NP-4565

E - NP-4450

Wo-33620

THE THEORY OF QUANTIZED FIELDS III

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DISCLAIMER

Abstract

In this paper we discuss the electromagnetic field, as perturbed by a prescribed current. All quantities of physical interest in various situations, eigenvalues, eigenfunctions, and transition probabilities, are derived from a general transformation function which is expressed in a non-Hermitian representation. The problems treated are: the determination of the energy-momentum eigenvalues and eigenfunctions for the isolated electromagnetic field, and the energy eigenvalues and eigenfunctions for the field perturbed by a time-independent current; the evaluation of transition probabilities and photon number expectation values for a time-dependent current that departs from zero only within a finite time interval, and for a time-dependent current that assumes non-vanishing time-independent values initially and finally. The results are applied in a discussion of the infra-red catastrophe and of the adiabatic theorem. It is shown how the latter can be exploited to give a uniform formulation for all problems requiring the evaluation of transition probabilities or eigenvalue displacements.

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INTRODUCTION

We shall approach the general problem of coupled fields through the simpler situation presented by a single field which is externally perturbed. In this paper we illustrate the treatment of a Bose-Einstein system by discussing the Maxwell field with a prescribed electric current. A succeeding paper will be devoted to the Dirac field.

The solution to all dynamical questions is obtained by constructing the transformation function linking two descriptions of the system that are associated with different space-like surfaces. Thus, for a closed system, the general transformation function can be expressed as

$$(\zeta_1'\sigma_1|\zeta_2''\sigma_2) = \sum_{\gamma'\gamma''}(\zeta_1'|\gamma')(\gamma'\sigma_1|\gamma''\sigma_2)(\gamma''|\zeta_2''),$$

where the γ's are a complete set of compatible constants of the motion, in terms of which the energy-momentum vector P_μ can be exhibited. In the γ representation, the effect of an infinitesimal translation of σ_1 is given by

$$\begin{aligned}\delta_\varepsilon(\gamma'\sigma_1|\gamma''\sigma_2) &= i(\gamma'\sigma_1|P_\mu\,\delta\varepsilon_\mu|\gamma''\sigma_2)\\ &= i P_\mu'\,\delta\varepsilon_\mu(\gamma'\sigma_1|\gamma''\sigma_2),\end{aligned}$$

where

$$P_\mu' = P_\mu(\gamma').$$

Accordingly, if σ_1 is parallel to σ_2, and is generated from the latter by the translation X_μ, we have

$$(\gamma'\sigma_1|\gamma''\sigma_2) = \delta(\gamma',\gamma'')\exp(iP_\mu'X_\mu),$$

and

$$(\zeta_1'\sigma_1|\zeta_2''\sigma_2) = \sum_{\gamma'}(\zeta_1'|\gamma')\exp(iP_\mu'X_\mu)(\gamma'|\zeta_2''). \tag{1}$$

This shows how a knowledge of the transformation function that relates two conveniently chosen representations on parallel surfaces yields all the eigenvalues and eigenfunctions of P_μ.

Another illustration of the utility of transformation functions relates to the situation in which the same system is externally perturbed, in the interior of the space-time region bounded by σ_1 and σ_2. The transformation function $(\gamma'\sigma_1|\gamma''\sigma_2)$, inferred from the knowledge of $(\zeta_1'\sigma_1|\zeta_2''\sigma_2)$, then yields the probability of a transition from the initial state γ'' to the final state γ',

$$p(\gamma',\gamma'') = |(\gamma'\sigma_1|\gamma''\sigma_2)|^2. \tag{2}$$

Representations of particular convenience are suggested by the characterization of the vacuum state for a complete system. The vacuum is the state of minimum energy. If this natural origin of energy is adjusted to zero, the vacuum can be described as that state presenting identical properties to all observers,

$$P_\mu \Psi_0 = J_{\mu\nu} \Psi_0 = 0$$

and is therefore independent of the surface σ. Now, if the general field component χ is analyzed into contributions of various frequencies, χ_{p_0}, we have

$$[\chi_{p_0}, P_0] = p_0 \chi_{p_0},$$

or

$$P_0 \chi_{p_0} = \chi_{p_0}(P_0 - p_0).$$

When this relation, involving a positive frequency, $p_0 > 0$, is applied to the vacuum state vector, we obtain

$$P_0(\chi_{p_0} \Psi_0) = -p_0(\chi_{p_0} \Psi_0)$$

Hence

$$\chi_{p_0} \Psi_0 = 0, \qquad p_0 > 0, \tag{3}$$

since this state, of energy less than that of the vacuum, must be non-existent.

A similar discussion yields

$$\Psi_0^+ \chi_{p_0} = 0, \quad p_0 < 0,$$

which is the statement adjoint to (3). The vector Ψ_0 is thus characterized as the right eigenvector of the positive frequency parts of the field components, $\chi^{(+)}$, with zero eigenvalues, and $\Psi_0^{\dagger}$ appears as the left eigenvector, with zero eigenvalues, of the $\chi^{(-)}$, the negative frequency parts of the field components. It should be noted that the decomposition into positive and negative frequency parts is invariant under orthochronous Lorentz transformations. The complete sets of eigenvectors of these types will evidently be of particular value for the construction of energy eigenstates.

THE MAXWELL FIELD

Elementary descriptions of the electromagnetic field on a given σ are provided by the alternative complete sets of commuting operators, the transverse potential $A_k(x)$, and the transverse electric field[1] $F_{0k}(x)$.

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[1] When no confusion is likely, we shall not employ the more complete notation $F^{(T)}_{(0)(k)}(x)$, which indicates that these are the transverse field components, relative to a local coordinate system based on σ.

Following the suggestion of the preceding section, we employ, instead, the non-Hermitian operators, $F_{0k}^{(+)}(x)$ and $F_{0k}^{(-)}(x)$, which in the absence of an external current, are the positive and negative frequency parts of $F_{0k}(x)$. The transverse field equations, for zero current, are

$$-\partial_0 A_k = F_{0k},$$

$$\partial_0 F_{0k} = \partial_l^2 A_k = \omega^2 A_k$$

where ω, as a coordinate operator, is defined by the matrix

$$(\underset{\sim}{x}|\omega|\underset{\sim}{x}') = \int \frac{(d\underset{\sim}{k})}{(2\pi)^3}\, |\underset{\sim}{k}|\, \exp\left(i\,\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right),$$

which is symmetrical and positive-definite. On writing

$$F_{0k} = F_{0k}^{(+)} + F_{0k}^{(-)},$$

$$A_k = A_k^{(+)} + A_k^{(-)},$$

where

$$F_{0k}^{(\pm)} = \pm i \omega A_k^{(\pm)}$$
$$= \tfrac{1}{2}(F_{0k} \pm i \omega A_k),$$

the equations of motion assume the form

$$\partial_0 F_{0k}^{(\pm)} = \mp i \omega F_{0k}^{(\pm)},$$

which, in virtue of the positive-definite nature of ω, confirms the interpretation of $F_{0k}^{(\pm)}$.

