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Current Algebra*

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I. ALGEBRA OF CURRENTS -- BACKGROUND

In the first part of these lectures we will be concerned with a review of the main ideas of the algebra of currents. Probably most of you are already familiar with them, as I understand that the subject has been treated in previous lectures, but perhaps that is not too bad. The subject of current algebras was ignored for many years, and lately it is receiving probably too much attention. It has to be emphasized that many of the underlying ideas are highly tentative, and could well be wrong; we will arrange them in a hierarchy such that the first are very simple and almost certainly true, and then afterwards one can accept each assumption without accepting the successive ones.

We start with the isotopic spin operators. As far as the strong interactions are concerned, we can think of them as constants of motion, obeying the following commutation rules:

$$[I_i, I_j] = i\, \epsilon_{ijk}\, I_k \quad . \tag{1}$$

However, we can also describe (or define) the isotopic spin in another way because of the principle of the conserved vector current (CVC).[1)]

Let's consider the hadronic weak current, which is coupled to the lepton pairs. We may write it as

$$J_\alpha^{\text{hadronic}} = J_\alpha^{(\Delta I=1,\ \Delta Y=0)} + J_\alpha^{(\Delta I=1/2,\ \Delta Y/\Delta Q\, =\, +1)} + \ldots \tag{2}$$

where the dots refer to other terms which may be present, but which in many cases we can show to be much smaller than the first two, if they are present at all.

We can further split each term in a vector and axial vector part, i.e.,

$$J_\alpha^{hadronic} = V_\alpha^{(\Delta Y=0)} + A_\alpha^{(\Delta Y=0)} + V_\alpha^{(\Delta Y=1)} + A_\alpha^{(\Delta Y=1)} + \ldots \qquad (3)$$

Now the idea of CVC is that the first term is equal to the isotopic spin raising current, and we can then define $I_1 + i\, I_2 = \int V_o^{(\Delta Y=0)}\, d^3x$. In the same manner we have for the electromagnetic current $J_\alpha^{em} = J_{3\alpha}$ + isoscalar, and we can define I_3 to be the integral over space of the isovector time component of J_α^{em}:

$$I_3 = \int J_{30}\, d^3x \qquad . \qquad (4)$$

We see then that we can define the isospin operators in terms of quantities which, at least in the lowest order of electricity and weak interactions, are measurable, as an alternative to defining them as a set of good quantum numbers of strong interactions, obeying the commutation rules (1).

This suggests that perhaps also the other portions of the vector weak current (and even axial current) have charge operators, i.e., space integrals of their time components, obeying some simple set of commutation rules. However, as we know very well, these charges are not conserved, so that they are not independent of time; we can talk only about equal time commutation relations. Only these can be simple, as we don't know how to handle different time commutation rules without some knowledge of dynamics.

The simplest possibility, as suggested five years ago,[2)] was that this algebra should close, after adding to the isospin generators the $\Delta I = 1/2$ strangeness-changing vector charges, with the inclusion of the smallest possible number of additional operators. In this way one is led to the algebra of SU(3), which consists of eight charges F_i $(i = 1,....8)$ obeying the equal time commutation rules

$$[F_i, F_j] = i f_{ijk} F_k \quad . \tag{5}$$

F_1, F_2, F_3 are identified with the isospin generators, F_8 with $\sqrt{3}/2\, Y$ (Y is the hypercharge) and the others are operators with $\Delta I = 1/2$ and $\Delta Y = \pm 1$. It may well be that these assumptions have to be still modified, in order to include in a larger algebra some possible new terms, which might be present in the vector currents; but, since at present there is no particular evidence for them, let us stick to SU(3) and go on to complete the algebraic structure of the theory by incorporating also the axial currents.

Thus we put in also charges for the axial vector currents and require them to obey the commutation rules of a simple, relatively small algebra. The simplest possibility[2,3] is to add eight axial charges F_i^5, with the following commutation relations:

$$\begin{aligned} [F_i, F_j^5] &= i f_{ijk} F_k^5 \quad , \\ [F_i^5, F_j^5] &= i f_{ijk} F_k \quad , \end{aligned} \tag{6}$$

such that the operators $1/2(F_i \pm F_i^5)$ generate two commuting SU(3) algebras, the so-called chiral $SU(3) \otimes SU(3)$.

At this point we can try the notion of combining the strangeness preserving and changing charge operators, to make something which has the same form and strength as the weak vector and axial charges for leptons, so as to give meaning to the concept of universality of weak interactions between leptons and hadrons.[4,5]

To this aim, we observe that the charge operators

$$(F_1 \pm i\, F_2) \cos\Theta + (F_4 \pm i\, F_5) \sin\Theta + (F_1{}^5 \pm i\, F_2{}^5) \cos\Theta + (F_4{}^5 \pm i\, F_5{}^5) \sin\Theta \quad , \tag{7}$$

together with a third operator, obey the commutation rule of an angular momentum, exactly in the same way as does the lepton operator:

$$L^+ = \int d^3x \; \frac{1}{2} [\nu_e{}^+ (1+\gamma_5)\, e + \nu_\mu{}^+ (1+\gamma_5)\, \mu] \tag{8}$$

together with $L^- = (L^+)^\dagger$ and a third one.

By assuming the form (7) for hadron currents, Cabibbo[5)] was able to explain consistently the relative rates of the hyperon and meson leptonic decays with an angle $\Theta \simeq 15^o$.

However, a comparison with more accurate data will provide in the future not only a check of the universality principle, but also of the idea that equal time commutators of the charges correspond to the algebra of SU(3), and of the assumption that the eight baryon ground states approximately form an eight-dimensional irreducible representation of the same algebra. We see that we are in fact testing simultaneously many independent assumptions, but it is comfortable that so far things seem to work quite well.

So much for the commutation rules of the charges. Let us go on and make further assumptions on the equal time commutation rules of densities.

Let's use $\mathcal{F}_{i\alpha}(x,t)$ ($\alpha = 1, \dots 4;$ $i = 1, \dots 8$) for the vector currents, and $\mathcal{F}_{i\alpha}{}^5(\underline{x},t)$ for the corresponding axial currents.

