An rSQP Approach for a Single-level Reliability Optimization

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Abstract

In reliability-based design optimization (RBDO) the evaluation of the reliability constraints involve the solution of an optimization problem. This bi-level nature of most RBDO formulations can be too expensive to be implemented in some practical optimization problems, where a single function evaluation can range from minutes to hours or even days. An interesting approach replaces the optimization problem required to evaluate each reliability constraint with its corresponding first-order optimality conditions. As a result the dimensionality of the problem suffers an increment in the number of design variables and equality constraints. This structure holds much similarity to the one that arises in a simultaneous analysis and design (SAND) formulation for optimization of problems governed by PDEs. In this paper, ideas from research on reduced-Hessian methods for SAND are applied for the solution of the unilevel reliability optimization formulation. The reduced approach is applied to both SQP-like and response surface based-like approaches A discussion of the appropriate formulation for this treatment and some numerical examples are presented.

Introduction

In general nonlinear optimization, a general inequality constraint can be given by

$$g_i(\mathbf{x}) \geq 0, \tag{1}$$

where $\mathbf{x}$ is the vector of design variables. Some of this quantities may have uncertainties associated with them, and can be associated in the vector of random variables $\mathbf{r}$. The random quantities $\mathbf{r}$ can be transformed to the standard-normal space $\mathbf{u} = \mathbf{T}(\mathbf{r})$.

In reliability analysis, the constraints are formulated such that the probability of a given constraint to be violated is less than a required value.

$$1 - P(g_i(\mathbf{x}) \geq 0) \leq P_{req}. \tag{2}$$

Using a FORM formulation, the probability of failure can be related to the reliability index β. The deterministic constraint (1) is replaced by a reliability constraint. The reliability constraint can be formulated based on the reliability index approach (RIA) or the performance measure approach (PMA) (see (Tu *et al.*, 1999)). In a PMA formulation the reliability constraint is

$$gr_i(\mathbf{x}, \mathbf{T}^{-1}(\mathbf{u}^*)) \geq 0, \tag{3}$$

where $\mathbf{u}^*$ is the solution to the inverse reliability problem

$$\begin{array}{ll} \text{minimize} & Gr_i(\mathbf{x}, \mathbf{u}) = gr_i(\mathbf{x}, \mathbf{r}) \\ \text{subject to} & \mathbf{u}^T\mathbf{u} = \beta_{req}^2. \end{array} \tag{4}$$

Note that for each constraint, a different instance of $\mathbf{u}^*$ exists. In the unilevel PMA approach, the reliability constraint (3) is substituted by the first order KKT conditions of problem (4).

The formulation for the single-level RBDO approach in (Agarwal *et al.*, 2004b) replaces problem (4) with its corresponding first order KKT equations. A manipulation of the equations reduces the system to two equations, independent of the number of uncertain quantities.

In this paper, however, a more convenient form of the equations is presented.

$$\mathbf{hr}_i(\mathbf{x}, \mathbf{u}_i) = \frac{\nabla Gr_i}{||\nabla Gr_i||}) + \frac{\mathbf{u_i}}{\beta_r eq}. \tag{5}$$

Each system i has m equations for the same number of reliability variables $\mathbf{u}_i$. It can be solved for $\mathbf{u}_i$ for a given value of the deterministic quantities using Newton steps.

Structure of the unilevel reliability optimization problem

The unilevel optimization problem takes the form:

$$\text{Minimize}_{\mathbf{x},\mathbf{u}} \qquad f(\mathbf{x}),$$

$$
\begin{aligned}
s.t. \quad & \mathbf{g}(\mathbf{x}) \geq 0, \\
& \mathbf{Gr}(\mathbf{x}, \mathbf{u}) \geq 0, \\
& \mathbf{hr}(\mathbf{x}, \mathbf{u}) = 0,
\end{aligned} \tag{6}
$$

Where $\mathbf{hr}(\mathbf{x}, \mathbf{u})$ is the system of equations derived from the KKT conditions for each reliability constraint.