The canonical form of the infinitesimal generators

$$G_A = \int d\sigma\, (-F_{0k})\, \delta A_k$$
$$G_F = \int d\sigma\, A_k\, \delta F_{0k},$$

can be extended to the generators of infinitesimal changes in the non-Hermitian operators $F_{0k}^{(\pm)}$, in the sense of the transformation equation

$$G - \bar{G} = \delta(W).$$

Thus

$$G_F = \int d\sigma\, (A_k^{(+)} + A_k^{(-)})(\delta F_{0k}^{(+)} + \delta F_{0k}^{(-)})$$
$$= \int d\sigma\, 2 A_k^{(-)} \delta F_{0k}^{(+)} + \delta\Big[\int d\sigma \big(\tfrac{1}{2} A_k^{(+)} F_{0k}^{(+)} + \tfrac{1}{2} A_k^{(-)} F_{0k}^{(-)} + A_k^{(+)} F_{0k}^{(-)}\big)\Big],$$

and

$$G_F = \int d\sigma\, 2 A_k^{(+)} \delta F_{0k}^{(-)} + \delta\Big[\int d\sigma \big(\tfrac{1}{2} A_k^{(+)} F_{0k}^{(+)} + \tfrac{1}{2} A_k^{(-)} F_{0k}^{(-)} + A_k^{(-)} F_{0k}^{(+)}\big)\Big],$$

which yields

$$G_{F^{(+)}} = \int d\sigma\, 2 A_k^{(-)} \delta F_{0k}^{(+)} = 2i \int d\sigma\, F_{0k}^{(-)} \omega^{-1} \delta F_{0k}^{(+)},$$

and

$$G_{F^{(-)}} = \int d\sigma\, 2 A_k^{(+)} \delta F_{0k}^{(-)} = -2i \int d\sigma\, F_{0k}^{(+)} \omega^{-1} \delta F_{0k}^{(-)},$$

The commutation relations on σ, implied by these generators, are

$$[F_{0k}^{(+)}(x), F_{0l}^{(+)}(x')] = [F_{0k}^{(-)}(x), F_{0l}^{(-)}(x')] = 0,$$

and

$$[F_{0k}^{(+)}(x), 2 A_{0l}^{(-)}(x')] = [F_{0k}^{(-)}(x), 2 A_{0l}^{(+)}(x')] = i(\delta_{kl}\, \delta_\sigma(x-x'))^{(T)}.$$

The latter can also be written

$$\left[F_{ok}^{(+)}(x),\ F_{ol}^{(-)}(x')\right] = \frac{1}{2}\left(\delta_{kl}\ (\underset{\sim}{x}|\omega|\underset{\sim}{x}')\right)^{(T)} . \tag{4}$$

These operator properties can be verified directly from those of F_{ok} and A_k.

It should be noted that there exists some freedom in choosing the generator for a given set of independent variables. Thus,

$$'G_{F^{(+)}} = \int d\sigma\ 2A_k\,\delta F_{ok}^{(+)}$$

is also a generator of infinitesimal changes in $F_{ok}^{(+)}$, since

$$G_{F^{(+)}} - 'G_{F^{(+)}} = -\int d\sigma\ 2A_k^{(+)}\delta F_{ok}^{(+)} = 2i\int d\sigma\ F_{ok}^{(+)}\,\omega^{-1}\delta F_{ok}^{(+)}$$
$$= \delta\left[i\int d\sigma\ F_{ok}^{(+)}\,\omega^{-1}\,F_{ok}^{(+)}\right].$$

Similarly,

$$'G_{F^{(-)}} = \int d\sigma\ 2A_k\,\delta F_{ok}^{(-)}$$

is an alternative generator of changes in $F_{ok}^{(-)}$,

$$G_{F^{(-)}} - 'G_{F^{(-)}} = \delta\left[-i\int d\sigma\ F_{ok}^{(-)}\,\omega^{-1}\,F_{ok}^{(-)}\right].$$

The eigenvector concept can be extended to non-Hermitian operators, with some limitations. We introduce the right eigenvector of the complete set of commuting operators, $F_{ok}^{(+)}(x)$ on σ,

$$F_{ok}^{(+)}(x)\,\Psi(F^{(+)\prime}\sigma) = F_{ok}^{(+)\prime}(x)\,\Psi(F^{(+)\prime}\sigma),$$

and the left eigenvector of the complete set, $F_{ok}^{(-)}(x)$ on σ,

$$\Phi(F^{(-)\prime}\sigma)\,F_{ok}^{(-)}(x) = \Phi(F^{(-)\prime}\sigma)\,F_{ok}^{(-)\prime}(x) .$$

In virtue of the relation

$$F_{ok}^{(-)}(x) = F_{ok}^{(+)}(x)^{\dagger},$$

these eigenvectors and eigenvalues are connected by

$$\Phi(F^{(-)\prime}\sigma) = \Psi(F^{(+)\prime}\sigma)^{\dagger},$$
$$F_{ok}^{(-)\prime}(x) = F_{ok}^{(+)\prime}(x)^{*} . \tag{5}$$

However, the right eigenvector of the $F_{0k}^{(-)}$ and the left eigenvector of the $F_{0k}^{(+)}$ do not exist. This can be inferred from the commutator (4), in the form

$$\left[{}'F_{0k}^{(-)}(x)^{\dagger},\ {}'F_{0\ell}^{(-)}(x') \right] = \frac{1}{2}\left(\delta_{k\ell}\,(\underset{\sim}{x}|\omega|\underset{\sim}{x}')\right)^{(T)},$$

where

$${}'F_{0k}^{(-)}(x) = F_{0k}^{(-)}(x) - F_{0k}^{(-)\prime}(x) .$$

When applied to the hypothetical eigenvector $\Psi(F^{(-)\prime}\sigma)$, this relation yields

$$-{}'F_{0\ell}^{(-)}(x')\ {}'F_{0k}(x)^{\dagger}\,\Psi(F^{(-)\prime}\sigma) = \frac{1}{2}\left(\delta_{k\ell}\,(\underset{\sim}{x}|\omega|\underset{\sim}{x}')\right)^{(T)}\Psi(F^{(-)\prime}\sigma) .$$

The contradiction between the negative-definite nature of the operator on the left, and the positive-definite character of the numerical quantity on the right establishes the non-existence[2] of $\Psi(F^{(-)\prime}\sigma)$, and similarly,

[2] It is evident from the discussion of the first section, that this is related to the non-existence of a state with maximum energy.

of $\Phi(F^{(+)\prime}\sigma)$.

