately answer these questions.

II. A Selective Review of the State-of-the Art

The purpose of the present paper is to explore the conditions under which the common practice of specifying a dynamically unstable macro-model and then arbitrarily selecting the stable arm of that model may be justified. This section provides a brief review of two commonly cited papers. The first of the two serves to motivate the very general question posed above. It provides an extremely simple example of a dynamically unstable rational expectations macro-model. The second addresses the question of the non-uniqueness of equilibrium in this model by exploring the same question in a maximizing framework that generates similar dynamics.

As noted in the introduction, examples abound of both open and closed economy macro-models that "fly off the handle" unless precisely correct inital conditions on prices and/or exchange rates are satisfied. Sargent and Wallace (1973) illustrate this phenomenon neatly in a simple model in which real growth is zero, income and the nominal money supply are fixed, and the price level is assumed to be consistent with equilibrium in the money market, satisfying at each moment in time a Cagan money demand function of the form.

$$(1) \qquad \ln\left(\frac{M}{P_t}\right) = \alpha\left(\frac{\dot{P}_t}{P_t}\right)^e.$$

As usual M, P, and $(\dot{P}_t/P_t)$ denote the nominal money stock, the price level, and the expected rate of inflation, respectively. The real rate of interest is omitted from this formulation since it is assumed to be a constant.

Under the assumption of rational expectations (which, in a deterministic model, is equivalent to the assumption of perfect foresight), the expected rate of inflation is equal to the realized rate of inflation. Under this assumption, the model is dynamically unstable. It has a unique steady state in which the value of real balances is one, the price level is equal to the nominal money stock, and the rate of inflation is zero. If the price level assumes any value other than it's steady state value, P*, prices will diverge ever further from their steady state value as time passes. The price path will be explosive if the price level initially exceeds P* and "implosive" (P_t goes to zero) if the price level is initially less than P*.

It will be convenient to study the dynamics of this model and subsequent ones in terms of the path of real balances rather than the price level. If the nominal money stock is fixed, the rate of change of real balances is simply the negative of the rate of change of the price level. In this model, then,

(2) $\dot{m}_t/m_t = -(1/\alpha)\ln(m_t)$, where $m_t = (M_t/P_t)$.

This relationship is depicted in figure 1.

It is common in dynamically unstable macro-models to arbitrarily select the "stable arm" of the model in calculating equilibrium. When the dynamics are one-dimensional, as they are in the Sargent and Wallace framework, this means admitting only the steady state (or states) as an equilibrium. As long as the admissable equilibria of a model are restricted to steady state paths, the solution of the model is typically

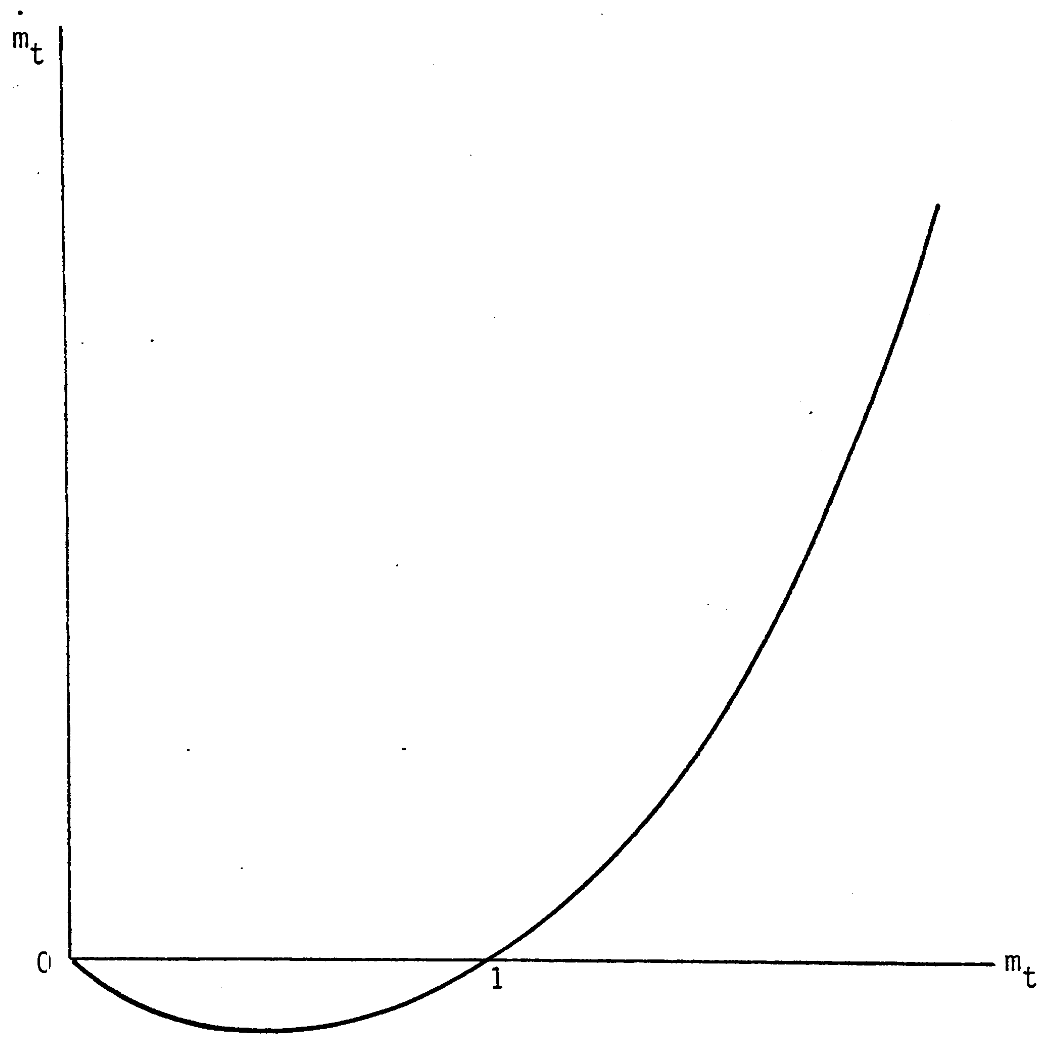


Figure 1

The Dynamics of the Sargent and Wallace Model straightforward, yielding a unique equilibrium as long as the steady state is unique. If stability is imposed on the Sargent and Wallace model, for example, the only admissable equilibrium solution to the model is the steady state, in which the value of real balances and the price level are constant. An unanticipated increase in the money supply on this model simply leads to a matching instantaneous jump in the price level -- real balances never deviate from their steady state value of unity. In the absence of the assumption of stability, one is left with a staggering infinity of admissable equilibria -- all but one of which involve ever increasing or ever decreasing holdings of money balances.

The arbitrary selection of the stable arm in the kind of model described above is sometimes justified by reference to an underlying framework involving intertemporal optimizing behavior. A frequently cited example of such a framework is provided by Brock (1975). However, contrary to the impression these citations generate, Brock's work does not justify the arbitrary selection of the stable arm of a dynamically unstable model. To the contrary, it is possible to delineate clearly in his model the conditions under which explosive price paths can not be ruled out as equilibrium solutions. Further, it is argued in section III that these conditions are the only economically relevant ones. We turn now to a brief summary of Brock's model and the conclusions it produces.

Consider an economy composed of a number of identical, infinitely lived households, each of which maximizes a utility function of the form[4/]

(3) $Z = \int_0^\infty e^{-\rho t}[u(c_t) + v(m_t)]dt$,

subject to the budget constraint

(4) $P_t y = P_t c_t + \dot{M}_t$.

Following conventional notation, ρ is an internal rate of discount, c_t and m_t are real household consumption and money balances at time t, P_t is the price level, and $\dot{M}_t$ is the instantaneous time rate of change of nominal money balances. Real income rains from the heavens on households at a fixed rate of y units per period. Consumption and the price level are contrained to be non-negative. Note that money yields utility directly in this formulation. Further, the utility function is additive in consumption and real balances. This is critical for Brock's results.

Formally, the solution to the household's problem is found by maximizing the Hamiltonian function, H, with respect to the household's control variable, c_t, where H is given by

(5) $H = [u(c_t) + v(m_t)]e^{-\rho t} + \lambda_t[P_t y - P_t c_t]$.

It is important to note that the household treats the path of the price level, P_t, as exogenous. From the maximum principle, we know that any interior consumption plan that maximizes H must satisfy equations (6) through (8) below.[5/]

(6) $\frac{\partial H}{\partial c_t} = u'(c_t)e^{-\rho t} - \lambda_t P_t = 0$, for all t.