The Jacobian of the equality constraints is given by

$$
A_h = \begin{bmatrix} A_{hru} & A_{hrx} \end{bmatrix}. \tag{7}
$$

Note that the Jacobian with respect to the design variables A_{hrx} is a dense matrix, but A_{hru} is sparse. Furthermore, the structure of A_{hru} is known since

$$
\mathbf{hr}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \mathbf{hr}_1(\mathbf{x}, \mathbf{u}_1) \\ \mathbf{hr}_2(\mathbf{x}, \mathbf{u}_2) \\ \vdots \\ \mathbf{hr}_p(\mathbf{x}, \mathbf{u}_p) \end{bmatrix}. \tag{8}
$$

So the Jacobian of the system with respect to the reliability variables $\mathbf{u}$ is

$$
A_{hru} = \begin{bmatrix} \frac{\partial \mathbf{hr}_1}{\partial \mathbf{u}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \mathbf{hr}_2}{\partial \mathbf{u}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{hr}_p}{\partial \mathbf{u}_p} \end{bmatrix} \tag{9}
$$

Both A_{hru} and A_{hrx} can be easily derived from (5):

$$
\nabla_u \mathbf{hr}_i = \frac{\nabla^2_{u_i} Gr_i}{(\nabla_{u_i} Gr_i^T \nabla_{u_i} Gr_i)^{1/2}} - \frac{\nabla_{u_i} Gr_i \nabla_{u_i} Gr_i^T \nabla^2_{u_i} Gr_i}{(\nabla_{u_i} Gr_i^T \nabla_{u_i} Gr_i)^{3/2}} + \frac{\mathbf{I}}{\beta_{req}}, \tag{10}
$$

$$
\nabla_x \mathbf{hr}_i = \frac{\frac{\partial^2 Gr_i}{\partial \mathbf{x} \partial \mathbf{u}_i}}{(\nabla_{u_i} Gr_i^T \nabla_{u_i} Gr_i)^{1/2}} - \frac{\nabla_{u_i} Gr_i \nabla_{u_i} Gr_i^T \frac{\partial^2 Gr_i}{\partial \mathbf{x} \partial \mathbf{u}_i}}{(\nabla_{u_i} Gr_i^T \nabla_{u_i} Gr_i)^{3/2}}. \tag{11}
$$

Note that this system of equations can be treated as p independent systems, with dense matrices, and its solution can be parallelize.

A reduced Hessian approach

In (Orozco and Ghattas, 1997) a reduced-space SQP framework is presented that can be applied to the solution of the unilevel RBDO problem. In the unilvevel reliability optimization problem (6) the vector of design variables is $\mathbf{w} = \{\mathbf{x}, \mathbf{u}\}$. A step p_w can be decomposed as:

$$p_w = \mathbf{Z}p_z + \mathbf{Y}p_y. \tag{12}$$

Following (Orozco and Ghattas, 1997),

$$\mathbf{Z} = \begin{bmatrix} -A_{hru}^{-1}\frac{\partial \mathbf{hr}}{\partial \mathbf{x}} \\ \mathbf{I} \end{bmatrix}, \tag{13}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}. \tag{14}$$

Note that $A_{hru}\mathbf{Z} = 0$ and $\mathbf{Y}$ spans the range space of A_{hru}^T.

A quadratic approximation for a function g is given by:

$$\tilde{g} = g_0 + \nabla g_0^T p_w + p_w^T H p_w, \tag{15}$$

where $p_w = \mathbf{w} - \mathbf{w}_0$ and the subindex 0 refers to the values at the current design point.

Substituting p_w from (12), the quadratic approximation is given by:

$$\begin{aligned} \tilde{g}(p_z, p_y) &= g_0 + \nabla_u g_0^T p_y + p_y^T H_y p_y + \ldots \\ &\quad (\nabla_x g_0^T - \nabla_u g_0^T A_{hru}^{-1}\frac{\partial h}{\partial x} + p_y^T \mathbf{Y}^T H_w \mathbf{Z}) p_z + p_z^T H_z p_z, \end{aligned} \tag{16}$$

where H_z is an approximation to the reduced Hessian $\mathbf{Z}^T H_w \mathbf{Z}$ and $H_y = \mathbf{Y}^T H_w \mathbf{Y} = \nabla_u^2 g$. The cross Hessian term $\mathbf{Y}^T H_w \mathbf{Z}$ is difficult to evaluate and is commonly neglected. For a given value of the step p_y, this is only a function of p_z.