Let us consider the significance of the change induced in the eigenvectors $\Psi(F^{(+)\prime}\sigma)$ and $\Phi(F^{(-)\prime}\sigma)$ by the respective generators $G_{F^{(+)}}$ and $G_{F^{(-)}}$, according to the mutually Hermitian conjugate equations,

$$\delta\Psi(F^{(+)\prime}\sigma) = -i\,G_{F^{(+)}}\,\Psi(F^{(+)\prime}\sigma)$$

$$\delta\Phi(F^{(-)\prime}\sigma) = i\,\Phi(F^{(-)\prime}\sigma)\,G_{F^{(-)}}$$

Now $\Psi(F^{(+)\prime}\sigma) + \delta\Psi(F^{(+)\prime}\sigma)$ is the eigenvector of the operator set $F_{0k}^{(+)} - \delta F_{0k}^{(+)}$, with the eigenvalues $F_{0k}^{(+)\prime}$. Since the $\delta F_{0k}^{(+)}$ are arbitrary infinitesimal numbers, this vector is also the eigenvector of the $F_{0k}^{(+)}$ with the eigenvalues $F_{0k}^{(+)\prime} + \delta F_{0k}^{(+)}$. Hence the alteration of the eigenvector $\Psi(F^{(+)\prime}\sigma)$ is that associated with the change of the eigenvalues by $\delta F_{0k}^{(+)}$. A similar statement applies to $\delta\Phi(F^{(-)\prime}\sigma)$.

The relation between the eigenvectors $\Psi(F^{(+)'}{}_\sigma)$ and ${}'\Psi(F^{(+)'}{}_\sigma)$, which are affected analogously by the respective generators $G_{F^{(+)}}$ and ${}'G_{F^{(+)}}$, can be deduced from

$$\begin{aligned}\delta\, {}'\Psi(F^{(+)'}{}_\sigma) &= -i\, {}'G_{F^{(+)}}\, {}'\Psi(F^{(+)'}{}_\sigma) \\ &= -i\, G_{F^{(+)}}\, {}'\Psi(F^{(+)'}{}_\sigma) - \delta\left(\int d\sigma\, F_{0k}^{(+)'}\omega^{-1} F_{0k}^{(+)'}\right) {}'\Psi(F^{(+)'}{}_\sigma),\end{aligned}$$

namely,

$$\begin{aligned}{}'\Psi(F^{(+)'}{}_\sigma) &= \exp\left(-\int d\sigma\, F_{0k}^{(+)'}\omega^{-1} F_{0k}^{(+)'}\right) \Psi(F^{(+)'}{}_\sigma) \\ &= \exp\left(-i\int d\sigma\, A_k^{(+)'} F_{0k}^{(+)'}\right) \Psi(F^{(+)'}{}_\sigma).\end{aligned} \tag{6}$$

The adjoint equation reads

$$\begin{aligned}{}'\Phi(F^{(-)'}{}_\sigma) &= \exp\left(-\int d\sigma\, F_{0k}^{(-)'}\omega^{-1} F_{0k}^{(-)'}\right) \Phi(F^{(-)'}{}_\sigma) \\ &= \exp\left(i\int d\sigma\, A_k^{(-)'} F_{0k}^{(-)'}\right) \Phi(F^{(-)'}{}_\sigma)\end{aligned} \tag{7}$$

We shall now discuss the Maxwell field under the influence of a prescribed current distribution $J_\mu(x)$. It is convenient, initially, to describe the relations between states on the two arbitrary plane surfaces, σ_1 and σ_2, by means of the transformation function

$${}'(F^{(-)'}{}_{\sigma_1} | F^{(+)'}{}_{\sigma_2}) = ({}'\Phi(F^{(-)'}{}_{\sigma_1})\; {}'\Psi(F^{(+)'}{}_{\sigma_2})). \tag{8}$$

The dependence of this transformation function on the eigenvalues $F_{(0)(k)}^{(-)'}$ and $F_{(0)(k)}^{(+)'}$ is indicated by

$$\begin{aligned}\delta_F\, {}'(F^{(-)'}{}_{\sigma_1} | F^{(+)'}{}_{\sigma_2}) &= i(F^{(-)'}{}_{\sigma_1} | [{}'G_{F^{(-)}}(\sigma_1) - {}'G_{F^{(+)}}(\sigma_2)] | F^{(+)'}{}_{\sigma_2}) \\ &= 2i\, {}'(F^{(-)'}{}_{\sigma_1} | [\int_{\sigma_1} d\sigma_\mu\, \delta F_{\mu\nu}^{(-)'}\, {}'A_\nu - \int_{\sigma_2} d\sigma_\mu\, \delta F_{\mu\nu}^{(+)'}\, {}'A_\nu] | F^{(+)'}{}_{\sigma_2}),\end{aligned} \tag{9}$$

while an infinitesimal change of the external current produces the alteration

$$\delta_J\, {}'(F^{(-)'}{}_{\sigma_1} | F^{(+)'}{}_{\sigma_2}) = i(F^{(-)'}{}_{\sigma_1} | \int_{\sigma_2}^{\sigma_1}(dx)\,\delta J_\mu A_\mu | F^{(+)'}{}_{\sigma_2}). \tag{10}$$

The current variations are subject to the restriction

$$\partial_\mu \delta J_\mu = 0$$

Accordingly, if we rewrite (10) in the notation

$$(\delta/\delta J_\mu(x))\, '(F^{(-)'}\sigma_1 | F^{(+)'}\sigma_2) = i\, '(F^{(-)'}\sigma_1 | A_\mu(x) | F^{(+)'}\sigma_2), \qquad (11)$$

we are at liberty to add an arbitrary gradient to $A_\mu(x)$. Since this coincides with the freedom of gauge transformations, we do not indicate it explicitly.