If we consider the equal time commutator, say, of two vector time components, microcausality and the requirement of getting Eq. (5) after integration over all the space variables enable us to write

$$[\mathcal{F}_{io}(\mathbf{x},t),\ \mathcal{F}_{jo}(\mathbf{x}',t)] \ = \ i\, f_{ijk}\ \mathcal{F}_{ko}(\mathbf{x},t)\ \delta(\mathbf{x} - x') + \dots \qquad (9)$$

Here the dots stand for terms containing higher derivatives of δ-functions,[6] about which we do not know anything but that only a finite number of them is allowed in order to have a local operator.

By including also the axial densities we have further

$$[\mathcal{F}_{io}(\mathbf{x},t),\ \mathcal{F}_{jo}{}^{5}(\mathbf{x}',t)] \ = \ i\, f_{ijk}\ \mathcal{F}_{ko}{}^{5}(\mathbf{x},t)\ \delta(\mathbf{x} - \mathbf{x}') + \dots \quad , \qquad (10)$$

$$[\mathcal{F}_{io}{}^{5}(\mathbf{x},t),\ \mathcal{F}_{jo}{}^{5}(\mathbf{x}',t)] \ = \ i\, f_{ijk}\ \mathcal{F}_{ko}(\mathbf{x},t)\ \delta(\mathbf{x} - x') + \dots \quad , \qquad (11)$$

which correspond to $SU(3) \otimes SU(3)$ for the charges. If we ignore the dots, we have an algebra of $SU(3) \otimes SU(3)$ at every point of space at the same time; that is, every granule of space carries its own $SU(3) \otimes SU(3)$.

Now let us make the drastic assumption that there is nothing in the place of the dots of the previous equations. This is in fact a much stronger assumption, which gives the possibility of making predictions about the matrix elements of currents at any momentum transfer, instead of confining them only to the zero spatial momentum transfer situation, as is the case for charges.

We may argue that this assumption is not in contradiction at least with the basic principles of field theory, relativity, and causality, by explicitly constructing a non-trivial relativistically covariant field theoretical model, i.e., the "Lagrangian quark model", in which it holds.

We introduce a triplet of spin 1/2 quarks, corresponding to the basic three-dimensional SU(3) representation, together with the corresponding triplet of antiquarks. We can write down a Lagrangian of the form

$$\mathcal{L} = \bar{q} \, (\gamma \partial + m_o) \, q + \text{interaction}$$

from which we can deduce the currents

$$\mathcal{J}_{i\alpha} = i \, \bar{q} \, \frac{\lambda_i}{2} \, \gamma_\alpha \, q$$

and

$$\mathcal{J}_{i\alpha}^{5} = i \, \bar{q} \, \frac{\lambda_i}{2} \, \gamma_\alpha \gamma_5 \, q \quad .$$

By using the canonical anti-commutation rules for q fields, we can compute, at least formally, the commutation relations of these currents, which turn out to be free from gradient terms.

It is important to stress that these commutation relations hold true, no matter how badly broken the symmetry is (for instance, by mass terms): SU(3) symmetry and the validity of equal time commutation relations are two quite independent things.

Anyway, as we said, the non-appearance of gradient terms in the theory may be only formally true, in the sense that we obtain this result if we do not care about the strongly singular nature of the commutators involved. Recently, Johnson and Low at MIT[7)] have been looking at this problem. They take a simple theory of quarks interacting with a scalar neutral boson, and compute in power series the commutators of currents to see whether the results they get are the same as those which can be formally derived in the quark model. Whereas in general they find funny extra terms, for the time components of the vector and axial vector currents they show a consistency with our assumption. Things are much more complicated with vector meson interaction and they are not sure at present whether the last conclusions are to be modified.

II. THE FRAME $P = \infty$

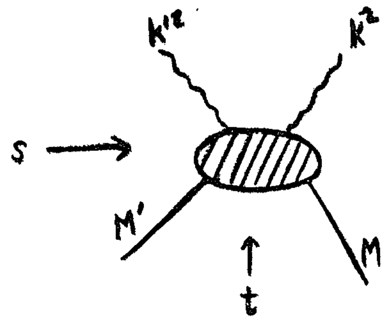
We will see that these commutation relations are best used by sandwiching them between states of infinite momentum along, say, the z-direction.[8,9]

This looks like a non-covariant procedure, but in fact we introduced from the beginning a non-covariant element in the theory, by considering the simplest kind of commutators of two space-like separated local operators, i.e., equal time commutators. This choice selects a particular Lorentz frame in which it is a physically sensible question to ask what is the momentum of the states between which the commutators are to be sandwiched, and we may expect that the results are quite different when the momenta are near zero or near infinity. In fact, there are a number of unpleasant features connected with the use of states with finite momentum. For the sake of simplicity, let us consider only commutators of charges, sandwiched between states of total space momentum zero and masses M and M':

$$\langle \underset{\sim}{P} = 0, M \,|\, [\int \mathcal{F}_{io}\, d^3x, \int \mathcal{F}_{jo}\, d^3x] \,|\, \underset{\sim}{P}' = 0, M' \rangle \qquad .$$

We can insert a complete set of intermediate states having total space momentum equal to zero and masses M"; the four momentum transfers to the intermediate states are $k'^2 = -(M' - M'')^2$ and $k^2 = -(M'' - M)^2$.

Now the equal time commutator can be considered as the energy integral of the imaginary part of a suitable scattering amplitude as shown in the following diagram:



+ crossed diagram

and in the case of $\underset{\sim}{P} = 0$ one very annoying thing is that the four momentum transfer to each intermediate state depends on its mass. This amounts in dispersion language to doing a dispersion integral not in the relativistic variable s with fixed external masses, but in the energy variable with the space momentum fixed and variable masses, which is well known to be a disgusting way to do dispersion theory.

Moreover, if the four-momentum transfer depends on the mass differences then we can get, for certain values of masses of intermediate states, resonances in the channel of the currents. For instance, in the case of vector currents, we could get k^2 or k'^2 equal to the square of the ρ mass, and this could terribly enhance the continuum contribution in that region, seriously interfering with the program of approximating the sum over intermediate states with few resonant states, a thing we will often like to do for practical purposes of calculation.