Note that $p_y^T H_y p_y = 0$ for the objective function and all the deterministic constraints, and $p_y^T H_y p_y = \mathbf{u}_i^T \nabla_{u_i}^2 Gr_i \mathbf{u}_i$ for the reliability constraints.

The reduced-Hessian H_z of the approximation, can be computed either by a quasi-Newton approximation, or as a response surface approximation by sampling the local region.

The reduced-space quadratic approximation (16) can be used to implement an rSQP or a reduced sequential approximate optimization (rSAO) approach, depending how the reduced-Hessian H_z of the approximation is computed. In rSQP a quasi-Newton update formula as BFGS or SR1 is used, while in rSAO the Hessian is computed by response surface techniques, sampling the region surrounding the current design point.

In either case, the general minimization subproblem in the reduced space, subject to a trust region is:

$$\begin{aligned} \text{Minimize}_{\mathbf{x}} \quad & \tilde{\theta}(p_z), \\ s.t. \quad & \tilde{\mathbf{g}}(p_z) \geq 0, \\ & \tilde{\mathbf{Gr}}(p_z, p_y) \geq 0, \qquad (17) \\ & p_z = \mathbf{x} - \mathbf{x}_0, \\ & ||p_z|| \leq \Delta, \qquad (18) \end{aligned}$$

where Δ is the trust region radius and p_y is previously computed by solving

$$A_{hru} p_y = -\mathbf{h}_0 \qquad (19)$$

Once both steps have been computed, one can update all the design variables by

$$\begin{aligned} \mathbf{x}^{k+1} &= \mathbf{x}^k + p_z, \qquad (20) \\ \mathbf{u}^{k+1} &= \mathbf{u}^k + p_y + A_{hru}^{-1} \frac{\partial h}{\partial x} p_z \end{aligned}$$

An rSQP formulation

In SQP, a sequence of quadratic programming problems is solved with a quadratic approximation to the Lagrangian as the objective function and linearized constraints. The Hessian of the Lagrangian is computed by using quasi-Newton approximations. The minimization subproblem (18) is solved with θ as the Lagrangian and the second order terms in the constraints neglected. The Hessian of the Lagrangian is computed using a quasi-Newton update formula.

A rSAO formulation

In (Pérez *et al.*, 2002; Pérez *et al.*, 2003) an optimization problem is solved by the use of sequential quadratic approximations. A similar procedure could be employed for solving the single level RBDO problem. However in SAO the number of design variables affects the overall performance of the algorithm as the Hessian of the approximations is built by sampling the design space. Applying a reduced Hessian approach would make SAO a viable option for this problems.

In SAO the approximate minimization subproblem requires a quadratic approximation of the objective function and constraints. The minimization subproblem (18) is solved with θ as the Lagrangian and the constraints are approximated by quadratic functions as (16). Note that the major difference with respect to SQP is the way the Hessian of the approximation is computed. A quasi-Newton update approximates the Hessian at the current design point $\mathbf{x}_0$, while a response surface based Hessian, computes the Hessian of an approximation over a region around $\mathbf{x}_0$.

Note that a variant of the rSQP formulation could use quadratic approximations in the constraints (and use a nonlinear solver for each minimization subproblem) and likewise the rSAO formulation may solve the problem with linearized constraints.

Procedure

In the present paper the reduced space approach is used within the interior-point, trust-region, sequential approximate optimization framework (IP-TR-SAO) developed by (Pérez *et al.*, 2003). The framework solves an optimization problem by constructing quadratic approximations of the objective function and constraints that are valid within a local region. The IP-TR-SAO framework is similar to a trust-region SQP algorithm except that the constraints are approximated by quadratic polynomials, instead of being linearized, and the second order information is approximated by sampling the design space instead of using quasi-Newton approximations (BFGS or SR1, for example).