The advantage provided by the transformation function (8) rests in the possibility of combining (9) and (11) into

$$\delta_F\, '(F^{(-)'}\sigma_1 | F^{(+)'}\sigma_2)$$
$$= \left[2\int_{\sigma_1} d\sigma_\mu\, \delta F_{\mu\nu}^{(-)'}(\delta/\delta J_\nu) - 2\int_{\sigma_2} d\sigma_\mu\, \delta F_{\mu\nu}^{(+)'}(\delta/\delta J_\nu)\right] '(F^{(-)'}\sigma_1 | F^{(+)'}\sigma_2),$$

which possesses the formal solution

$$'(F^{(-)'}\sigma_1 | F^{(+)'}\sigma_2)$$
$$= \exp\left[2\int_{\sigma_1} d\sigma_\mu\, F_{\mu\nu}^{(-)'}(\delta/\delta J_\nu) - 2\int_{\sigma_2} d\sigma_\mu\, F_{\mu\nu}^{(+)'}(\delta/\delta J_\nu)\right] '(0\sigma_1 | 0\sigma_2) \qquad (12)$$

The problem is thus reduced to the construction of the transformation function referring to null eigenvalues.

We shall write[3]

[3] The dash is omitted, since there is no distinction between the eigenvectors $\Psi(F^{(+)'}\sigma)$ and $'\Psi(F^{(+)'}\sigma)$, for zero eigenvalues.

$$(0\sigma_1 | 0\sigma_2) = \exp(iW_0),$$

AND

$$(0\sigma_1 | A_\mu(x) | 0\sigma_2)/(0\sigma_1 | 0\sigma_2) = \langle A_\mu(x) \rangle$$

In this notation, the dependence of the null eigenvalue transformation function upon the external current is described by

$$(\delta/\delta J_\mu(x))\, W_0 = \langle A_\mu(x) \rangle,$$

or

$$\delta W_0 = \int_{-\infty}^{\infty} (dx)\, \delta J_\mu(x) \langle A_\mu(x) \rangle, \qquad (13)$$

in which we have extended the integration over the entirety of space-time by supposing that the current vanishes externally to the region of interest, the volume bounded by σ_1 and σ_2. According to the operator field equation

$$\partial_\nu F_{\mu\nu} = -\partial_\nu^2 A_\mu + \partial_\mu \partial_\nu A_\nu = J_\mu$$

the numerical quantity $\langle A_\mu(x) \rangle$ obeys the differential equation

$$-\partial_\nu^2 \langle A_\mu \rangle + \partial_\mu \partial_\nu \langle A_\nu \rangle = J_\mu \tag{14}$$

Now the gauge ambiguity of $\langle A_\mu \rangle$ is completely without effect in (13), since J_μ vanishes on the boundary of the extended region. Therefore, for the purpose of constructing W_0, we can replace the differential equation (14) with

$$-\partial_\nu^2 \langle A_\mu \rangle = J_\mu \tag{15}$$

We are concerned with the solution of this equation that is compatible with the boundary conditions

$$\langle F^{(-)}_{(0)(k)} \rangle = 0, \quad \text{on } \sigma_1$$

$$\langle F^{(+)}_{(0)(k)} \rangle = 0, \quad \text{on } \sigma_2$$

which follow from the nature of the null eigenvalue states on σ_1 and σ_2. Since the current vector is zero in the external region, we can rephrase these boundary conditions as the requirement that the field shall contain only positive frequencies in the domain constituting the future of σ_1, and only negative frequencies in the region prior to σ_2. This excludes a possible homogeneous solution of (15), whence

$$\langle A_\mu(x) \rangle = \int_{-\infty}^{\infty} (dx') D_+(x-x') J_\mu(x'), \tag{16}$$

in which $D_+(x-x')$ is the Green's function defined by

$$-\partial_\nu^2 D_+(x-x') = \delta(x-x'),$$

together with the statement that it contains only positive frequencies for $x_0 > x_0'$, and only negative frequencies for $x_0 < x_0'$. It therefore satisfies the temporal analogue of the outgoing wave or radiation condition familiar in the spatial description of a harmonic source.[4]

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[4] Green's functions of this type have been discussed by E. C. G. Stueckelberg, Helv. Phys. Acta, 19,242 (1946), and by R. P. Feynman, Phys. Rev. 76,749 (1949).

The expression of (16) provided by

$$(\delta/\delta J_\nu(x'))\langle A_\mu(x)\rangle = (\delta/\delta J_\mu(x))(\delta/\delta J_\nu(x'))W_0$$
$$= \delta_{\mu\nu} D_+(x-x'),$$

indicates that $D_+(x-x')$ is a symmetrical function of x and x'. Accordingly, the integral of (13) is

$$W_0 = \frac{1}{2}\int_{-\infty}^{\infty}(dx)(dx')\, J_\mu(x) D_+(x-x') J_\mu(x'), \tag{17}$$

apart from the additive constant which is the value of W_0 for the isolated electromagnetic field $(J_\mu = 0)$. It is an advantage of the representation we have been employing that this integration constant has the value zero. Indeed, the null eigenvalue states of the complete system provided by the electromagnetic field with no external current are just the σ- independent vacuum state, whence

$$J_\mu = 0: \qquad (0\sigma_1|0\sigma_2) = 1$$
$$W_0 = 0$$

The differential operator appearing in (12) has the effect of inducing the substitution

$$J_\mu(x) \rightarrow J_\mu(x) + i\left[\delta_\mu(x,\sigma_1) F_{\mu\nu}^{(-)'}(x) - \delta_\mu(x,\sigma_2) F_{\mu\nu}^{(+)'}(x)\right]$$

in W_0. Here $\delta_\mu(x,\sigma)$ represents a one-dimensional delta function, which is defined by