On the contrary, when we sandwich the commutators between states of infinite momentum, the situation changes entirely.

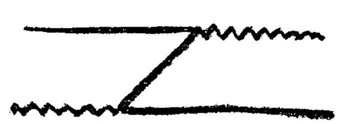
First of all, by going to $P_z = \infty$, we can make the four-momentum transfers k^2 and k'^2 vanish without regard to the masses of the intermediate states (in the case of the commutators of particular Fourier components of the currents, they go to fixed limits), and this corresponds in the dispersion language to having a dispersion integral in the s variable with fixed external masses, which is a much more sensible way of doing things.

Also from the point of view of comparison with the experiments, it is much easier to measure things with fixed k^2 and k'^2, and this is another advantage, though rather minor, of taking $P_z = \infty$. Far more important is the problem of convergence of the sum rules obtained from the commutators. At $P_z = \infty$ we get sum rules of the typical form[8,9)]

$$\int ds \, \mathrm{Im}\, A(s, t; k^2, k'^2) = f(t) \qquad ,$$

and from Reggeism in the crossed channel we know the asymptotic behavior of Im A in the variable s, thus having an idea of the convergence of the sum rule. On the other hand, if k^2 and k'^2 are not fixed, we have no idea whatsoever what the convergence is.

Another very crucial point is that, at finite momentum, an important contribution to the sum over intermediate states is given by diagrams of the following type:



These z-diagrams correspond to complicated three-particle intermediate states, where the initial current creates a pair and one member of the pair annihilates with the initial particle to give the final current. On the contrary, at infinite momentum we get rid of all these diagrams as well as all disconnected pair diagrams, like

and the reason is that they all correspond to intermediate states of infinite mass. In fact, in the z-diagrams the **pair** has to have zero or finite total space momentum since it is created by the current from the vacuum, but each member of the pair has infinite momentum, thus resulting in a state of infinite mass; in the completely disconnected diagrams, the relative momentum between the initial particle and the pair is infinite and we end up again with an intermediate state of infinite mass.

Now, whether these infinite mass states contribute or not to the relations we get is a question of whether the dispersion relations have any subtractions.

We assume explicitly that there are no subtractions and in this way we get rid of all these complicated intermediate states, and we are left only with genuine intermediate states in the s-channel.

Finally we mention another very important feature of the $P_z = \infty$ frame; that is, the possibility of using for the axial vector current the PCAC hypothesis to approximate the integral of the fourth component of the axial current with a one pion state. This has meaning only when we are dealing with small and fixed k^2 and k'^2, so that they are reasonably close to the pion pole. On the other hand, at $\underset{\sim}{P} = 0$ there is no justification of such an approximation since k^2 and k'^2 get bigger and bigger with the mass of the intermediate states. Let's summarize finally the advantages of the $P = \infty$ frame:

i) Only things which are easily measurable appear;

ii) We have relations whose convergence can be estimated by Reggeism;

iii) No variations of k^2 and k'^2 resulting in non-appearance of unwanted resonances in the current channel;

iv) No contribution from disconnected or semi-disconnected graphs (z-diagrams);

v) It is possible to use PCAC for axial vector current.

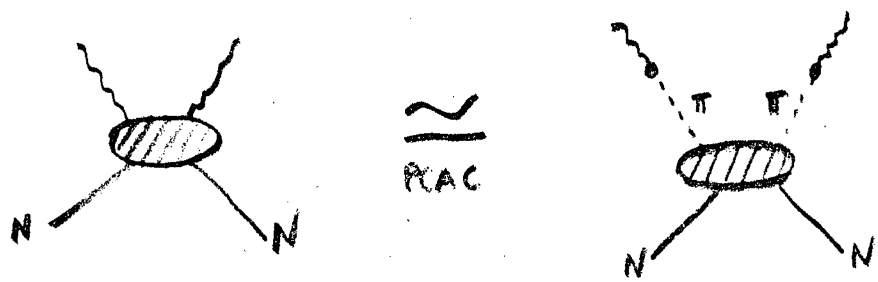
III. EXPERIMENTAL TESTS

We want to discuss now what are the experimental tests of all the previous assumptions. To begin with, we consider the commutation relations of the axial charges with themselves, Eq. (6), sandwiched between nucleon states taken at $P_z \to \infty$. Since we are dealing with space integrals, we have in this special case $k^2 = k'^2 = 0$. Inserting a complete set of intermediate states in the commutator, and extracting the nucleon pole, we end up with the Adler-Weisberger relation:[10)]

$$1 = G_A{}^2 + \int ds\, f(s)\, [\sigma^{\text{forward}}_{\nu \to e}(s) - \sigma^{\text{forward}}_{\bar{\nu} \to \bar{e}}(s)] \qquad , \qquad (13)$$

where G_A is the axial neutron β-decay coupling constant, $f(s)$ is a known kinematical factor, and $\sigma^{\text{forward}}_{\nu \to e}(s)$ $(\sigma^{\text{forward}}_{\bar{\nu} \to \bar{e}}(s))$ is the forward differential cross section for the scattering $\nu + p \to e^- + \beta$ $(\bar{\nu} + p \to e^+ + \beta')$, β and β' being any possible hadron final state. Equation (13) is a prefectly rigorous consequence of the considered commutation relations.

In this form, the Adler-Weisberger relation is not very useful, because of the rather poor experimental information on neutrino cross sections. On the other hand, it is possible to put it in a more directly useful form by using the PCAC hypothesis, in which the neutrino (antineutrino) forward cross section is replaced by a $\pi^+ p$ $(\pi^- p)$ total cross section. This corresponds to the approximate identity



We get then:

$$1 = G_A^2 + \int ds\ g(s) \left[\sigma^{total}_{\pi^+ p}(s) - \sigma^{total}_{\pi^- p}(s)\right] \quad , \tag{14}$$

where again $g(s)$ is a known kinematic factor, including the proportionality constant which appears in the Goldberger-Treiman relation.