(7) $$\frac{\partial H}{\partial M_t} = -\dot{\lambda}_t = \lambda_t v'(m_t)(1/P_t)e^{-\rho t}, \quad \text{for all } t.$$

(8) $$\frac{\partial H}{\partial \lambda_t} = \dot{M}_t = P_t y - P_t c_t, \quad \text{for all } t.$$

Given an arbitrary price path, equations (6) through (8) can be solved for the set of consumption plans that meet the necessary conditions for an optimum. However, we are not interested in all price paths and their associated optimal consumption plans. We are interested only in equilibrium price and consumption paths. Equilibrium in the markets for goods and money requires that the fixed stock of nominal money balances be demanded by households at each point in time. For the representative household this means that planned net increments to its nominal money balances must be zero, or

(9) $$\dot{M}_t = 0, \quad \text{for all } t.$$

Combining equation (19) with equations (6) through (8) allows us to solve for the set of equilibrium price and consumption paths that meet the necessary conditons for an optimum.[6/] This set consists of all price and consumption paths that satisfy both equation (10) and equation (11) below.

(10) $$\frac{\dot{P}_t}{P_t} = \frac{v'(m_t)}{u'(y)} - \rho, \quad \text{for all } t.$$

(11) $c_t = y$, for all t.

Equation (10) is often referrred to as the Euler equation. It summarizes the dynamic characteristics of the model's candidate solution paths. Intuitively, it is the condition that insures the absence of profitable arbitrage opportunities along any finite segment of an optimal consumption plan. This interpretation of the Euler equation is discussed in greater detail in section III below.

Again, it will be convenient to study the dynamics of this model in terms of the path of real balances rather than the pricel level. Given a fixed stock of nominal money, equations (10) and (11) imply equation (12).

(12) $$\frac{\dot{m}_t}{m_t} = \rho - \frac{v'(m_t)}{u'(y)} .$$

This relationship is depicted in figure 2. The steady state level of real money balances, m*, is found by setting $\dot{m}_t$ equal to zero in equation (12). If real balances initially exceed m*, equation (12) states that m_t will increase indefinitely over time. Under the assumption of diminishing marginal utility of money, it is also true that the rate of increase of m_t will increase as m_t rises. The case in which real balances initially fall short of m* is somewhat more complicated. Equation (12) can produce, under different specifications of the function v(m), any one of the three implosive real balance paths shown. At issue is the limiting value of $\dot{m}_t$ as m_t approaches zero. From (12), this limit can be written as

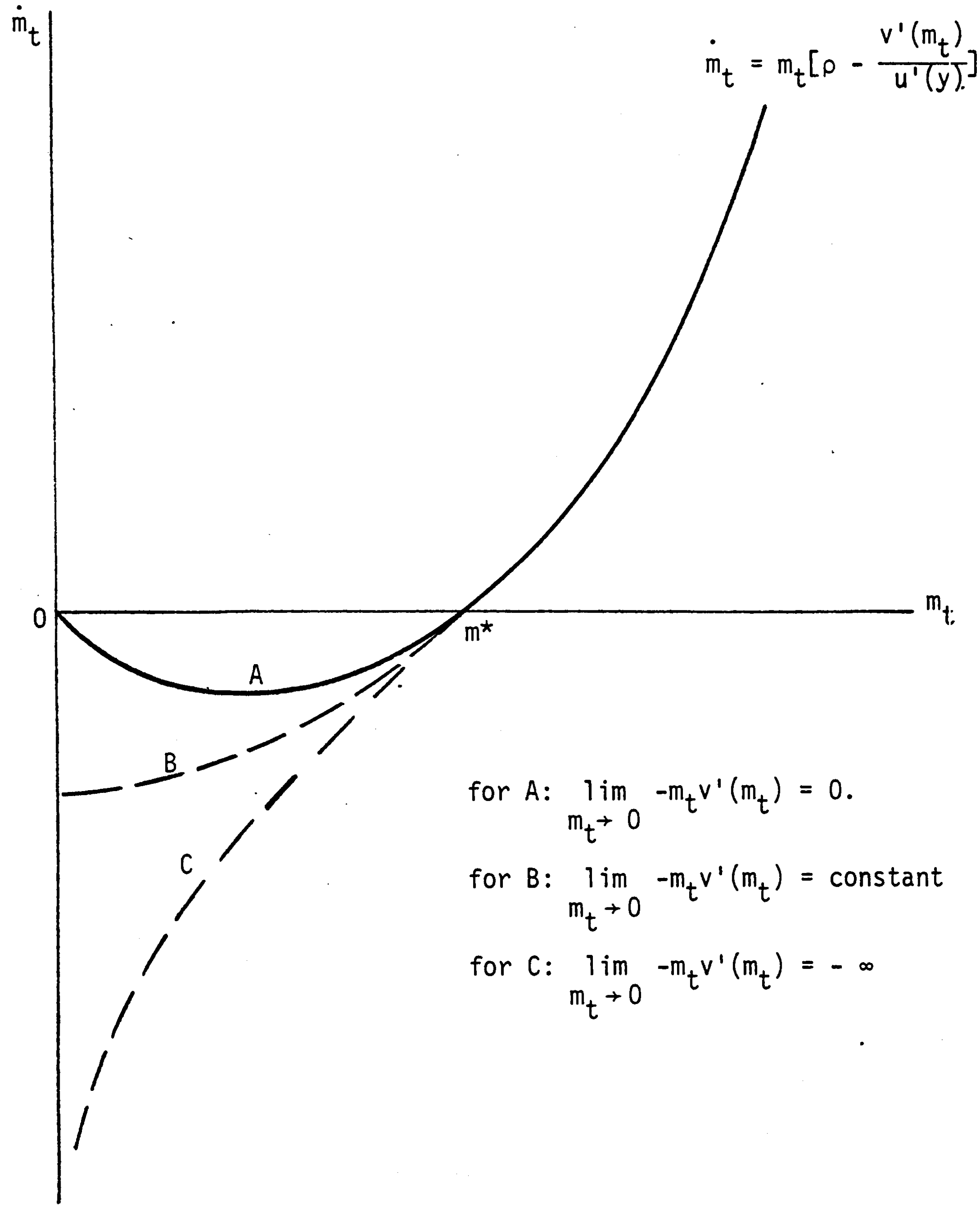


Figure 2

The Dynamics of Brock's Model

(13) $$\lim_{m_t \to 0} \dot{m}_t = \lim_{m_t \to 0} m_t[\rho - \frac{v'(m_t)}{u'(y)}] = \lim_{m_t \to 0} - \frac{m_t v'(m_t)}{u'(y)} .$$

If this limit is zero, $\dot{m}_t$ goes to zero as m_t goes to zero, resulting in path A. If the limit is a constant, path B results. And if the limit doesn't exist (it is negative infinity), path C results.

All of the real balance paths described above may be regarded as candidate equilibrium solutions. Following Brock, we will divide them into three groups and study them further. Consider first all the paths along which real balances are growing over time. It is possible to show that none of these paths are true equilibrium solutions. Formally, they may be ruled out because they violate a transversality condition. Intuitively, it can be shown that households can increase their utility by running down their money balances along such paths. These arguments are examined in greater detail in section IV. For now we simply note that it is possible to exclude as true eqilibria explosive real balance paths (or, equivalently, implosive price paths). This is true in Brock's model as well as the transactions cost framework of the next section.

Consider next all the paths along which real balances are shrinking over time. Some of these paths can be rule out because they imply that real balances turn negative at some (finite) point in time. This is true of all paths of type B and type C. Along these paths, the time rate of change of real balances goes to a negative constant or negative infinity as the level of real balance approaches zero. At some point, then, the equations of motion that generate these paths imply that real balances pass through zero and become negative. Given a fixed (positive) nominal money stock, this, in turn, implies a negative price level, which is infeasible. The remaining paths -- all type A paths -- can <u>not</u> be ruled out. Along these paths, the time rate of change of real balance slows as the level of real balances falls, nearing zero as real balances go to zero. As time goes to infinity, the level of real balances asymtotically approaches zero. Such path can not be ruled out on the grounds that they are infeasible. Nor do they present profitable arbitrage opportunities as explosive real balance paths do (alternatively, they violate no known transversality condition.) It follows, then, that it is possible to eliminate as equilibrium solutions only those implosive real balance paths that satisfy the following condition:

$$(14) \qquad \lim_{m_t \to 0} \left[- m_t \left(\frac{v'(m_t)}{u'(y)}\right)\right] < 0.$$ [7/]

Finally, we turn to the steady state solution -- $m_t = m^*$ for all t. This solution is feasible and violates no known transversality condition. Therefore it can not be ruled out as an equilibrium path of real balances.

The results of Brock's model may be summarized as follows: If equation (14) holds, the steady state is the unique equilibrium solution to his model. If equation (14) does <u>not</u> hold (the limiting value of $m_t v'(m_t)$ is equal to zero), there exists a multiplicity of equilibrium solutions. These consist of the steady state and an infinite number of implosive real balance paths -- one corresponding to each possible level of initial real balances. This model, then, does not provide unqualified support for the practice of arbitrarily selecting the stable arm of macro-models with similar dynamic characteristics.