$$\int(dx)\,\delta_\mu(x,\sigma) f_\mu(x) = \int_\sigma d\sigma_\mu f_\mu(x).$$

Hence

$$(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2) = \exp(iW),$$

where

$$
\begin{aligned}
W = {} & W_0 + 2\int_{\sigma_1} d\sigma_\mu\, F_{\mu\nu}^{(-)\prime}(x)\langle A_\nu(x)\rangle - 2\int_{\sigma_2} d\sigma_\mu\, F_{\mu\nu}^{(+)\prime}(x)\langle A_\nu(x)\rangle \\
& + 2\int_{\sigma_1} d\sigma_\mu \int_{\sigma_1} d\sigma_\nu{}'\, F_{\mu\lambda}^{(-)\prime}(x)\, D_+(x-x')\, F_{\nu\lambda}^{(-)\prime}(x') + 2\int_{\sigma_2} d\sigma_\mu \int_{\sigma_2} d\sigma_\nu{}'\, F_{\mu\lambda}^{(+)}(x)\, D_+(x-x')\, F_{\nu\lambda}^{(+)\prime}(x') \\
& - 4\int_{\sigma_1} d\sigma_\mu \int_{\sigma_2} d\sigma_\nu{}'\, F_{\mu\lambda}^{(-)\prime}(x)\, D_+(x-x')\, F_{\nu\lambda}^{(+)\prime}(x').
\end{aligned}
\tag{18}
$$

The following symbolic form of the Green's function $D_+(x-x')$

$$
D_+(x-x') = \tfrac{1}{2} i\, \omega^{-1} \exp(-i\omega|x_0 - x_0'|)\, \delta(\underset{\sim}{x} - \underset{\sim}{x}'),
\tag{19}
$$

shows that

$$
x_0 = x_0' : \quad D_+(x-x') = \tfrac{1}{2} i\, (\underset{\sim}{x}|\omega^{-1}|\underset{\sim}{x}'),
$$

which identifies the double surface integrals, referring to a single surface, in (18), with the factors appearing in (6) and (7). Our result is therefore expressed more simply as

$$
(F^{(-)\prime}{}_{\sigma_1} | F^{(+)\prime}{}_{\sigma}) = \exp(iW)
\tag{20}
$$

with

$$
\begin{aligned}
W = {} & W_0 + 2\int_{\sigma_1} d\sigma_\mu\, F_{\mu\nu}^{(-)\prime}(x)\langle A_\nu(x)\rangle - 2\int_{\sigma_2} d\sigma_\mu\, F_{\mu\nu}^{(+)\prime}(x)\langle A_\nu(x)\rangle \\
& - 4\int_{\sigma_1} d\sigma_\mu \int_{\sigma_2} d\sigma_\nu{}'\, F_{\mu\lambda}^{(-)\prime}(x)\, D_+(x-x')\, F_{\nu\lambda}^{(+)\prime}(x')
\end{aligned}
$$

In particular,

$$
\begin{aligned}
J_\mu = 0 : \quad & (F^{(-)\prime}{}_{\sigma_1} | F^{(+)\prime}{}_{\sigma_2}) \\
& = \exp\left[-4i\int_{\sigma_1} d\sigma_\mu \int_{\sigma_2} d\sigma_\nu{}'\, F_{\mu\lambda}^{(-)\prime}(x)\, D_+(x-x')\, F_{\nu\lambda}^{(+)\prime}(x')\right]
\end{aligned}
\tag{21}
$$

APPLICATIONS

Explicit forms of the Green's function $D_+(x-x')$ are required for further work. The Fourier integral version of the three-dimensional delta function in (19) yields

$$
D_+(x-x') = \tfrac{1}{2} i \int \frac{(d\underset{\sim}{k})}{(2\pi)^3} \frac{1}{k_0} \begin{cases} e^{ik(x-x')}, & x_0 > x_0' \\ e^{-ik(x-x')}, & x_0 < x_0' \end{cases}
\tag{22}
$$

where

$$
k_0 = |\underset{\sim}{k}|
$$

is a <u>positive</u> frequency. The invariance of this structure is more evident in the four-dimensional transcription

$$
D_+(x-x') = i\int \frac{(dk)}{(2\pi)^3}\, \delta(k^2)\, e^{ik(x-x')},
$$

in which the integration is restricted to positive frequencies for $x_0 > x_0'$ and to negative frequencies for $x_0 < x_0'$. No conditions on the domain of integration are involved in the alternative four-dimensional form.

$$D_+(x-x') = \int \frac{(dk)}{(2\pi)^4} \frac{1}{k^2 - i\epsilon} e^{ik(x-x')}, \qquad \epsilon \to +0 .$$

We shall express the tensor Green's function, $\delta_{\mu\nu} D_+(x-x')$, with the aid of four orthonormal vectors associated with each plane wave,

$$\delta_{\mu\nu} = \sum_{\lambda=1}^{4} e_\mu(\lambda k)\, e_\nu(\lambda k) .$$

We choose the first two vectors to obey the conditions

$$\lambda = 1,2: \quad n_\mu e_\mu(\lambda k) = k_\mu e_\mu(\lambda k) = 0 ,$$

in which n_μ is an arbitrary time-like unit vector,

$$n_\mu^2 = -1$$

The remaining two are given explicitly by

$$e_\mu(3k) = n_\mu + k_\mu/(n_\nu k_\nu) , \qquad n_\mu e_\mu(3k) = 0 ,$$

and

$$e_\mu(4k) = i n_\mu .$$

Thus, employing the three-dimensional form (22), we have

$$\delta_{\mu\nu} D_+(x-x') = \tfrac{1}{2} i \sum_{\lambda=1}^{4} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{1}{k_0}\, e_\mu(\lambda k)\, e^{\pm ikx}\, e_\nu(\lambda k)\, e^{\pm ikx'} \tag{23}$$

For applications referring to parallel surfaces, there is a useful alternative form of W_0, which corresponds to the construction of $\langle A_\mu \rangle$ in the radiation gauge common to both surfaces,

$$\langle A_0(x) \rangle_{r.g.} = \int (dx')\, \mathcal{D}(x-x')\, J_0(x') ,$$

$$\langle A_k(x) \rangle_{r.g.} = \int (dx')\, D_+(x-x')\, J_k^{(T)}(x') ,$$

where

$$\mathcal{D}(x-x') = \delta(x_0 - x_0')\, \mathcal{D}(\mathbf{x} - \mathbf{x}') ,$$

and

$$\mathcal{D}(\mathbf{x} - \mathbf{x}') = (1/4\pi)\, |\mathbf{x} - \mathbf{x}'|^{-1}$$

Thus,

$$W_0 = \tfrac{1}{2} \int (dx)(dx') \left[J_k^{(T)}(x)\, D_+(x-x')\, J_k^{(T)}(x') - J_0(x)\, \mathcal{D}(x-x')\, J_0(x') \right] \tag{24}$$