This formula has been successfully compared with data and it seems to work very well. We want however to emphasize that again, as in the case of hyperon leptonic decays, actually we are checking more than one assumption; so that the agreement might be not too significant. However, let us assume the optimistic point of view that Eq. (14) is indeed a proof of the validity both of the algebra of charges and of the PCAC.

Let us go on now to the algebra of the densities. As a generalization of the previous procedure, we sandwich the commutators of the Fourier transforms of densities

$$F_i(\underset{\sim}{k}_\perp) = \int d^3x \; e^{i\underset{\sim}{k}_\perp \cdot x} \; \mathcal{F}_i(\underset{\sim}{x},0)$$

taken at a fixed momentum k , perpendicular to the z-direction, between states of finite P_x and P_y and infinite P_z; i.e., we consider the matrix elements

$$\langle P_z = \infty, P_x' \, P_y' \; |[F_i(\underset{\sim}{k}_\perp), F_j(\underset{\sim}{k}_\perp')]| \; P_z = \infty, P_x \, P_y \rangle$$

where

$$\underset{\sim}{k}_\perp + \underset{\sim}{k}_\perp' = \underset{\sim}{P}_\perp' - \underset{\sim}{P}_\perp \qquad .$$

In dispersion language, this gives rise to a relation of the typical form Eq. (12), where $k^2 = \underset{\sim}{k}_\perp^2$, $k'^2 = \underset{\sim}{k}_\perp'^2$, and $t = -(\underset{\sim}{k}_\perp + \underset{\sim}{k}_\perp')^2$, and by analyticity we can extend the values of the three independent variables to any value. Needless to say, the set of relations that can be derived in this way is far richer than what can be obtained by the commutation relations of charges. The only question is whether these relations can be actually compared with the presently existing experimental information.

For instance, consider the commutator of two vector currents sandwiched between nucleon states. What we get is an equality between an integral over energy of a bilinear form in photoproduction amplitudes (or, alternatively, an integral of the imaginary part of a Compton scattering amplitude) and a nucleon vector current form factor. If we fix

$k^2 = k'^2 = 0$ and $t \neq 0$, what is involved are the physical photoproduction amplitudes and we could obtain a whole set of relations for each t, but nobody knows at present how to exploit them, because photoproduction is a very complicated process as soon as one goes beyond the single pion production. The simplest relation to be extracted from this set is the one which was first given by Bjorken[11] and by Cabibbo and Radicati,[12] who consider the first moments of the isovector, vector currents.

This relation looks as follows:

$$\frac{1}{3} \langle r^2 \rangle_{F_1}^V = (\mu_{anom}^V)^2 + \frac{1}{2\pi^2 \alpha} \int_{M_N^2}^{\infty} \frac{(2\sigma_{1/2}^V - \sigma_{3/2}^V)}{s - M_N^2} ds \quad , \qquad (15)$$

where $\langle r^2 \rangle_{F_1}^V$ is the mean square radius of the isovector Dirac form factor, (μ_{anom}^V) is the isovector part of the anomalous nucleon magnetic moment, $\sigma_{1/2,3/2}^V$ are the total photoproduction isovector cross sections in the $I = 1/2,\ 3/2$ channels, and M_N is the mass of the nucleon.

What can we say about this formula? We know the photoproduction cross section fairly well up to $\sqrt{s} = 1500$ MeV, which is not too far above single pion production, and the conclusions depend very much on whether you are optimistic or pessimistic. Bjorken's conclusion was the following: he writes Eq. (15) in the form

$$(\mu_A^V)^2 - \frac{1}{3} \langle r^2 \rangle_{F_1}^V = - \frac{1}{2\pi^2 \alpha} \int_{M_N^2}^{\infty} \frac{ds}{s - M_N^2} (2\sigma_{1/2}^V - \sigma_{3/2}^V) \quad . \qquad (15')$$

The left-hand side is a very tiny number, but it has a definitely known sign and the contribution from the lowest energy part of the photoproduction cross section turns out to be of the wrong sign; so he concluded that this formula is not very good.

Cabibbo and Radicati came to the opposite conclusion by saying that $(\mu_A{}^V)^2$ and $\frac{1}{3}\langle r^2\rangle^V_{F_1}$ are roughly equal and the contribution of the integral is small. Anyway, in order to make more conclusive statements, we need more precise high energy data, and what is encouraging is that in this region the most important contributions come from $I = 1/2$ resonances which might reverse the sign of the integral as it is given by the low-energy portion dominated by the N^*_{33}.

IV. GOING BEYOND $SU(3) \otimes SU(3)$

One can try to extend the chiral $SU(3) \otimes SU(3)$ algebra by including the space integrals of all the vector and axial vector components, and if one evaluates the commutators following the formal quark model we introduced before, one gets the chiral $U(6) \otimes U(6)$ algebra. One may go even further by introducing additional currents, like scalar, pseudoscalar, and tensor currents, which have never been seen but which might be there, and one gets the compact $U(12)$ algebra; i.e., the algebra of all the sixteen Dirac matrices and the nine λ-matrices.

However, if we follow the previous philosophy of taking matrix elements at $P_z = \infty$, many of these operators have vanishing matrix elements between single particle states, and many others become equal.[9,13]

Let us look at this point in more detail.

We list the currents in the quark model and the behavior of their matrix elements between single particle states at $P_z = \infty$. (As Fubini and I did, we call "good" the operators whose matrix elements do not vanish, and "bad" the others.)