It is interesting to note that the dynamics of the Sargent and Wallace model (pictured in figure 1) are qualitatively identical to the dynamics of the Brock model for the case in which equation (14) does not hold (shown by path A of figure 2). This is, of course, exactly the case in which it is not possible to identify the steady state as the unique equilibrium of the model.

For those interested in insuring uniqueness of equilibrium in their models, the preceding discussion suggests an obvious strategy: Employ only models whose dynamic properties satisfy equation (14) or its equivalent. Unfortunately, this strategy is flawed. In section IV we will show that the conditions under which equation (14) holds are not economically relevant conditions. Under any reasonable set of assumptions about the nature and role of real money balances, the limit set out in equation (14) is equal to zero. The practice of selecting the stable arm of dynamically unstable models remains, in general, a practice that can claim no theoretical support. Price bubbles are presently alive and well, in theory at least.

III. A Transactions Cost Approach

In this section we develop and analyze a framework in which money is held because it reduces transactions costs. Money does not enter the household's utility function directly, as it does in Brock's model. There are several reasons for choosing to introduce such an alternative to Brock's model. First, it will serve to demonstrate that the dynamics of Brock's model are not special to his model and, in particular, are not due to the fact that money enters the utility function directly. Explicit modelling of the household's incentives to hold money balances does not lead to price or real balance dynamics that differ in any important way from those produced by Brock. Specifically, in the transactions cost model studied below it is generally not possible to rule out a multiplicity of equilibrium solutions, all but one of which involves an ever decreasing level of real money balances. Further, the conditions under which uniqueness of equilibrium can be insured in the model are precisely analogous to those of the Brock model.

A second reason for exploring a transactions cost approach is that it makes it easier to judge the "economic relevance" of the kind of restriction discussed above. We will see in section IV, for example, that the conditions which gurantee a unique equilibrium in the transactions cost model of this section also imply that transactions costs become unbounded as real money balances approach zero. This is not an economically relevant specification since it implies negative consumption at some point along any implosive real balance path.

Finally, this approach will prove useful in section V, which develops a condition that guarantees saddle-point dynamics (as opposed to local stability) in more general versions of both Brock's model and the transactions cost model of the present section. In Brock's model, this condition (which is sufficient, but not necessary) will be met under one set of (apparently) reasonable restrictions on the household's utility function but violated under another. In the transactions cost model discussed below, this condition is violated under <u>any</u> reasonable specification of the household's utility function. It is pointed out, however, that the ambiguity associated with Brock's framework can be duplicated in an alternative transactions cost framework. Thus, the answers to some questions may depend critically on the specific story one chooses to tell about why households hold money. Models in which money enters the utility function directly cannot adequately address these questions.

One of the tasks undertaken in this section is an intuitive development of the Euler equation using an "arbitrage" argument. This argument is, in essence, a simple application of calculus of variations. Readers already familiar with this ground may wish to skip the bulk of the material appearing between equations (18) and (27). They are advised, however, to acquant themselves with the particular arbitrage argument (or perturbation) employed, since it will play a role in the discussion of transverality conditions in section IV.

We turn now to the specification of a model in which money is held for the explicit purpose of reducing transactions costs. There are variety of ways in which transactions costs might be modelled. The approach taken in this section was choosen for its simplicity. Some alternate approaches are touched on in section V. Again, we begin by describing the representative household's problem, which is to maximize a utility function of the form

$$(15) \quad Z = \int_0^\infty e^{-\rho t} u(c_t)dt,$$

subject to the budget constraint

$$(16) \quad P_t y = P_t c_t - P_t \pi(m_t) - \dot{M}_t,$$

and the restrictions

$$u' > 0,\ u'' = 0,$$

$$\pi' < 0,\ \pi'' > 0.$$

Notation is the same as in the previous section. Note that real money balances do not appear in the utility function. Lifetime utility is a function only of the household's consumption stream. Instantaneous utility, $U(c_t)$, is a linear function of the level of consumption. This assumption is critical to the conclusions of this section in exactly the same way that the assumption of separable utility is critical to the conclusions drawn from Brock's model in the preceding section. Both assumptions are relaxed in section V. Real balances influence lifetime utility through their affect on the level of real resources available for consumption. Specifically, the conversion of income into consumption involves a transactions technology in which real resources and real money balances are substitutable. Real resource transactions costs per household are given by the function $\pi(m_t)$. As real balances increase, transactions costs decrease. But the reduction in transactions costs is smaller, the higher the initial level of real balances; the technology is subject to diminishing returns. The representative household, then, allocates its (fixed) income to consumption, transactions costs, and increments (positive or negative) to its money balances, as shown in equation (16).

Following the procedures outlined in section II for Brock's problem, we can solve for the set of price and consumption paths that satisfy both the usual necessary conditions for an optimum and the requirements of goods and money market equilibrium. This set consists of all price and consumption paths that satisfy both the Euler equation and the budge constraint for this problem -- that is, equations (17) and (18) below.

$$(17) \qquad \frac{\dot{P}_t}{P_t} = -\pi'(m_t) - \rho , \qquad \text{for all } t.$$

$$(18) \qquad c_t = y - \pi(\dot{m}_t), \qquad \text{for all } t.$$

It will prove useful later, in our discussion of transversality conditions, to have established an arbitrage interpretation of equation (17). The Euler equation can be usefully regarded as a restriction on the path of prices that must hold if various finite horizon arbitrage schemes (or "perturbations") are to be unprofitable along that path. This restriction must be met if the path is optimal; accordingly, it is a necessary (but not sufficient) condition for an optimum. For the purposes of illustration, we will be considering a very specific arbitrage scheme. The Euler equation can, however, be developed from any feasible finite horizon perturbation.

Consider a candidate equilibrium solution to the problem posed in equations (15) and (16) above -- an arbitrary price path and an associated consumption path that satisfy the budget constraint given by equation (16). Given the consumption path, the price path is an equilibrium price path if $\dot{M}_t = 0$ at every instant in time along the path. The household must be satisfied with its money holdings at every point in time. Thus, for equilibrium price and consumption paths, consumption must be equal to real income less real transactions costs. That is, equation (18) must be satisfied. In addition, however, the proposed consumption path must be optimal for the household, given the proposed price path. The consumption path will be optimal if all perturbations of the path that satisfy the household's budget constraint produce no increase in the household's utility. An example of such a perturbation would be a decumulation of the household's nominal money balances for some period of time, followed by a subsequent accumulation that just restores the household's nominal balances to their initial level. For the household's budget constraint to be satisfied, this requires a corresponding initial increase in the household's rate of consumption followed by a decrease, all relative to the original proposed path of consumption. Because this perturbation involves an "exchange" of consumption during one time period for consumption during another, it is referred to as an arbitrage scheme, or simply an arbitrage, in what follows.

If an arbitrage scheme like the one proposed above raised the household's utility, the household would exploit that opportunity and attempt to decumulate money for a period of time. This, however, would mean that the household is not satisfied with its money holdings at every point in time along the proposed price and consumption paths. Accordingly, the proposed paths could not constitute an equilibrium solution.

Formalizing the arguments outlined above is relatively straightforward. We begin with an arbitrary price path and consumption stream along which households are satisfied holding the (fixed) nominal stock of money existence. An example of such a proposed solution, involving constant prices and consumption, is depicted in figure 3. The paths of variables corresponding to this candidate optimal solution are identified by the superscript "0". The arbitrage scheme to be evaluated is also shown in figure 3, the associated paths identified by the superscript "A". Note, again, that the household takes the path of prices as given. The arbitrage scheme shown in figure 3 may be characterized as follows. The household consumes according to the proposed solution, neither adding to nor subtracting from its initial holdings of nominal money, up to some arbitrary time T. At time T, it begins to decumulate nominal money at a constant rate μ, where μ is small. The decumulation continues until time T + 1, giving a total change in nominal money balances at that point in time of μ units. From T + 1 to T + 2, the household accumulates nominal money at the same rate, u, returning the household to its initial money stock at time T + 2. From time T + 2 on, the household again consumes according to its original plan and holds a constant stock of nominal money. Thus, the paths of M_t^A and $\dot{M}_t^A$ may be written as