The algorithm:

1. Initialize $\mathbf{x}_0$, $\mathbf{u}_0$, Δ_x
2. Compute f_0, $\mathbf{g}_0$, h_0, $\nabla_x g_0$, $\nabla_u g_0$, $\frac{\partial h}{\partial x}$, $A_{hru} = \frac{\partial h}{\partial u}$.
3. Solve for p_y as in Eq. (19).
4. Sample the design space in $\mathbf{x}$ and compute the Hessian matrices for $\tilde{f}$ and $\tilde{\mathbf{g}}$.
5. Solve the approximate optimization problem (18), obtain new design $\mathbf{x}$.
6. Compute f, $\mathbf{g}$, h, $\nabla_x g$, $\nabla_u g$, $\frac{\partial h}{\partial x}$, $A_{hru} = \frac{\partial h}{\partial u}$ for $\mathbf{x}$.

7. Accept/reject point, update trust region size.
8. Update design variables according to (20).
9. Check convergence.
   - If converged, stop
   - else, go to 3.

Numerical experiments

A simple numerical problem is implemented to demonstrate the procedure. This analytical problem was presented in (Agarwal *et al.*, 2004a) and consists of two design variables, two random parameters and two reliability constraints. The problem is given by

In the unilevel reliability formulation, the problem has 6 design variables, 2 inequality constraints and 4 equality constraints. In mathematical form the problem is:

$$
\begin{aligned}
\text{Minimize:} \quad & x_1^2 + 10x_2^2 + y_1, \\
\text{subject to:} \quad & g_1 = y_1/8 - 1 \geq 0, \\
& g_2 = 1 - y_2/5 \geq 0, \\
& -10 \leq x_1 \leq 10, \\
& 0 \leq x_2 \leq 10, \\
& y_1(\mathbf{x}, \mathbf{p}) = x_1^2 + x_2 - 0.2y_2 + p1, \\
& y_2(\mathbf{x}, \mathbf{p}) = x_1 - x_2^2 + \sqrt{y_1} + x_2.
\end{aligned}
$$

Both parameters p_1 and p_2 are random quantities with a uniform distribution over the intervals $[-1, 1]$ and $[-0.75, 0.75]$. Using the reliability index β as a measure of failure (FORM formulation), both inequalities can be substituted by either a PMA or RIA formulation. In the PMA formulation (inverse reliability) the values of p_1 and p_2 correspond to those of the MPP point for a given reliability index. Their correspondent values in the standard normal space $\mathbf{u}_1$ and $\mathbf{u}_1$ can be used.

Applying the single level formulation described before, the problem has 6 design variables, 2 inequality constraints and 4 equality constraints.

The system of 4 equality constraints can be used to solve for the reliability variables $u_1 - u_4$ given values for the deterministic d. v. x_1, x_2. As pointed out in the paper, this system is formed by two systems of equations that can be solved separately.

The RBDO problem is solved using two different approaches. A full space unilevel approach and the proposed reduced-space unilvel approach.

In the full-space unilevel approach, the optimizer controls both the deterministic design variables and the reliability variables at the same time and no distinction is made in the way it solves for either group. In the reduced-space approach, the optimizer controls the deterministic design variables, while the values of the reliability variables are updated at each iteration as described above.

Results for the rSQP method

The results presented in Table 1 show the number of iterations and the number of function calls for each approach. The reduced Hessians are computed using a rank one quasi-Newton update formula (SR1). No positive definite Hessian is required due to the use of a trust region method as globalization strategy. In SQP the constraints are linearized, and in QSQP, quadratic approximations are used.

	Full-space	Reduced-space
SQP	13/42	13/48
QSQP	8/27	19/62

Table 1: Comparison of full vs. reduced space in SQP

As expected, the full-space approach performs better than the reduce-space one, The use of quadratic approximations to the constraints, contributed to a significant improvement in the full-space, while increasing the cost in the reduced space. Analyzing the iteration history for the reliability variables the problem is identified. The Newton step quadratically convergent close to the actual value of the variables, but away from it may create oscillations or diverge. In this case it is oscillating and converging very slowly, as shown in Figure 1.

A way to avoid this problem is to implement a globalization technique in the Newton step. We implemented a simple backtracking technique. If the Newton step p_y is too large ($p_y > a$) the algorithm checks if a reduction in the value of the equality KKT constrains is observed. If not, the algorithm takes only a fraction of the step γp_y. Note that this simple approach does not guarantee convergence from points far away, as does not perform a proper line search, but at least reduces the probability of divergence for large steps. For the test problem we used $a = 1$, $\gamma = 0.7$.