S	:	$q^+ \beta \lambda_i q \approx \frac{1}{P_z}$	"bad"	
P	:	$q^+ \beta \gamma_5 \lambda_i q \approx \frac{1}{P_z}$	"bad"	
V	:	$q^+ \lambda_i q \approx 1$	"good"	identical
		$q^+ \alpha_z \lambda_i q \approx 1$	"good"	
		$q^+ \alpha_{x,y} \lambda_i q \approx \frac{1}{P_z}$	"bad"	
A	:	$- q^+ \gamma_5 \lambda_i q \approx 1$	"good"	identical
		$q^+ \sigma_z \lambda_i q \approx 1$	"good"	
		$q^+ \sigma_{x,y} \lambda_i q \approx \frac{1}{P_z}$	"bad"	

$$T : \quad i\, q^{+} \beta \alpha_x \lambda_i q \sim 1 \qquad \text{"good"}$$

$$q^{+} \beta \sigma_y \lambda_i q \sim 1 \qquad \text{"good"}$$

(identical)

$$-i\, q^{+} \beta \alpha_y \lambda_i q \sim 1 \qquad \text{"good"}$$

$$q^{+} \beta \sigma_x \lambda_i q \sim 1 \qquad \text{"good"}$$

(identical)

$$q^{+} \beta \alpha_z \lambda_i q \sim \frac{1}{P_z} \qquad \text{"bad"}$$

$$q^{+} \beta \sigma_z \lambda_i q \sim \frac{1}{P_z} \qquad \text{"bad"}$$

From the preceding list, we see that by restricting ourselves to V and A currents, i.e., to the $U(6) \otimes U(6)$ chiral algebra, we get at $P_z = \infty$ two identical $U(3) \otimes U(3)$ containing two more operators with respect to $SU(3) \otimes SU(3)$, the time component of the baryon current and the axial vector analogue of it. On the other hand, the algebra of U(12) reduces to the $[U(6)]_W$ algebra which is obtained from $U(3) \otimes U(3)$ by adjoining the "good" tensor current components.

Out of the whole set of commutation relations of the compact U(12), we can pick commutators of three different types:

1) good-good commutators. They involve two good operators to give another good one. We get from them sensible sum rules (in the way we explained before), apart from the physical interpretation of tensor currents which is still dubious.

2) good-bad commutators. The right-hand side is again a bad operator, and since both members go to zero like $1/P_z$, we can expect to extract meaningful information from these rules when they converge.

3) bad-bad commutators. These are quite awful things. Since the left-hand side would go like $1/P_z^2$ and the right-hand side is of the order of 1, being the matrix element of a good operator, there must appear terribly divergent integrals so that the rules we get look like $\infty/\infty = 1$. Nobody has yet succeeded in giving any meaning to them.

As for the physical significance of the U(12) algebra, only the V and A currents have been clearly identified with measurable quantities in weak and electromagnetic processes. For the S, P, and T currents, the interpretation is at present highly tentative. For example, it could well be that the S density appears as a part of the energy density, for instance in the mass difference term. Another possibility is that they appear in new interactions if they exist at all.

However, we can define these currents in still another way[14)] which perhaps is the only way to relate them with physical observable quantities. It could in fact be that they are local operators with a very simple analytical structure, i.e., with as few singularities as possible and, if this is the case, their matrix elements are indirectly connected with S-matrix elements in the sense that they are the least singular solutions of linear homogeneous integral equations having as coefficients the relevant on-shell S-matrix elements.

These matrix elements could even admit single pole approximations (for instance, we can have a partial conservation of the tensor current) so that they would be related to scattering processes involving the appropriate mesons.

V. TRYING TO REPRESENT THE LOCAL $U(3) \otimes U(3)$ ALGEBRA

The advantages of considering the local chiral algebra commutation rules, Eqs. (9), (10), (11), sandwiched between states with $P_z = \infty$ have already been emphasized in the preceding sections.

I want to present now some further investigations which Dashen and I have made at Caltech[15] on the possibility of finding an infinite dimensional representation of the complete set of local commutators.

The motivations which may give physical significance to the problem are the following. We started considering the $U(3) \otimes U(3)$ algebra of vector and axial vector charges, and one may ask whether it is possible to use it as an approximate symmetry for hadrons, i.e., whether hadrons can be described approximately with irreducible or small reducible representations of $U(3) \otimes U(3)$ in the same way as we know it is true for $U(3)$.

The answer we get from experiment is that $U(3) \otimes U(3)$ cannot be a symmetry in any accurate sense. In fact, we know from the Adler-Weisberger relation that the baryon octet and decimet do not form a single irreducible $U(3) \times U(3)$ representation, because they are quite strongly connected through the axial charges to other states, mainly resonances.

Many people[16] however have looked at the question of what happens if we consider them as a part of a reducible representation, that is if we assume baryons to be a mixture of a small number of irreducible $U(3) \times U(3)$ multiplets incorporating besides the N and N^* some other higher resonances which are known to contribute to the Adler-Weisberger sum rule.

This seems to work quite well, with the baryon ground states mixing predominantly with a few excited states, those excited states presumably mixing predominantly with the ground state and with some still higher excited states, and so forth. The known electromagnetic and weak matrix elements between the lowest baryon states can be fitted with an admixture of $U(3) \times U(3)$ representations corresponding mathematically to three-quark configurations.

We should like to describe in a unified way this presumably infinite chain of representation mixings. This is why, instead of representing first the algebra of charges and then trying to extend it to higher moments of the currents, we tried to represent at once the whole local chiral algebra in an infinite dimensional space. The hope is to find in this way an approximate model for hadrons depending on a possibly small number of continuous parameters and giving a rough description of the hadron spectrum and of its physical properties (form factors, coupling constants, etc.).

Of course it is possible that this program cannot be accomplished unless one builds up a complete relativistic theory of all the world, but we feel it nevertheless interesting to investigate it and see if this is the case.

In order to make the problem more precise, we have to consider further the general angular momentum properties of the matrix elements of the currents taken at $P_z = \infty$.

Since we will deal with matrix elements between eigenstates of total momentum, it is natural to consider in the place of Eqs. (9), (10), (11), their Fourier transforms taken, as we did before, at a space momentum perpendicular to the z-direction; i.e.,

$$[F_i(\underset{\sim}{k}_\perp),\ F_j(\underset{\sim}{k}'_\perp)] = i\, f_{ijk}\, F_k(\underset{\sim}{k}_\perp + \underset{\sim}{k}'_\perp) \quad , \tag{16}$$

$$[F_i(\underset{\sim}{k}_\perp),\ F_j{}^5(\underset{\sim}{k}'_\perp)] = i\, f_{ijk}\, F_k{}^5(\underset{\sim}{k}_\perp + \underset{\sim}{k}'_\perp) \quad , \tag{17}$$

$$[F_i{}^5(\underset{\sim}{k}_\perp),\ F_j{}^5(\underset{\sim}{k}'_\perp)] = i\, f_{ijk}\, F_k(\underset{\sim}{k}_\perp + \underset{\sim}{k}'_\perp) \quad . \tag{18}$$

These operators will be sandwiched between states of the following type:

$$|N,\ h,\ P_z = \infty,\ P_x\ P_y\rangle$$

where h is the helicity (the z-component of total angular momentum) and N describes all the possible additional quantum numbers we need to characterize the state.