$$(19) \quad M_t^A = M_t^0 - \mu n(t)$$

and

$$(20) \quad \dot{M}_t^A = -\mu\dot{n}(t),$$

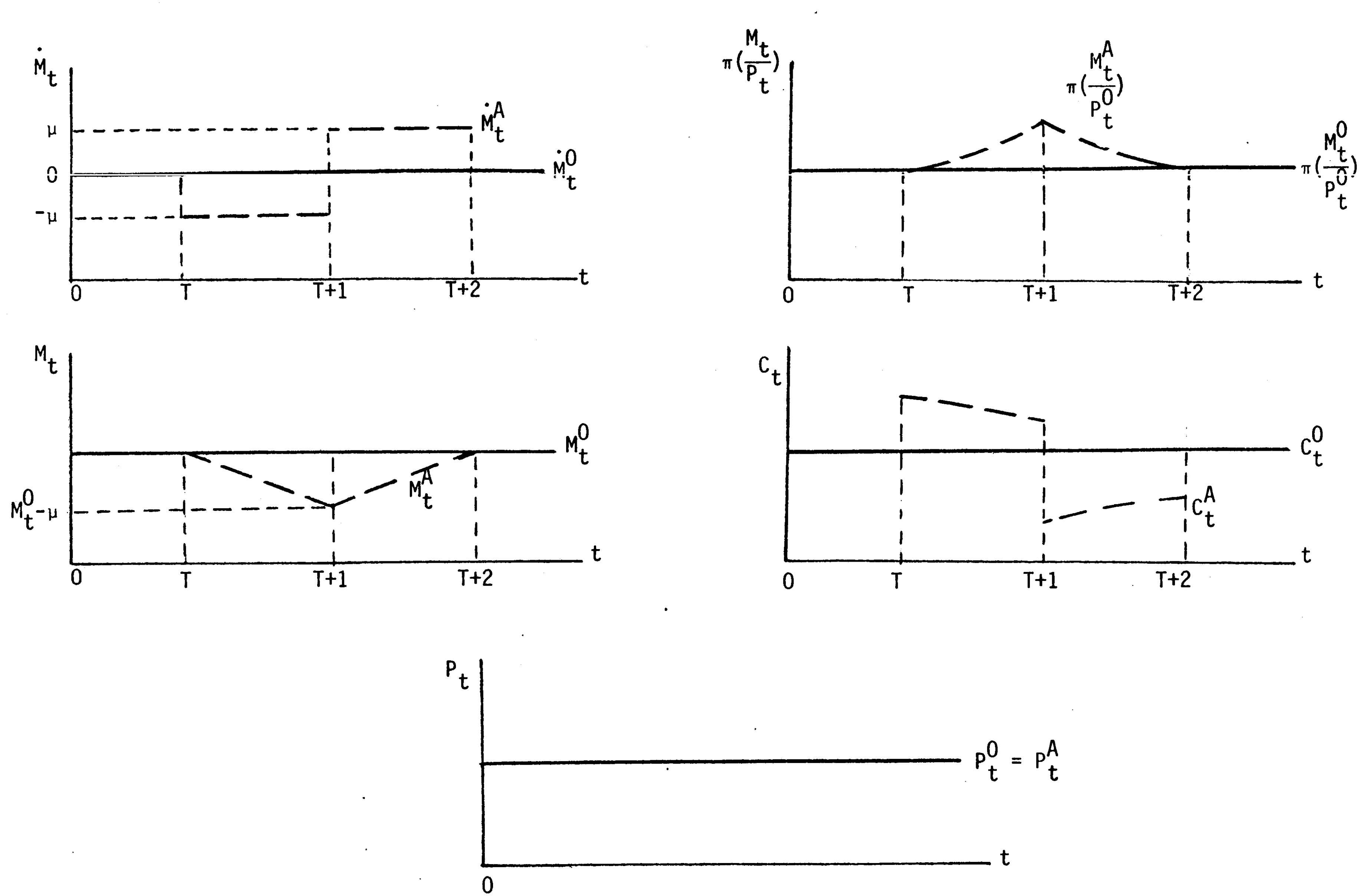


Figure 3

An Example of a Finite Horizon Perturbation where

$$n(t) = \begin{cases} 0 \\ t - T \\ T + 2 - t \\ 0 \end{cases} \quad \text{and} \quad \dot{n}(t) = \begin{cases} 0 & \text{for } t < T. \\ 1 & \text{for } T \leq t < T + 1. \\ -1 & \text{for } T + 1 < t \leq T + 2. \\ 0 & \text{for } t \geq T + 2. \end{cases}$$

The corresponding path of consumption is given by

$$(21) \quad C_t^A = y - \pi\left(\frac{M_t^A}{P_t^0}\right) - \frac{\dot{M}_t^A}{P_t^0} = y - \pi\left(\frac{M_t^0 - \mu n(t)}{P_t^0}\right) + \frac{\mu \dot{n}(t)}{P_t^0} .$$

Note that (19) through (21) are correct for <u>any</u> proposed solution, not just the constant price and consumption paths depicted in figure 3.

It is now possible to evaluate the change in household utility associated with this arbitrage scheme. Begin by substituting equations (19) through (21) into the expression for lifetime household utility given by equation (15). This produces a characterization of lifetime utility involving μ, prices, the time T, and other exogenous variables:

$$(22) \quad Z(\mu) = \int_0^T U\left[y - \pi\left(\frac{M_t^0}{P_t^0}\right)\right] e^{-\rho t} dt$$
$$+ \int_T^{T+2} U\left[y - \pi\left(\frac{M_t^0 - \mu n(t)}{P_t^0}\right) + \frac{\mu \dot{n}(t)}{P_t^0}\right] e^{-\rho t} dt$$
$$+ \int_{T+2}^{\infty} U\left[y - \pi\left(\frac{M_t^0}{P_t^0}\right)\right] e^{-\rho t} dt.$$

Given an arbitrary path of prices and an initial money stock M_0 that satisfy equation (18), we wish to know if household utility can be increased by a small perturbation of the original path -- that is, by increasing μ from a value of zero to some small poitive value. The change in household utility associated with such an exercise is just the derivative of $Z(\mu)$ with respect to μ, evaluated at μ equal to zero. It is given by equation (23).

$$(23) \quad \left.\frac{dZ(\mu)}{d\mu}\right|_{\mu=0} = \int_T^{T+2} [U'(C_t^0)\pi'(M_t^0)\left(\frac{n(t)}{P_t^0}\right)e^{-\rho t} + U'(C_t^0)\left(\frac{\dot{n}(t)}{P_t^0}\right)e^{-\rho t}]dt.$$

Integration by parts of the second term in the integral allows us to rewrite equation (23) in the following way:[8/]

$$(24) \quad \left.\frac{dZ(\mu)}{d\mu}\right|_{\mu=0} = \int_T^{T+2} U'(C_t^0)n(t)\left[\frac{\pi'(m_t^0)}{P_t^0} + \frac{\rho}{P_t^0} + \frac{\dot{P}_t^0}{(P_t^0)^2}\right]e^{-\rho t}dt.$$

If the originally proposed price and consumption paths are optimal, it must be true that

$$(25) \quad \left.\frac{dZ(\mu)}{d\mu}\right|_{\mu=0} = 0$$

This, in turn, will be true if the expression in brackets on the right-hand-side of equation (24) is equal to zero, or

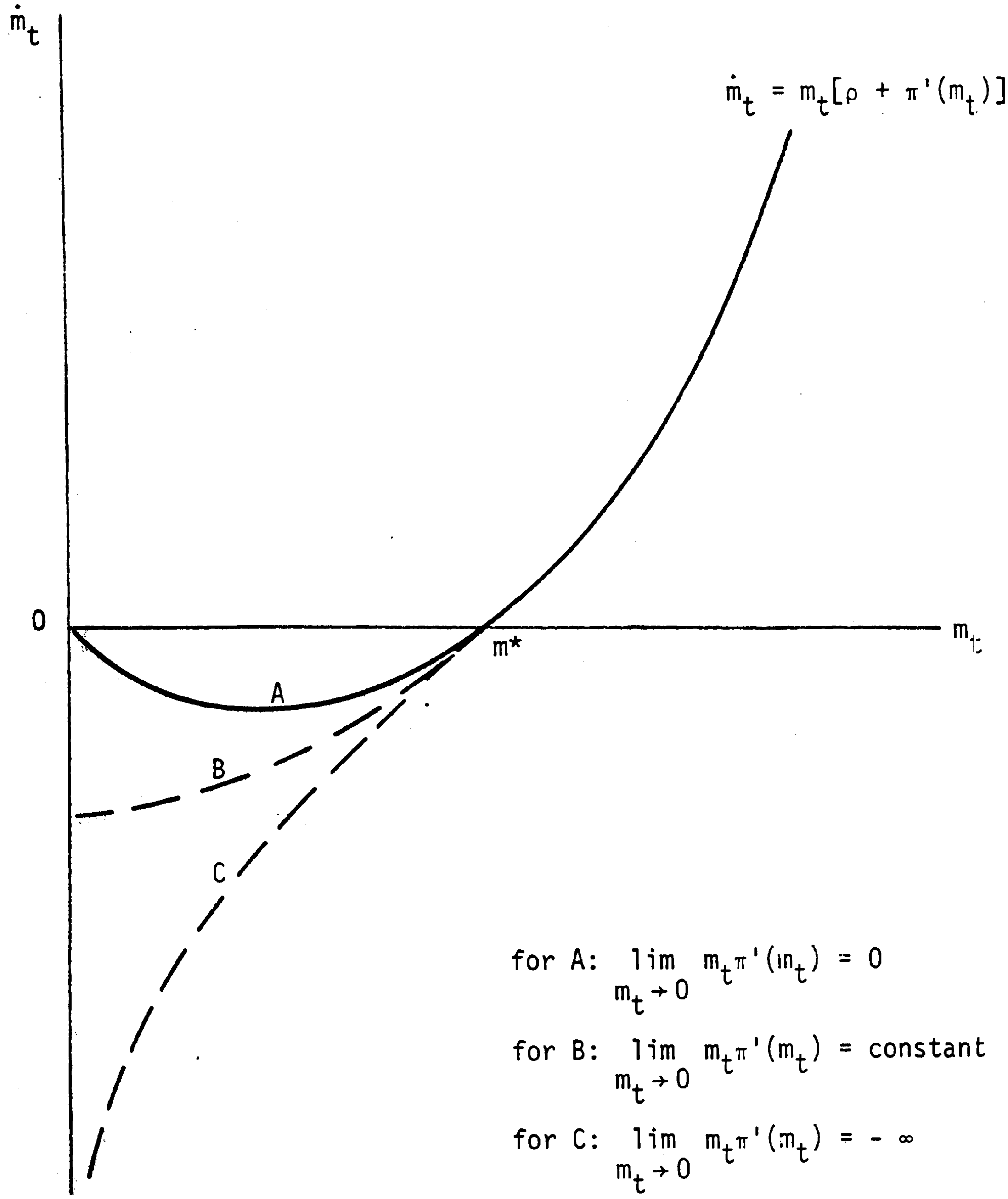
$$(26) \quad \dot{P}_t^0/P_t = -\pi'(m_t^0) - \rho \qquad \text{for } T \leq t \leq T + 2.$$