The direct proof of equivalence with (17) employs the expression for the longitudinal current,

$$J_k^{(L)}(x) = \partial_k \int (dx') \mathcal{D}(x-x') \partial_0' J_0(x'),$$

and the identity

$$D_+(x-x') = \mathcal{D}(x-x') + \partial_0 \partial_0' \int (dx'') D(x-x'') \mathcal{D}(x''-x'). \tag{25}$$

The latter, incidentally, can be expressed in the symbolic form

$$\frac{1}{2} i \omega^{-1} e^{-i\omega|x_0-x_0'|} \delta(\underset{\sim}{x}-\underset{\sim}{x}') = \omega^{-2} \delta(x_0-x_0') \delta(\underset{\sim}{x}-\underset{\sim}{x}') + \partial_0 \partial_0' \left[\frac{1}{2} i \omega^{-3} e^{-i\omega|x_0-x_0'|} \delta(x-x')\right] \tag{26}$$

Zero Current

We shall use the appropriate form of the transformation function, (21), to illustrate the construction of the eigenvalues and eigenfunctions of P_μ for a complete system. It is supposed that the surface σ_1 is obtained from σ_2 by a translation X, which brings the point X_2 of σ_2 into the point X_1. Since the surfaces are parallel, and the eigenvalues refer to transverse fields, one can write (21) as

$$(F^{(-)'}_{\sigma_1} | F^{(+)'}_{\sigma_2}) = \exp\left[-4i \int_{\sigma_1} d\sigma \int_{\sigma_2} d\sigma' F^{(-)'}_{0m}(x) \left(\delta_{mn} D_+(x-x')\right) F^{(+)'}_{0n}(x')\right].$$

With the time-like vector n_μ identified with the common normal to both surfaces, we see that the $e_\mu(\lambda k)$, $\lambda = 1, 2, 3,$ are pure space vectors, while the fourth vector possesses only a time component. Furthermore, the first two vectors are orthogonal to $\underset{\sim}{k}$, which the third vector parallels. Hence,

$x_0 > x_0'$: $\left(\delta_{mn} D_+(x-x')\right)^{(T)}$

$$= \frac{1}{2} i \sum_{\lambda=1,2} \sum_{k} \left(\frac{(d\underset{\sim}{k})}{(2\pi)^3} \frac{1}{k_0}\right)^{1/2} e_m(\lambda k) e^{ikx} \left(\frac{(d\underset{\sim}{k})}{(2\pi)^3} \frac{1}{k_0}\right)^{1/2} e_n(\lambda k) e^{-ikx'},$$

where we have also replaced the integration with respect to $\underset{\sim}{k}$ by a summation over cells of volume $(d\underset{\sim}{k})$.

On defining

$$a^{(-)}_{\lambda k}(\sigma_1) = -i \left(\frac{(d\underset{\sim}{k})}{(2\pi)^3} \frac{2}{k_0}\right)^{1/2} \int_{\sigma_1} d\sigma F^{(-)}_{0m}(x) e_m(\lambda k) e^{ik(x-X_1)}, \tag{27}$$

and

$$a^{(+)}_{\lambda k}(\sigma_2) = i \left(\frac{(d\underset{\sim}{k})}{(2\pi)^3} \frac{2}{k_0}\right)^{1/2} \int_{\sigma_2} d\sigma e^{-ik(x-X_2)} e_m(\lambda k) F^{(+)}_{0m}(x), \tag{28}$$

which are correspondingly constructed linear combinations of the $F_{om}^{(-)}$ on σ_1, and of the $F_{om}^{(+)}$ on σ_2, we obtain

$$(F^{(-)'}\sigma_1 | F^{(+)'}\sigma_2) = \exp\left[\sum_{\lambda k} e^{ikX} a_{\lambda k}^{(-)'} a_{\lambda k}^{(+)'}\right]$$

$$= \prod_{\lambda k} \exp\left[e^{ikX} a_{\lambda k}^{(-)'} a_{\lambda k}^{(+)'}\right]$$

$$= \prod_{\lambda k} \sum_{n_{\lambda k}=0}^{\infty} e^{inkX} \frac{(a^{(-)'})^n}{(n!)^{1/2}} \frac{(a^{(+)'})^n}{(n!)^{1/2}},$$

or

$$(F^{(-)'}\sigma_1 / F^{(+)'}\sigma_2) = \sum_n \left[\prod_{\lambda k} \frac{(a^{(-)'})^n}{(n!)^{1/2}}\right] \exp\left[i\left(\sum_{\lambda k} nk_\mu\right) X_\mu\right]\left[\prod_{\lambda k} \frac{(a^{(+)'})^n}{(n!)^{1/2}}\right]$$

A comparison with (1) shows that

$$P_\mu' = P_\mu(n) = \sum_{\lambda k} n_{\lambda k} k_\mu, \qquad n_{\lambda k} = 0, 1, 2, \ldots, \tag{29}$$

where, in particular,

$$P_0' = \sum_{\lambda k} n_{\lambda k} k_0 \geq 0,$$

and that

$$(F^{(-)'}/n) = \prod_{\lambda k} \frac{(a^{(-)'})^n}{(n!)^{1/2}},$$

$$(n/F^{(+)'}) = \prod_{\lambda k} \frac{(a^{(+)'})^n}{(n!)^{1/2}}$$

The occupation numbers $n_{\lambda k}$ provide the complete set of constants of the motion.

Note that if the eigenvalues at corresponding points are in the relation

$$F_{om}^{(-)'} = F_{om}^{(+)'*}$$

we have

$$a_{\lambda k}^{(-)'} = a_{\lambda k}^{(+)'*}$$

and therefore

$$(F^{(-)'}/n) = (n/F^{(+)'})^*$$

as required by (5). With the knowledge of these simple eigenfunctions, one can construct eigenfunctions for any other representation of interest. We can also present our results without reference to a representation. On remarking that the vacuum state eigenfunction is

$$(F^{(+)\prime}\sigma|0) = 1,$$

we can write

$$(F^{(+)\prime}\sigma|n\sigma) = \prod_{\lambda k}\frac{(a^{(+)\prime})^{n}}{(n!)^{1/2}}(F^{(+)\prime}\sigma|0) = \left(F^{(+)\prime}\sigma\left|\prod_{\lambda k}\frac{(a^{(+)}_{\lambda k}(\sigma))^{n}}{(n!)^{1/2}}\right|0\right)$$

Therefore,

$$\Psi(n\sigma) = \prod_{\lambda k}\frac{(a^{(+)}(\sigma))^{n}}{(n!)^{1/2}}\Psi_0 ,$$

and

$$\Psi(n\sigma)^{\dagger} = \Psi_0^{\dagger}\prod_{\lambda k}\frac{(a^{(-)}(\sigma))^{n}}{(n!)^{1/2}},$$

are the eigenvectors of the state with photon occupation numbers, $n_{\lambda k}$.