A very important feature of the resulting matrix elements

$$\langle N',\ h',\ \underset{\sim}{P}'_\perp\ |F_i(\underset{\sim}{k}_\perp)|\ N,\ h,\ \underset{\sim}{P}_\perp\rangle \qquad (\text{with } \underset{\sim}{k}_\perp = \underset{\sim}{P}'_\perp - \underset{\sim}{P}_\perp)$$

is that they do not depend on the average momentum $\frac{1}{2}(\underset{\sim}{P}_\perp + \underset{\sim}{P}'_\perp)$ perpendicular to the z-direction. We will not demonstrate it in general but we will check it later in a particular example. This is quite interesting because it means that at infinite momentum we don't need momentum indices to label the states, the difference of momenta $\underset{\sim}{k}_\perp$ being already in the argument of the operators.

This is very analogous to the situation one runs into in the very primitive commutation relations in non-relativistic quantum mechanics, when doing the atomic sum rules. In fact, in this case one does not have to deal with the gross state of the motion of the atom, which is held fixed, but only with relative momenta. What is not similar to non-relativistic atomic physics is that here we have to match up the commutation relations with relativistic angular momentum properties, which guarantee that we are dealing with particles of definite spin. This is very difficult, and has prevented us up to now from finding a complete solution of the problem. We have, however, some preliminary results which hold in general and whose content we are going to discuss.

The first thing to do is to define $\underset{\sim}{\mathcal{J}}$, the spin operator at $P_z = \infty$, and this will be done in the following way: we define

$$\mathcal{J}_z = J_z = h \quad \text{(the helicity operator)} \quad ,$$

while $\mathcal{J}_x$ and $\mathcal{J}_y$ have non-vanishing matrix elements only between states with $N = N'$ and equal to the usual angular momentum matrix elements appropriate to the spin of the state N.

To deduce the angular momentum properties of $F_i(\underset{\sim}{k}_\perp)$, it is most useful to express the matrix elements

$$\langle N'\ h'\ |F_i(\underset{\sim}{k}_\perp)|\ N\ h\rangle$$

in terms of matrix elements of currents in the Breit frame. We start from

$$\langle N'h'\ |F_i(\underset{\sim}{k}_\perp)|\ Nh\rangle$$

$$= \langle N'h',\ P'_x = \frac{k}{2}\ ,\ P'_y=0,\ P'_z=\infty\ |\mathcal{F}_{i0}(0)|\ Nh,\ P_x = -\frac{k}{2}\ ,\ P_y=0,\ P_z=\infty\rangle \quad , \tag{19}$$

where we have taken $\underset{\sim}{k}_\perp$ in the x-direction, and $P_x = -P'_x = -\frac{k}{2}$.

To go to the Breit frame, we apply the relevant pure Lorentz transformation G, ending up with

$$\langle N'h' \,|G^{-1}\, G\, \mathcal{J}_{io}(0)\, G^{-1}\, G|\, Nh\rangle \qquad .$$

Now we have to express the states $G\,|N'h'\rangle$ and $G\,|Nh\rangle$ in terms of helicity states in the Breit frame. It is easily shown[17] that $G\,|Nh\rangle$ can be obtained by applying a suitable spin rotation, depending on the masses of the states and the modulus of $\underset{\sim}{k}_\perp$, to $|N, h, P_x = -\frac{k}{2}, P_z = -\frac{\ell_z}{2}, P_y = 0\rangle$ where $\ell_z/2$ is the transformed z-momentum and the same applies to $G\,|N'h'\rangle$, in general with a different rotation, and with $P_z = +\frac{\ell_z}{2}$.

Apart from a factor which includes the normalization of the states and the relevant Lorentz contraction factor γ, we may write

$$G\,\mathcal{J}_{io}(0)\, G^{-1} \;\propto\; \mathcal{J}_{io}(0) + \mathcal{J}_{iz}(0) \qquad .$$

At this point we have the matrix elements of a combination of the current components between helicity eigenstates in the Breit frame with momenta along a certain direction different from the z-direction, which, in the case of equal masses, is the x-direction. We then insert in the matrix element a further rotation around the y-axis to align the momenta along the x-axis.

The result of all these operations is the following:

$$\langle N'h' \,|F_i(\underset{\sim}{k}_\perp)|\, Nh\rangle \;=\; \eta\, \langle N', h', P'_x = \frac{q}{2}\,, P'_y = 0,$$

$$P'_z = 0\, |e^{-i\,\mathcal{J}_y \phi'}\;\; Y_i(0)\;\; e^{i\,\mathcal{J}_y \phi}\,|\, N, h, P_x = -\frac{q}{2}\,, P_y = 0, P_z = 0\rangle \qquad , \tag{20}$$

where η is the overall factor mentioned before and is given by

$$\eta = 2 \frac{\epsilon\epsilon'}{\epsilon + \epsilon'} ,$$

ϵ and ϵ' being the energies in the Breit frame; $q/2$ is the space momentum of each state in the same frame; ϕ and ϕ' are the total angles of the rotations performed,

$$\phi = \arctan \frac{M' - M}{k} + \arctan \frac{k}{M' + M} ,$$

$$\phi' = \arctan \frac{M' - M}{k} - \arctan \frac{k}{M' + M} ,$$

and finally

$$Y_i(0) = \mathcal{F}_{io}(0) + \cos\theta\, \mathcal{F}_{iz}(0) - \sin\theta\, \mathcal{F}_{ix}(0)$$

where θ is the angle between the original Breit frame momentum of N' and the x-direction, such that $\cos\theta = k/q$.