To show that equation (26) is a necessary condition for an optimum requires one more step. For a particular arbitrage scheme, denoted by a particular function of time, $n(t)$, the derivative $dZ(\mu)/d\mu$ could, in principle, equal zero even if equation (26) did not hold. While this could occur for a particular function, $n(t)$, it could not, however, occur for all possible functions, $n(t)$, that met the boundary conditions $n(T) = 0$ and $n(T + 2) = 0$. That is, if the change in utility associated with all arbitrage schemes on the interval $(T, T+2)$ is to be zero, equation (26) must hold. Finally, the interval $(T, T+2)$ was selected arbitrarily. The arguments that produced equation (26) could be made for any choice of T. Accordingly, equation (26) must hold at every point in time along any equilibrium solution. This completes our proof that the Euler equation is a necessary condition for an optimum.[9/]

The Euler equation given by equation (17) describes the set of price paths that are candidates for equilibrium in the model of this section. As before, however, it will be convenient to study the dynamics of the model in terms of the path of real balances rather than the price level. Given a fixed nominal stock of money, equation (17) implies equation (27).

$$(27) \quad \frac{\dot{m}_t}{m_t} = \rho + \pi'(m_t) \qquad \text{for all } t.$$

This relationship is depicted in figure 4. Comparison of figure 4 with figure 2 shows that the dynamics of this model are identical to the dynamics of Brock's model. In both models the steady state level of real balances, m^*, is found by equating marginal value of real balances in terms of the



<u>Figure 4</u>

The Dynamics of the Transactions
Cost Model

consumption good to the internal rate of discount, ρ. If real balances initially exceed m^*, they will increase at an increasing rate over time. If real balances initially fall short of m^*, one of three implosive real balance paths will result. Which of three occurrs depends on the limiting value of $\dot{m}_t$ as m_t approaches. In the transactions costs model of the present section, this limit is given by

$$(28) \quad \lim_{m_t \to 0} \dot{m}_t = \lim_{m_t \to 0} m_t \pi'(m_t).$$

The economic interpretation of the term $m_t \pi'(m_t)$ is identical to the interpretation of the term $-m_t v'(m_t)/u'(y)$ in Brock's model. Both represent the stock of real balances times the marginal value of real balances in terms of the consumption good. If, as m_t goes to zero, the limiting value of this term is zero, path A results. If the limit is a constant, path B results. And if the limit is negative infinity, path C results.

We have developed, in this section, a model in which money demand is motivated by transactions costs. This model produces price and real balance dynamics that are identical to those of Brock's model. As before, we are confronted with a multiplicity of candidate solutions to the model, all of which satisfy the Euler equation and the household's budget constraint. The Euler equation guarantees the absence of profitable arbitrage opportunities along any finite segment of these solutions.

IV. A Closer Look at Explosive and Implosive Paths

The transactions cost model of the preceding section produced a multiplicity of candidate solution paths for prices and real balances. The characteristics of these paths are identical to those of the candidate solutions produced by Brock's model. As before, they can be divided into three groups -- those along which real balances increase over time, those along which real balances decrease over time, and those along which real balances are constant over time.

In this section, we substantiate and provide the intuition behind the claims made in section II concerning explosive and implosive real balance paths. It is verified that explosive real balance paths are not equilibrium solutions. Along such paths, the household can increase it's lifetime utility by exploiting an open-ended arbitrage opportunity -- one not ruled out by the Euler equation, which all these paths necessarily satisfy. This arbitrage scheme is shown to produce a transversality condition which must be satisfied by all optimal real balance paths. It is violated by all the explosive real balances paths examined in this paper. Thus, we are able to provide an arbitrage interpretation of the use of transversality conditions in infinite horizon maximization problems.[10/]

Implosive real balance paths are shown to automatically satisfy the transversality condition appropriate for the models discussed in this paper. They can, nevertheless, be ruled out as equilibrium solutions if they lead to infeasible (negative) values of the price level. This will occurr if certain restrictions on the transactions cost function, $\pi(m_t)$, (or, in Brock's model, on the function $v(m_t)$) are met. However, these restrictions are shown to be economically irrelevant in that they imply an unbounded level of transactions costs, (or infinitely negative utility) at some small but finite level of real balances. Implosive real balance paths can also be ruled out if they lead to negative levels of consumption. This can not occur in Brock's model but can occur in the particular transactions costs model described in the previous section. The major conclusion of the section is that uniqueness of equilibrium can not be guaranteed in either model under any reasonable set of assumptions about the nature and role of real money balances.

A. Explosive Real Balance Paths

Along real balance paths of this kind, prices decrease, asymtotically approaching zero, and real money balances increase, becoming arbitrarily large. Transactions costs decrease and consumption increases. If these paths are optimal, the household must be unable to increase its lifetime utility by varying its consumption plan or money holdings in any way. There can exist no profitable arbitrage opportunities of any kind along an optimal solution. All the paths under consideration in this section satisfy the Euler equation and the household's budget constraint. Accordingly, we are guaranteed the absence of profitable arbitrage opportunities along any finite segment of these paths. The arbitrage schemes already precluded, then, are those that are eventually "reversed". At some (finite) point in time, real money balances and consumption are returned to their original paths. There is no assurance, however, that "unreversed" arbitrage schemes are unprofitable along these paths. In fact, they are profitable. In demonstrating this point, we will use the unreversed version of the arbitrage scheme that was employed in the preceding section to establish the Euler equation.