Once the backtracking is applied, the reduced-space approach required only 8 iterations and 31 function evaluations to converge. This is comparable to the full-space approach. Only one iteration (the first one) opted for the backtracking option. Most important, the reliability variables converged to their actual value without oscillations as shown in Figure 1.

reduced Sequential Approximate Optimization results

The reduced-space technique was also applied to the SAO framework as discussed above. Constructing the quadratic approximation requires at least $n(n+1)/2$ function evaluations. As with the SQP approach, two versions were tested. One with the

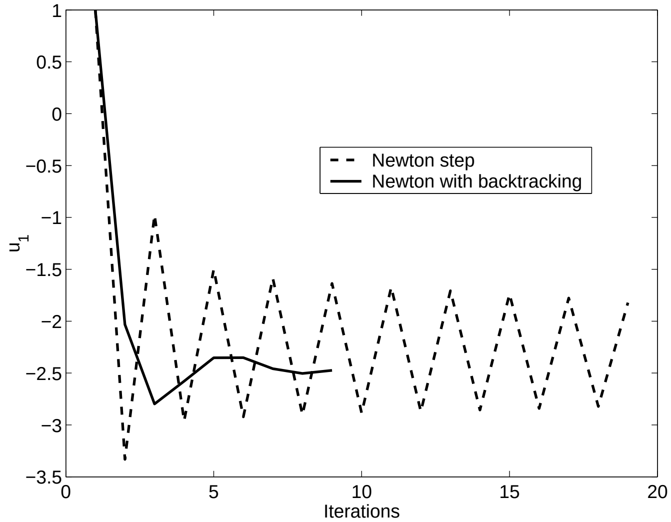

Figure 1: Oscillations in u_1

linearized constraints in the minimization subproblem (lSAO) and the other with quadratic approximations to the constraints (SAO). For sampling in the full space, an I-optimal array with 27 designs generated by the software Gosset(Hardin and Sloane, n.d.) was used. The results are shown in Table 2

	Full-space	Reduced-space	Reduced w/backtracking
lSAO	13/1095	21/360	17/296
SAO	8/675	15/258	5/92

Table 2: Comparison of full vs. reduced space in SAO

In the case of the SAO implementation, the reduced-order approach shows considerable savings due to the reduction in the number of designs to be sampled at every iteration. This result was expected as one of the important issues in SAO is the *curse of dimensionality*. Also it is noted that the use of quadratic approximations in the minimization subproblem reduces the number of iterations required to converge, as was observed in the SQP approach. The use of the simple backtracking scheme again avoided oscillation of the reliability variables in the reduced-space technique, improving the convergence of the algorithm.

Concluding remarks

A unilevel implementation of the reliability-based optimization problem presents several advantages over its bi-level counterpart. Its structure, suggests the use of reduced order methods for its solution when large number of random quantities and reliability constraints occur. In this paper an implementation of a reduced-space SQP method was applied to the solution of this type of problems. The paper analysis the structure of the problem and the reduced-space formulation. The formulation was applied to an SQP and an SAO formulations. The standard SQP formulation was complemented with optional explicit quadratic approximations to the constraints. A simple test problem demonstrates the behavior of the rSQP applied to the unilevel reliability-based optimization formulation.

The results demonstrate the ability of the reduced-space approach to converge under certain circumstances. It is noted that a simple Newton step may fail to converge the reliability variables if it is not accompanied by a globalization technique. A simple selective backtracking method was used with success.

The behavior of the problem is as expected with a reduced-Hessian implementation. In general is more expensive in the number of function evaluations, but allows the optimizer to handle only a small portion of the variables. In the rSAO approach, however, the number of function evaluations grows quadratically with the number of design variables. The use of a reduced space, improves the performance of the algorithm significantly.

One interesting observation is that in the reduced-space formulation, convergence on the reliability variables $\mathbf{u}$ is indirectly enforced through the constraints. As a result, the method computes the reliability variable $\mathbf{u}$ up to the necessary precision, while solving the full system would enforce convergence to the same precision in both the design variables and the reliability variables.

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