In the Breit frame the properties of the matrix elements of $Z_i(0)$ are well-known and expressible in terms of the analogues of the famous Sachs form factors G_E and G_M. Moreover, by inverting Eq. (20), they are equal to the matrix elements

$$\langle N'h' | e^{+i \mathcal{J}_y \phi'} F_i(\underset{\sim}{k}_\perp) e^{-i \mathcal{J}_y \phi} | Nh \rangle$$

and these are thus objects with known angular momentum behavior; most important, we have for these matrix elements the property

$$\Delta \mathcal{J}_x = 0, \pm 1 \quad \text{for} \quad \langle N'h' | e^{i \mathcal{J}_y \phi'} F_i(\underset{\sim}{k}_\perp) e^{-i \mathcal{J}_y \phi} | Nh \rangle \quad . \tag{21}$$

We can also find, for fixed N' and N, the following properties of $\langle N'h' | F_i(\underset{\sim}{k}_\perp) | Nh \rangle$:

i) The part odd in $\underset{\sim}{k}_\perp$ has $\Delta \mathcal{J}_z$ odd, while the even part has $\Delta \mathcal{J}_z$ even.

ii) It is invariant under $\mathcal{P}\, e^{-i \mathcal{J}_y \pi}$ and $\mathcal{T}\, e^{-i \mathcal{J}_y \pi}$ where $\mathcal{P}$ and $\mathcal{T}$ are the parity and time-reversal operators respectively.

iii) For $N = N'$, so that the Breit frame momentum q is equal to k, a multipole expansion shows that the coefficient of k^j contains only $|\Delta \underset{\sim}{\mathcal{J}}| = 0, 2, \ldots j$ for j even, and $|\Delta \underset{\sim}{\mathcal{J}}| = 1, 2, \ldots j+1$ for j odd (only odd $|\Delta \underset{\sim}{\mathcal{J}}|$ for odd j when the current is conserved).

In the case of $F_i^5(\underset{\sim}{k}_\perp)$, the only difference is that it is odd under $\mathcal{P}\, e^{-i \mathcal{J}_y \pi}$, and the coefficient of k^j contains only $|\Delta \underset{\sim}{\mathcal{J}}| = 1, 2, 3, \ldots j+1$ for j even and $|\Delta \underset{\sim}{\mathcal{J}}| = 1, 3, \ldots j$ for j odd.

An interesting thing to note is that for degenerate states $N = N'$ we can make simple statements about $|\Delta \underset{\sim}{\mathcal{J}}|$ properties and not only about $\Delta \mathcal{J}_x$, as in the general case; and the reason is that in this case we have only "allowed transitions" with the $|\Delta \underset{\sim}{\mathcal{J}}|$ properties mentioned, while in the general case, as we put in higher and higher powers of $(M - M')$, we get higher and higher "forbidden" transitions in addition to the allowed ones.

We want to give now two illustrations which will clarify how all such things work. Let us first consider the very simple example of the matrix element of $F_i(\underset{\sim}{k}_\perp)$ taken between two states with equal masses and spin and parity $1/2^+$ at $P_z = \infty$. Then we have in terms of Dirac-Pauli form factors:

$$\left\langle U' \left| \frac{(1 + \alpha_z)}{\sqrt{2}} \left[F_1(\underset{\sim}{k}_\perp^{\,2}) + \frac{F_2}{2M} (\underset{\sim}{k}_\perp^{\,2})\, \beta\, \alpha_x\, k\right] \frac{(1 + \alpha_z)}{\sqrt{2}} \right| U \right\rangle$$

where $|U\rangle$ and $|U'\rangle$ are Dirac spinors. Since $|U\rangle$ is such that $(1+\beta_z)/2\,|U\rangle = |U\rangle$, we may write this matrix element in the form

$$\langle U' \,|[F_1(\underset{\sim}{k}_\perp{}^2) - i\,\sigma_y\,\frac{k}{2M}\,F_2(\underset{\sim}{k}_\perp{}^2)]|\,U\rangle \quad .$$

Now if we want to evaluate the matrix element (21) appropriate to the Breit frame, we have to insert the matrices $e^{+i\,\mathcal{J}_y\phi}$, thus obtaining

$$N(k)\,\left\langle U' \left| e^{-i\sigma_y \arctan\frac{k}{2M}}\,[F_1 - i\,\sigma_y\,\frac{k}{2M}\,F_2]\right| U\right\rangle \quad ,$$

where $N(k)$ is the normalization factor pertaining to the Breit frame, i.e.,

$$N(k) = \frac{1}{\sqrt{1+\frac{k^2}{4M^2}}} \quad .$$

By noting that

$$e^{-i\sigma_y \arctan\frac{k}{2M}} = \frac{1}{\sqrt{1+\frac{k^2}{4M^2}}}\,[1 - i\,\sigma_y\,\frac{k}{2M}] \quad ,$$

we immediately get

$$\frac{1}{\left(1+\frac{k^2}{4M^2}\right)}\,\left\langle U' \left|\left\{[F_1 - \frac{k^2}{4M^2}\,F_2] - i\,\sigma_y\,\frac{k}{2M}\,[F_1 + F_2]\right\}\right|\,U\right\rangle$$

which is the well-known Sachs form of the current, provided we identify in the usual manner

$$G_E(k^2) = \left(F_1 - \frac{k^2}{4M^2} F_2\right) (1 + k^2/4M^2)^{-1} \quad ,$$

$$G_M(k^2) = (F_1 + F_2) (1 + k^2/4M^2)^{-1} \quad .$$

So we see that by considering the matrix elements (21), in this simple case we found the form of the current which is appropriate to the Breit system and which has well-defined angular momentum properties. Moreover, this example clearly shows that the matrix elements of the currents at $P_z = \infty$ do not depend on the average transverse momentum.

Let us look at another illustration of the general angular momentum properties of the current matrix elements that we listed before. Now we will choose the case of a system which is intrinsically non-relativistic, in the sense that the interesting part of the spectrum, which saturates the commutation rules of our algebra, has a level spacing which is small compared with the mass of the system. This is true for atoms, for nuclei, and it is also very roughly true for baryons, but it is not so for mesons.

In this case $(M' - M) \ll (M' + M)$; a situation in which one, for instance, ignores the Dirac magnetic moment as compared with the anomalous one, and also the very complicated "Dirac" effects in the matrix elements, which we would have for general spins.