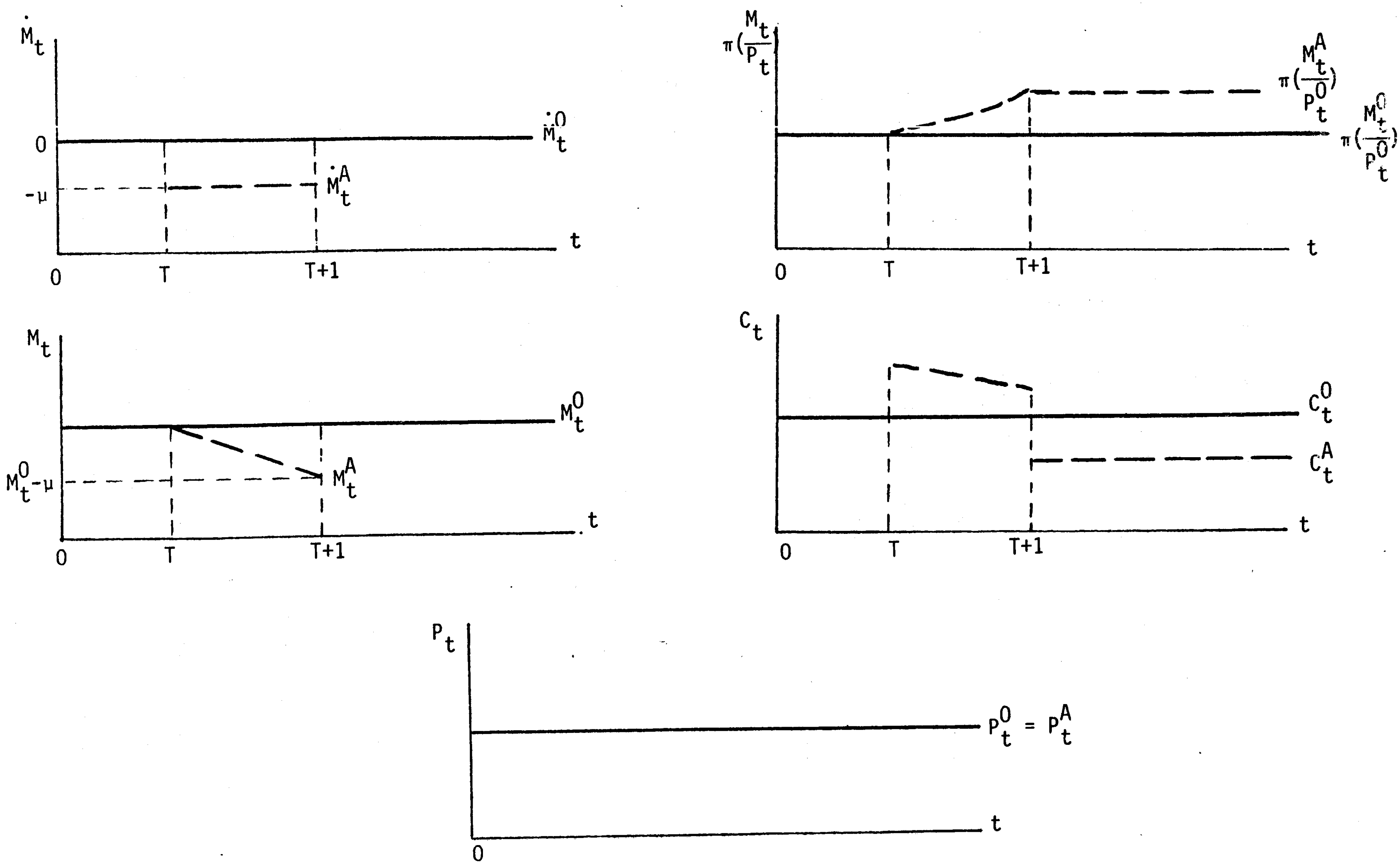


Figure 5

An Example of an Infinite Horizon Perturbation

Consider a particular candidate solution along which real money balances are increasing through time. Suppose that at some arbitrary time T, the household begins to decumulate nominal money balances at a rate μ, continues to decumulate money at this rate until time T+1, and then maintains the resulting (lower) level of nominal money balances thereafter. The essential feature of this arbitrage scheme is that money balances are never returned to their original path. It is an unreversed arbitrage. It can be shown that the gains to this arbitrage are positive along any explosive real balance path. Intuitively, the net gain to converting a small amount of money into consumption along such paths is positive and increases with time. (Net gains, here, are discounted to the time at which the arbitrage is initiated.) As real money balances become arbitrarily large, the discounted stream of increased transactions costs that would be incurred if nominal money balances were decreased by a unit becomes arbitrarily small. But the benefit to reducing nominal balances by a unit, measured by the amount of consumption that could be purchased by a unit of money, or $1/P_t$, becomes arbitrarily large. It follows that the net gain, discounted to the time the arbitrage is initiated, must increase over time and must be positive for sufficiently distant future points in time. As the analysis below will demonstrate, this net gain, when discounted back to the beginning of the household's planning horizon, is not only positive but is independent of when the arbitrage is initiated.

Formally, these results can be obtained by explicitly evaluating the change in household utility resulting from the arbitrage scheme outlined above. For clarity, the scheme under consideration is illustrated in figure 5. Notation is the same as in the previous section. The superscript "0" identifies the paths of variables associated with the proposed optimal solution, and the superscript "A" identifies paths associated with the arbitrage scheme. As before the household takes the path of prices as given. To facilitate comparison with the results of section III, the transactions costs model of that section is used. The paths of M_t^A $\dot{M}_t^A$, and C_t^A may be written as follows:

$$(29) \quad M_t^A = M_t^0 - \mu n(t),$$

$$(30) \quad \dot{M}_t^A = -\mu\dot{n}(t),$$

$$(31) \quad C_t^A = y - \pi(\frac{M_t^A}{P_t^0}) - \frac{\dot{M}_t^A}{P_t^0} = y - \pi(\frac{M_t^0 - \mu n(t)}{P_t^0}) + \frac{\mu\dot{n}(t)}{P_t^0},$$

where

$$n(t) = \begin{cases} 0 \\ t - T \\ 1 \end{cases} \quad \text{and } \dot{n}(t) = \begin{cases} 0 & \text{for } t < T, \\ 1 & \text{for } T \leq t < T + 1, \\ 0 & \text{for } t \geq T + 1. \end{cases}$$

Substituting equation (31) into the equation (15) produces an expression for household utility involving μ, prices, the time T, and other exogenous variables:

$$Z(u) = \int_0^T U[y - \pi(\frac{M_t^0}{P_t^0})]e^{-\rho t}dt$$

$$(32) \qquad + \int_T^{T+1} U[y - \pi(\frac{M_t^0 - \mu n(t)}{P_t^0}) + \frac{\mu \dot{n}(t)}{P_t^0}]e^{-\rho t}dt$$

$$+ \int_{T+1}^{\infty} U[y - \pi(\frac{M_t^0 - \mu}{P_t^0})]e^{-\rho t}dt$$

Given the proposed solution, we wish to know if the household's utility can be increased by a small perturbation of its consumption plan -- that is, by increasing μ from zero to some small value in equation (32). The resulting change in household utility is given by equation (33).

$$(33) \quad \left.\frac{dZ(\mu)}{d\mu}\right|_{\mu=0} = \int_T^{T+1} [U'(C_t^0)\pi'(m_t^0)(\frac{n(t)}{P_t^0})e^{-\rho t} + U'(C_t^0)(\frac{\dot{n}(t)}{P_t^0})e^{-\rho t}]dt$$

$$+ \int_{T+1}^{\infty} U'(C_t^0)\pi'(m_t^0)(\frac{1}{P_t^0})e^{-\rho t}dt.$$

Integration by parts of the second term in the first integral allows us to rewrite equation (33) as follows:

$$(34) \quad \left.\frac{dZ(\mu)}{d\mu}\right|_{\mu=0} = \int_T^{T+1} U'(C_t^0)n(t)(1/P_t^0)[\pi'(m_t^0) + \rho + \frac{\dot{P}_t^0}{P_t^0}]e^{-\rho t}dt$$

$$+ \frac{U'(C_{T+1}^0)e^{-\rho(T+1)}}{P_{T+1}} + \int_{T+1}^{\infty} U'(C_t^0)\pi(m_t^0)(1/P_t^0)e^{-\rho t}dt.$$

Along any optimal path of prices and consumption it must be true that

$$(35) \quad \left.\frac{dZ(\mu)}{d\mu}\right|_{\mu=0} = 0 \qquad \text{for all } T.$$

Since all the paths under consideration in this section satisfy the Euler equation (equation (17)), the first integral on the right hand side of equation (34) is equal to zero. Thus, for equation (35) to hold, it must be true that

$$(36) \quad \frac{U'(C^0_{T+1})e^{-\rho(T+1)}}{P^0_{T+1}} = -\int_{T+1}^{\infty} U'(C^0_t)\pi'(m^0_t)(1/P^0_t)e^{-\rho t}dt \qquad \text{for all } T.$$

Along any optimal solution, then, equation (36) must hold. The interpretation of this condition is closely related to the intuitive arguments set out earlier. The left-hand-side of equation (36) gives the present discounted value (in terms of utility) of the additional consumption the household can obtain by giving up one unit of nominal money balances at time T+1. The right-hand-side of the equation gives the present discounted value (in terms of utility)of the stream of increased transactions costs incurred if money balances are lowered by one unit from time T+1 on. If the two sides are equal, the proposed arbitrage is unprofitable.

In order to establish the existence of profitable arbitrage opportunities along explosive real balance paths, it is sufficient to demonstrate that equation (36) does not hold along such paths. To do this, consider the following conceptual exercise. Suppose that for some time T+1 equal to t_0 equation (36) holds. Now think about values of T+1 greater than t_0. As T+1 increases, the left side of equation (36) decreases, but approaches a positive constant (a lower limit) as T+1 becomes arbitrarily large. This follows from the fact that U' is a constant and P_{T+1} decreases with time at a rate that is less than ρ at any finite point in time, but approaches ρ as T+1 becomes arbitrarily large. (See equation (17).) As T+1 becomes arbitrarily large, the rate of change of $e^{-\rho(T+1)}$ and P_{T+1} both have limiting values of ρ. Consequently, their ratio approaches a positive constant.

By contrast, the right side of equation (36) approaches zero as T+1 becomes large. This is shown by evaluating the integrand as time (and real balances) become infinite. It has just been established that the first term in the integrand is a constant and that the product of the last two terms in the integrand goes to a constant as time becomes infinite. That leaves the term $\pi'(m_t^0)$, which, under any economically relevant specification,[11/] must approach zero as m_t^0 becomes large. The limiting value of the integrand, then, is also zero along any explosive real balance path. It follows that as T+1 becomes arbitrarily large, the integral on the right side of equation (36) becomes arbitrarily small.