Time Independent Currant

In this situation,

$$J_\mu(x) = J_\mu(\underset{\sim}{x}),$$

the energy operator P_0 is still a constant of the motion, and its eigenvalues and eigenfunctions are obtained from the transformation function that characterizes the time translation

$$T = t_1 - t_2$$

where t_1 and t_2 are the time coordinates that label σ_1 and σ_2 .

On employing the form (24) for W_0 we get

$$W_0 = -E(0)T + \tfrac{1}{2}\int(d\underset{\sim}{x})(d\underset{\sim}{x}')J_k(\underset{\sim}{x})J_k(\underset{\sim}{x}')\int_{t_2}^{t_1}dx_0\,dx_0'\left[D_+(x-x') - \mathcal{D}(x-x')\right],$$

where

$$E(0) = -\tfrac{1}{2}\int(d\underset{\sim}{x})(d\underset{\sim}{x}')J_\mu(\underset{\sim}{x})\,\mathcal{D}(x-x')\,J_\mu(\underset{\sim}{x}') .$$

According to the symbolic form (26)

$$\int_{t_2}^{t_1}(dx_0)(dx_0')\left[D_+(x-x') - \mathcal{D}(x-x')\right] = \int_{t_2}^{t_1}dx_0\,dx_0'\,\partial_0\partial_0'\left[\tfrac{1}{2}i\omega^{-3}\,e^{-i\omega|x_0-x_0'|}\right]\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$= i\omega^{-3}(1-e^{-i\omega T})\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

so that,

$$W_0 = -E(0)T + \tfrac{1}{2}i\int(d\underset{\sim}{x})\,J_k\,\omega^{-3}(1-e^{-i\omega T})\,J_k .$$

Furthermore

$$x_0 = t_1: \quad \langle A_k(x)\rangle = \int_{t_2}^{t_1} dx_0' \, \tfrac{1}{2} i \, \omega^{-1} e^{+i\omega(x_0' - t_1)} J_k(\mathbf{x})$$

$$= \tfrac{1}{2}\omega^{-2}(1 - e^{-i\omega T}) J_k(\mathbf{x}) .$$

and

$$x_0 = t_2: \quad \langle A_k(x)\rangle = \int_{t_2}^{t_1} dx_0' \, \tfrac{1}{2} i \, \omega^{-1} e^{-i\omega(x_0' - t_2)} J_k(\mathbf{x})$$

$$= \tfrac{1}{2}\omega^{-2}(1 - e^{-i\omega T}) J_k(\mathbf{x}) .$$

The transformation function (20) is thus obtained as

$$\frac{(F^{(-)'}\sigma_1 | F^{(+)'}\sigma_2)}{(F^{(-)'} | F^{(+)'})} = \exp\left[-i E(0) T - 2\int (d\mathbf{x}) (F_{0k}^{(-)'} + \tfrac{1}{2} i \omega^{-1} J_k) \frac{1 - e^{-i\omega T}}{\omega} (F_{0k}^{(+)'} - \tfrac{1}{2} i \omega^{-1} J_k)\right] \tag{30}$$

in which we have divided by the transformation function referring to a common surface,

$$(F^{(-)'} | F^{(+)'}) = \exp\left[-4i \int_\sigma d\sigma \, d\sigma' \, F_{0k}^{(-)'}(x) D_+(x - x') F_{0k}^{(+)'}(x')\right]$$

$$= \exp\left[2 \int d\sigma \, F_{0k}^{(-)'} \omega^{-1} F_{0k}^{(+)'}\right] .$$

It is evidently desirable to employ a new description, characterized by the eigenvalues $\bar{F}_{0k}^{(\mp)'}$, where

$$\bar{F}_{0k}^{(\mp)} = F_{0k}^{(\pm)} \mp \tfrac{1}{2} i \omega^{-1} J_k$$

and

$$\bar{A}_k^{(\mp)} = A_k^{(\mp)} - \tfrac{1}{2}\omega^{-2} J_k \tag{31}$$

The relation between the eigenvectors $\Psi(\bar{F}^{(+)'}\sigma)$ and $\Psi(F^{(+)'}\sigma)$ can be inferred from the generator

$$G_{\bar{F}(+)} = \int d\sigma \; 2 \bar{A}_k^{(-)} \delta \bar{F}_{0k}^{(+)} = \int d\sigma \, (2 A_k^{(-)} - \omega^{-2} J_k) \, \delta F_{0k}^{(+)}$$

$$= G_{F(+)} - \delta\left[\int d\sigma \, J_k \, \omega^{-2} \, \tfrac{1}{2} (F_{0k}^{(+)} + \bar{F}_{0k}^{(+)})\right]$$

where we have maintained the symmetry between $F_{0k}^{(+)}$ and $\bar{F}_{0k}^{(+)}$ that accompanies the substitution $J_k \to -J_k$. Thus,

$$\Psi(\bar{F}^{(+)'}\sigma) = \exp\left[i \int d\sigma \, J_k \, \omega^{-2} \, \tfrac{1}{2} (F_{0k}^{(+)'} + \bar{F}_{0k}^{(+)'})\right] \Psi(F^{(+)'}\sigma)$$

$$= \exp\left[i \int d\sigma \, J_k \, \omega^{-2} F_{0k}^{(+)'} + \tfrac{1}{4}\int d\sigma \, J_k \omega^{-3} J_k\right] \Psi(F^{(+)'}\sigma)$$

and

$$\Phi(\bar{F}^{(-)\prime}_{\sigma}) = \exp\left[-i\int d\sigma\, J_k\, \omega^{-2} F_{0k}^{(-)\prime} + \frac{1}{4}\int d\sigma\, J_k\, \omega^{-3} J_k\right] \Phi(F^{(-)\prime}_{\sigma}) .$$