In this approximation, let us consider instead of equal time commutators taken at $P_z = \infty$, commutators at a lightlike interval between states approximately at rest. This is obtained simply by a pure Lorentz transformation from $P_z = \infty$ to rest, provided the mass differences are negligible.

What we pick up is the following:

$$F_i(k\ ) = \int [\mathcal{F}_{io}(xyz,z) + \mathcal{F}_{iz}(xyz,z)]\ e^{+i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}}\ d^3x \qquad (22)$$

and this is the same as

$$\int e^{-iHz}\ [\mathcal{F}_{io}(xyz,0) + \mathcal{F}_{iz}(xyz,0)]\ e^{iHz}\ e^{+i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}}\ d^3x \quad ,$$

where H is the energy operator. Noting that for our essentially non-relativistic system the energy is approximately equal to the mass, what we finally get by sandwiching these operators between two states with masses M and M' is

$$\langle N'M'\ |F_i(\underset{\sim}{k}_\perp)|\ NM\rangle = \int e^{-i\Delta Mz}\ e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}}$$

$$\langle N'M'\ |[\mathcal{F}_{io}(\underset{\sim}{x},0) + \mathcal{F}_{iz}(\underset{\sim}{x},0)]|\ NM\rangle\ d^3x \quad . \qquad (23)$$

This clearly shows why we have such complicated angular momentum properties for these matrix elements; the reason being that in addition to charge and current densities and the $e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}}$ factor, we have an extra retardation factor $e^{-i\Delta Mz}$ which introduces an amount of z-dependence that depends on the mass differences.

To evaluate the matrix element (21) in this case, we have to perform a rotation of an angle $\phi' = \phi = \arctan \Delta M/k$ around the y-axis (the rest of each rotation vanishing, since $k/(M+M') \simeq 0$), and we end up with an overall exponential factor $e^{-ix\sqrt{\Delta M^2 + k^2}}$, i.e., we get

$$\langle N'M' | e^{+i \mathcal{G}_y \phi} \; F_i(\underset{\sim}{k}_\perp) \; e^{-i \mathcal{G}_y \phi} | NM \rangle = \int d^3x \; e^{-ix\sqrt{\Delta M^2 + k^2}}$$

$$\langle N'M' | [\mathcal{F}_{io}(\underset{\sim}{x},0) + \cos\phi \; \mathcal{F}_{iz}(\underset{\sim}{x},0) + \sin\phi \; \mathcal{F}_{ix}(\underset{\sim}{x},0)] | NM \rangle \quad . \qquad (24)$$

From this it is clear that the $\Delta \mathcal{G}_x$ selection rule comes now entirely from the current indices, so that we have $\Delta \mathcal{G}_x = 0, \pm 1$; the exponential factor has $\Delta \mathcal{G}_x = 0$. The multipole expansion properties and parity and time-reversal properties of the matrix element are all easily read off from this expression.

The sum rules that can be obtained in this way differ from the usual sum rules used in atomic and nuclear physics in that, first of all these are supposed to be exact in the strong interaction, and secondly, because of the presence of the retardation factor these rules are evaluated at fixed four-momentum transfer $= +k^2$, while in ordinary theoretical atomic physics one deals with momentum transfers depending on the masses. And the price for that is that the angular properties are now somewhat more complicated.

Since the Hamiltonian is just the time translation operator, we can expand Eq. (23) in the following way:

$$\langle N'h' | F_i(\underset{\sim}{k}_\perp) | Nh \rangle = \sum_n \frac{(-1)^n}{n!} \frac{d^n}{dt^n} \int z^n \; e^{ikx}$$

$$\langle N'h' | \mathcal{F}_{io}(\underset{\sim}{x},0) + \mathcal{F}_{iz}(\underset{\sim}{x},0) | Nh \rangle \; d^3x \qquad . \qquad (25)$$

To have a concrete example, let us specialize this formula to the case of a single non-relativistic quark in a potential whose energy is given by

$$M \approx \text{const.} + \frac{p^2}{2m} + V(\underset{\sim}{x}) \quad .$$

Now it is very easy to write down the currents for this object in such a way that they almost satisfy the relativistic commutation rules; i.e.,

$$F_i(\underset{\sim}{k}_\perp) = \frac{\lambda_i}{2} \sum_n \frac{(-1)^n}{n!} \frac{d^n}{dt^n} \left[\left(\frac{z}{c}\right)^n e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}} \left(1 + \frac{\dot{z}}{c}\right)\right] \quad , \qquad (26)$$

where z and x are now the Heisenberg operators for the location of the particle, λ_i are the usual SU(3) matrices and inside the parentheses there is the sum of the charge and of the current density.

The reason why this is only approximately a representation of the local algebra is that, apart from corrections from coordinate-velocity commutators, the left-hand side of Eq. (26) is just the operator

$$\frac{\lambda_i}{2} \; e^{i\underset{\sim}{k}_\perp \cdot \underset{\sim}{x}(t_o)}$$

(where t_o is defined to be such that $z(t_o) = -t_o$), by which the algebra is trivially obeyed. Quantum mechanically, the difficulty is that one has to neglect the order in which z and $\dot{z}$ appear. However, this introduces only corrections of fairly high order in (v/c) in the commutation rules.

As a final remark, we note that for the axial vector charges, in this approximation we have

$$\langle N'h' | F_i^{5}(0) | Nh \rangle = \int d^3x \ e^{-i\Delta Mz} \langle N'h' | \mathcal{F}_{io}^{5}(\underset{\sim}{x},0) + \mathcal{F}_{iz}^{5}(\underset{\sim}{x},0) | Nh \rangle$$

and this has the features we have learned from the application of the Adler-Weisberger relation. Indeed, the retardation factor $e^{-i\Delta Mz}$ allows the axial charges to couple the nucleon to states with higher spins, because every power of ΔM introduces one more power of z which brings one more unit of $\Delta \mathcal{J}$ into the transition. In terms of pion coupling, this is equivalent to saying that the pion couples the nucleon to higher spin resonances through its orbital angular momentum.

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