From the preceding arguments, the following may be concluded: If equation (36) holds for some T+1 equal to t_0 along an explosive real balance path, there will exist a time t_1 greater than t_0 for which it will be true that equation (36) is volated. Thus, along any explosive real balance path, there is at least one point in time at which it will be profitable for the household to permanently decrease it holdings of nominal money balances. Accordingly explosive real balance paths -- or, alternatively, implosive price paths -- offer the household at least one profitable arbitrage opportunity.

Differentiation of equation (34) with respect to T demonstrates an additional point: The discounted gains associated with the decumulation of μ units of money are independent of when the decumulation takes place. The derivative calculated in equation (36) does not vary with T. Thus, if such an arbitrage is profitable at any one point in time, it is equally profitable at all points in time. In the case of explosive real balance paths, then, the household has an incentive to reduce its money holdings, but is indifferent with regard to timing. This result is also implicit in the Euler equation given by equation (17). The relationship shown there could equally well have been derived by writing down the conditions under which the household would be indifferent between exchanging a unit of money for goods today, and the same action carried out at a later date.

Finally, from equation (36) we can obtain a transversality condition that is a necessary condition for an optimum in this model. Consider once again the expression on the left-hand-side of equation (36). It is well-defined for any finite value of T+1 along any path of prices and real balances that imply feasible (non-negative) levels of consumption. It's limiting value as T+1 goes to infinity is a positive constant for all explosive real balance paths and is zero for all implosive real balance paths and the steady state. For equation (36) to hold, then, the integral on its right-hand-side must have a finite value -- that is, it must converge. This means that as T+1 becomes arbitrarily large, the value of the integral must approach zero. This, in turn, implies that the left-hand-side of equation (36) must go to zero as T+1 goes to infinity. Thus, a necessary condition for an optimum in this problem is

$$\text{(37)} \quad \lim_{t\to\infty} \frac{U'(C_t^0)e^{-\rho t}}{P_t^0} = 0.$$

This can be rewritten in a more familiar form.

$$\text{(38)} \quad \lim_{t\to\infty} \lambda_t e^{-\rho t} = 0, \quad \text{where } \lambda_t = U'(C_t^0)/P_t^0.$$

This condition is often imposed in infinite horizon optimization problems, frequently without justification. It requires that the discounted utility value of an additional unit of nominal money balances go to zero as time goes to infinity -- that the cost of reversing our proposed arbitrage go to zero as the time of reversal goes to infinity.[12/] As demonstrated above, it is a necessary condition for an optimum in the transactions cost model explored in this section because it insures the absence of profitable arbitrage opportunities that are not excluded by the Euler equation. An analogous condition is also necessary in Brock's model. It does not follow that the transversality condition developed in this section is appropriate for all problems. It is not. It is possible, however, to develop the correct transversality condition (if it exists) for most problems by using arbitrage arguments similar to the ones employed here.[13/]

Explosive real balance paths do, of course, violate the transversality conditon set out in equation (37). As discussed earlier, the limit in question is a positive constant for such paths, not zero.

The preceding analysis shows that along any explosive real balance path households can make themselves better off by reducing their money balances and increasing their consumption at some point in their lives. At the moment the decumulation of money begins, there will be an excess demand for goods and an excess supply of money. Therefore for no explosive real balance path -- or, alternatively, no implosive price path -- can be an equilibrium path. Of the total set of price paths consistent with equations (17) and (18), we may now exclude from further consideration all real balance paths that increase over time. It is left to the interested reader to proove that the same arguments can be used to exclude explosive real balance paths in Brock's model.

Two classes of candidates for equilibrium real balance paths remain -- the steady state path and the set of paths that diverge from the steady state toward zero. We turn now to the second of these.

B. Implosive Real Balance Paths

Along real balance paths of this kind, prices increase, becoming arbitrarily large, and real money balances decrease. Transactions costs increase with time, implying continual decreases in the level of household consumption. If these paths are optimal, there can be no profitable arbitrage opportunities available to the household along them. Since they all satisfy the Euler equation, we are guaranteed the absence of profitable arbtirage opportunities along any finite segment of these paths. Further, the transversality condition given by equation (37) is automatically satisfied by any implosive real balance path. Thus, households can not make themselves better off along such paths by permanently increasing their nominal money balances. Unreversed arbitrage schemes of the sort examined in the preceding subsection are unprofitable along implosive real balance paths. This does not mean, however, that such paths are necessarily admissable equilibrium solutions. They can be ruled out if they lead to infeasible (negative) values of the price level or consumption.

From the results of section III, we know implosive real balance paths can be divided into three classes. Examples of each are shown in figure 3. The Euler equation can produce, under different specifications of the transactions technology, any one of the three types of paths shown there. At issue is the limiting value of $\dot{m}_t$ as m_t goes to zero. This limit is given by equation (28) of section III and is repeated here for convenience

$$(28) \quad \lim_{m_t \to 0} \dot{m}_t = \lim_{m_t \to 0} m_t \pi'(m_t).$$

If this limit is zero, paths of type A result. If the limit is a constant, type B paths result. And if the limit is negative infinity, type C paths result.

Paths of types B and C -- those for which the time rate of change of real balances goes to a negative constant or negative infinity as real balances approach zero -- can be ruled out as equilirium solutions. The equation of motion that generates these paths (the Euler equation) implies that at some finite point in time real balances pass through zero and become negative. Given a fixed (positive) nominal money stock this, in turn, implies a negative price level, which is infeasible. The remaining paths --

all type A paths -- can not be ruled out on the same grounds. Along these paths, the time rate of change of real balances slows as the level of real balances falls, nearing zero as real balances go to zero. As time goes to infinity, then, the level of real balances asymtotically approaches zero.

From the foregoing discussion, it is evident that we can eliminate as an equilibrium solution any implosive real balance path of type B or C. Accordingly, it is possible to rule out all paths that satisfy the following condition:

$$(39) \quad \lim_{m_t \to 0} m_t \pi'(m_t) < 0.$$

Equation (14) of section II gives the analogous condition for the Brock model.

The preceding discussion suggests an obvious strategy for those interested in insuring uniqueness of equilibrium in the models they work with: Employ only models whose dynamic properties satisfy equation (39) or its equivalent. The problem with this strategy is that the conditions under which equation (39) holds are not economically relevant conditions. The same is true of equation (14) for Brock's model. This argument is made by Obstfeld and Rogoff in a recent paper on hyperinflationary equilibria.[14/] Using Brock's model, they show that the conditions under which implosive real balance paths can be excluded also imply that the utility of real balances necessarily goes to negative infinity as real balances go to zero. This, it may argued, does not reflect a reasonable view of the economic role of real money balances.

It is, however, easier to discuss the economic role of money in a model in which money is assigned an explicit economic function. The Obstfeld-Rogoff argument can, of course, be made in the context of the transactions cost framework introduced in this paper. It can be shown that if equation (39) holds, it must be true that transactions costs become infinite as real balances go to zero. This is infeasible for an economy in which resources are limited since it implies negative levels of consumption.

It has been established that the only economically relevant specifications of transactions costs (or utility, for Brock's model) are those that produce implosive real balance paths of type A. As already shown these paths can not be ruled out on the grounds that they lead to infeasible (negative) values of the price level. Nor do they provide unexploited arbitrage opportunities. They can, nevertheless, be ruled out as equilibrium solutions if they lead to infeasible (negative) levels of consumption. This can not occur in Brock's model, but it can occur in a transactions cost model. In particular, it will occur for any specification of the transactions technology that generates transactions costs in excess of household income as real balances become arbitrarily small. It is possible, then, to rule out implosive real balance paths for specifications under which it is true that

$$(40) \quad \lim_{m_t \to 0} \pi(m_t) > y.$$

Accordingly, for the transactions cost model of this paper, there are conditions under which it is possible to rule out implosive real balance path of type A. Again, however, these conditions are not economically relevant. Any economically relevant specification of transactions costs would, presumably, include the restriction that transactions costs never exceed the household's resources, which implies that

$$(41) \quad \lim_{m_t \to 0} \pi(m_t) \leq y.$$

If equation (41) holds, consumption remains non-negative at all points along any implosive real balance path.

We may conclude, then, that implosive real balance paths -- or, alternatively, explosive price paths -- can not be ruled out as equilibrium solutions in either of the models studied in this paper. Under any reasonable set of assumptions concerning the economic role of real money balances, these paths fail to violate any known necessary condition for an optimum or feasibility requirement.

We turn, finally, to the steady state path of real balances. This path satisfies the transversality condition given by equation (37). It violates no feasibility requirements. It is an admissable equilibrium solution.

The results of this section may now be summarized as follows: Uniqueness of equilibrium can not be guaranteed in either of the models studied in this paper so far under any reasonable set of assumptions concerning the economic role of real money balances.

V. Stability Versus Instability

Both of model's studied so far have been dynamically unstable; the unique steady-state of each has been a saddle-point. For neither of these models have we been able to prove that the steady-state is its only equilibrium solution. Even so, there is an advantage to working with such models relative to working with models that are stable in the neighborhood of the steady-state.

As discussed earlier, it is common practice to specify a model with saddle-point properties and to then arbitrarily select the stable arm of that model. This procedure rules out implosive and explosive solutions by assumption -- and, as shown above, may not be justifiable. The advantage to such a modelling strategy is that it leads to unique solution paths for all the variables of the model. This is of particular interest when analyzing the effects of exogenous disturbances. Consider, for example, the effects of an unanticipated increase in the nominal supply of money in either of the models studied so far. If the steady-state is assumed to be the only equilibrium solution in these models, it follows that the price level must instantaneously increase in the same proportion as the money supply, leaving real balances and consumption unchanged. The only effect of an unanticipated change in the money supply is a jump in the price level.

Of course, not all models are dynamically unstable. More general versions of both of the models of this paper can be shown to be stable under some conditions. However, stability is not necessarily a desirable property of such a model. It can lead to a multiplicity of solution paths, all of which converge to the steady-state. Consider again an unanticipated increase in the nominal money supply, this time in a stable version of one of the models discussed above. Without additional assumptions, there is no way to tell what will happen to the price level at the time of the increase. It could rise, fall, or remain unchanged. Regardless of what happens to the price level in the first instant, it will proceed to converge monotonically to its new steady-state value thereafter. As a result, there will be a multiplicity of solution paths following such shocks, one corresponding to each feasible level of the price level at the instant of the shock.

In some stable models this problem is solved by assuming that the price in question -- here, the price level -- can not jump. This assumption identifies the solution path leading from the pre-shock price to the new steady-state price as the unique equilibrium solution to the model. In models in which the price in question is the price level, the assumption of sticky prices may be appropriate. In models where the relevant price is the exchange rate or an interest rate such an assumption is likely to be inappropriate. If, then, one is interested in explaining the behavior of a variable that is observed to "jump" in response to some shocks, a model with saddle-point properties is preferable to a model that is stable in the neighborhood of the steady-state.

In the remainder of this section, we will examine more general versions of Brock's model and the transactions cost model of section III. For both models there is a condition that is sufficient (but not necessary) to insure that the model has saddle-point properties. That condition can be met in Brock's model but can not be met in the transactions costs framework. It is argued, however, that an alternative transactions technology can produce results analogous to those of the generalized Brock model.

Thus, identifying the specific economic role of money is shown to be important for some questions.

A critical feature of Brock's model is the assumption of separable utility. Suppose instead that the household's utility function takes the more general form given by equation (3')

$$(3') \quad Z = \int_0^{\infty} e^{-\rho t} U(c_t, m_t) dt,$$

where

$$U_c > 0, \ U_{cc} < 0,$$

$$U_m > 0, \ U_{mm} < 0.$$

If equation (3) of section II is replaced by equation (3'), the resulting optimization problem produces an Euler equation of the form

$$(42) \quad \frac{\dot{P}_t}{P_t} = \frac{(U_m/U_c) - \rho}{1 + m_t(U_{cm}/U_c)} \ .$$

In the case of separable utility, the cross-term U_{cm} is equal to zero and equation (42) reduces to equation (10) of section II. The dynamic properties of this more general model will be the same as those of the simpler model as long as U_{cm} is not negative and large enough to make the denominator of the right side of equation (42) negative. Thus, a condition sufficient to guarantee a saddle-point in the general model is

$$(43) \quad U_{cm} > 0.$$

That is, consumption and money balances must be regarded by the household as complements.[15/] *A priori*, there is little to suggest whether such a restriction is reasonable or not.

Consider next the transactions cost model developed in section III. In that model it is assumed that the household's utility at a point in time is a linear function of the level of its consumption -- that the marginal utility of consumption is a constant. Suppose instead that the more common assumption of diminishing marginal utility is adopted. In that case, the correct Euler equation for the model is given by

$$(44) \quad \frac{\dot{P}_t}{P_t} = \frac{-\pi'(m_t) - \rho}{1 - m[\pi'(m_t)U''/U']} .$$

In the special case of constant marginal utility of consumption, U" is equal to zero and equation (44) reduces to equation (17) of section III. The dynamics of the more general model will be the same as the dynamics of the simpler model as long as the denominator of the right side of equation (44) is positive. A condition sufficient to insure this result would be the analog of equation (43) -- the requirement that $-\pi'(m_t)U''$ be positive. By assumption, this condition is not be met. Both $\pi'(m_t)$ and U" are negative. It is interesting to note the interpretation of the negative of the product of these two terms. It gives the effect on the marginal utility of consumption of the change in consumption generated by a change in real money balances. More concisely, it represents the effect on the marginal utility of consumption of a change in real balances. It is the conceptual equivalent of the term U_{cm} in Brock's model. In our transactions cost model this effect is necessarily negative. An increase in real balances lowers transactions costs, increasing consumption. Higher consumption, in turn, generates a lower marginal utility of consumption. Thus real balances and cosumption are necessarily substitutes in this model.

It would be incorrect to assume, on the basis of the foregoing discussion, that consumption and money act as substitutes in all transactions cost models of money demand. Consider, for example, the following problem: The household maximizes a utility function that includes both consumption goods and leisure,

$$Z = \int_0^\infty U(c_t, \ell_t)e^{-\rho t}dt,$$

subject to the constraints

$$p_t y = p_t c_t + \dot{M}_t, \qquad U_c > 0,\ U_\ell > 0,$$

$$\ell_t = \overline{\ell} - T(m_t), \qquad U_{cc} < 0,\ U_{\ell\ell} < 0,$$

$$U_{c\ell} > 0.\underline{16/}$$

This model specifies a transactions technology in which leisure and real money balances are substitutable. An increase in money lowers transactions costs, which increases the household's leisure time. If consumption goods and leisure time are complements (as assumed here), the increase in leisure raises the marginal utility of consumption goods. In this transactions cost model, then, money and consumption act as complements.

The Euler equation for this problem is given by

$$\frac{\dot{P}_t}{P_t} = \frac{-\left(\frac{U_L T_m}{U_c}\right) - \rho}{1 - m_t\left(\frac{T_m U_{cL}}{U_c}\right)} .$$

Under the assumptions outlined above, this model is necessarily dynamically unstable.

Footnotes

1/ See Sargent and Wallace (1973), Penti Kouri (1976), Rudiger Dornbush (1976), to mention only a few.

2/ See Lucas (1975).

3/ See also Brock (1974) and Brock (mimeo.)

4/ Brock's original work is formulated in discrete time. The continuous time analog of the simplest of his models is presented here.

5/ See Intriligator (1971), Chapter 14.

6/ Equation (11) is obtained by setting $\dot{M}_t = 0$ in (8). To derive equation (10), first differentiate (6) totally with respect to time and manipulate the resulting expression to get

$$\frac{U''(c_t)\dot{c}_t}{U'(c_t)} - \rho = \frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{P}_t}{P_t}$$

Substituting (7) and (11) into this expression gives (10).

7/ Brock overlooked this condition in his original work (Brock (1974) and (1975)). This omission is corrected in a later note (Brock (mimeo.)). I am indebted to Brock for pointing this condition out to me in an early conversation pertaining to his original papers.

8/ Using integration by parts, we have

$$\int_T^{T+2} U'(C_t^0)\left(\frac{\dot{n}(t)}{P_t^0}\right)e^{-\rho t}dt =$$

$$\left.\frac{U'C_t^0)e^{-\rho t}n(t)}{P_t^0}\right]_T^{T+2} - \int_T^{T+2} U'(C_t^0)n(t)e^{-\rho t}\left[\frac{-\rho P_t^0 - \dot{P}_t^0}{(P_t^0)^2}\right]dt.$$

The first term on the right-hand-side of this equation is zero since $n(T+2) = n(T) = 0$. Substituting the integrand of the remaining term into the right-hand-side of equation (23) produces equation (24).

9/ Our proof follows closely the development of the Euler equation presented in Chapter 12 of Intriligator (1971).

10/ This is not to say that the transversality condition developed in this paper -- which is the usual condition imposed in such problems -- is appropriate for all problems. It is not. For a detailed discussion of how to develop the transversality condition appropriate for particular classes of problems, see Gray and Salant (1981).

11/ For $\pi'(m_t^0)$ to approach zero as m_t^0 becomes arbitrarily large, it is sufficient to assume that transactions costs have a lower bound. To assume otherwise would imply that transactions costs become infinitely negative as real balances increase.

12/ See Gray and Salant (1981) for a more detailed discussion of this interpretation of the standard transversality condition.

13/ Again, see Gray and Salant (1981).

14/ See Obsfeldt and Rogoff (1981).

15/ The importance of this restriction for more general versions of Brock's model is discussed in Calvo (1979).

16/ This model, according to Bob Flood (who first described it to me), is known to some as the "Virginia monetary model."

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