In further confirmation, observe that

$$(F^{(-)\prime}|F^{(+)\prime}) = \exp\left[2\int d\sigma\, \bar{F}_{0k}^{(-)\prime}\, \omega^{-1}\, \bar{F}_{0k}^{(+)\prime}\right]$$

$$= \exp\left[-i\int d\sigma\, J_k\, \omega^{-2} F_{0k}^{(-)\prime} + \frac{1}{4}\int d\sigma\, J_k\, \omega^{-3} J_k\right](F^{(-)\prime}|F^{(+)\prime}) \cdot$$

$$\cdot \exp\left[i\int d\sigma\, J_k\, \omega^{-2} F_{0k}^{(+)\prime} + \frac{1}{4}\int d\sigma\, J_k\, \omega^{-3} J_k\right],$$

which results in the same eigenvector transformation properties. Since these conversion factors do not refer explicitly to the surface, the transformation function ratio in (30) preserves its structure on introducing the new representations. Therefore,

$$(\bar{F}^{(-)\prime}_{\sigma_1}|\bar{F}^{(+)\prime}_{\sigma_2}) = \exp(i\bar{W}) \tag{32}$$

where

$$\bar{W} = -E(0)T - 2i\int (d\underset{\sim}{x})\, \bar{F}_{0k}^{(-)\prime}\, \omega^{-1} e^{-i\omega T}\, \bar{F}_{0k}^{(+)\prime}$$

$$= -E(0)T - 4\int_{\sigma_1} d\sigma \int_{\sigma_2} d\sigma'\, \bar{F}_{0k}^{(-)\prime}(x)\, D_+(x-x')\, \bar{F}_{0k}^{(+)\prime}(x') .$$

Apart from the factor $\exp(-iE(0)T)$, this transformation function is identical in form with (21). The expanded version of (32) is, therefore,

$$(\bar{F}^{(-)\prime}_{\sigma_1}|\bar{F}^{(+)\prime}_{\sigma_2}) = \sum_{n} (\bar{F}^{(-)\prime}|n) \exp(-iP_0' T)(n|\bar{F}^{(+)\prime}),$$

where

$$P_0' = E(0) + \sum_{\lambda k} n_{\lambda k}\, k_0 , \qquad n_{\lambda k} = 0, 1, 2, \ldots ,$$

and

$$(n|\bar{F}^{(+)\prime}) = \prod_{\lambda k} \frac{(\bar{a}^{(+)\prime})^n}{(n!)^{1/2}} ,$$

$$(\bar{F}^{(-)\prime}|n) = \prod_{\lambda k} \frac{(\bar{a}^{(-)\prime})^n}{(n!)^{1/2}} .$$

We see that $E(0)$ is to be interpreted as the energy of the photon vacuum, the state of minimum energy, $n_{\lambda k} = 0$. With respect to this displaced origin, the energy eigenvalues are the same as in the absence of a current. One may say that the field $\bar{F}_{0k}$ describes pure radiation, which is without coupling to the external current. Indeed, (31), written as

$$\bar{A}_k^{(\pm)}(x) = A_k^{(\pm)}(x) - \frac{1}{2}\int (dx')\, \mathcal{D}(x-x') J_k(x') ,$$

represents the removal from $A_k(x)$ of the time-independent potential produced by the static current, which is allocated equally to $A_k^{(+)}$ and to $A_k^{(-)}$. In view of this uncoupling of the static field and the radiation field, one can assign momentum as well as energy eigenvalues to the radiation quanta, as given by (29).

Time Dependent Currents

We shall now discuss the class of problems in which the physical information contained in the general transformation function (20) refers to transition probabilities rather than eigenvalues. Let us first suppose that the current is zero on σ_2, varies in an arbitrary manner in the region between the parallel surfaces σ_1 and σ_2, but again reduces to zero on σ_1. Thus, the physical states on σ_1 and σ_2 are those of the isolated electromagnetic field, and we wish to compute the probabilities of the transitions induced by this perturbing current.

Now,

$$2\int_{\sigma_1} d\sigma\, F_{0m}^{(-)\prime}(x) \langle A_m(x)\rangle = 2\int_{\sigma_1} d\sigma\, F_{0m}^{(-)\prime}(x) \int_{\sigma_2}^{\sigma_1} (dx') \left(\delta_{mn} D_+(x-x')\right)^{(T)} J_n(x')$$

$$= \sum_{\lambda k} a_{\lambda k}^{(-)\prime} e^{ikx_1} J_{\lambda k} ,$$

and

$$2\int_{\sigma_2} d\sigma\, F_{0m}^{(+)\prime}(x) \langle A_m(x)\rangle = 2\int_{\sigma_2} d\sigma\, F_{0m}^{(+)\prime}(x) \int_{\sigma_2}^{\sigma_1} (dx') \left(\delta_{mn} D_+(x-x')\right)^{(T)} J_n(x')$$

$$= \sum_{\lambda k} a_{\lambda k}^{(+)\prime} e^{-ikx_2} J_{\lambda k}^{*} ,$$

where

$$J_{\lambda k} = \left(\frac{(d\mathbf{k})}{(2\pi)^3} \frac{1}{2k_0}\right)^{1/2} \int_{-\infty}^{\infty} (dx)\, e_m(\lambda k)\, e^{-ikx} J_m(x) . \tag{33}$$

The quantity $\mathcal{W}$ determining the transformation function $(F^{(-)'}_{\sigma_1}/F^{(+)'}_{\sigma_2})$, is thus obtained in the form

$$\mathcal{W} = \mathcal{W}_0 - \sum_{\lambda k} \left[a^{(-)'}_{\lambda k} e^{i k x_1} J_{\lambda k} + a^{(+)'}_{\lambda k} e^{-i k x_2} J^*_{\lambda k} + i a^{(-)'}_{\lambda k} e^{i k x_1} a^{(+)'}_{\lambda k} e^{-i k x_2} \right].$$

The transformation function then serves, according to

$$(F^{(-)'}_{\sigma_1}/F^{(+)'}_{\sigma_2}) = \sum_{n,n'} (F^{(-)'}/n)(n\sigma_1/n'\sigma_2)(n'/F^{(+)'}),$$

as a generating function for $(n\sigma_1/n'\sigma_2)$, from which the transition probabilities are found in the manner of (2).

It is somewhat more convenient to deal with the elements of the matrix $(n/S/n') = e^{-iP(n)x_1}(n\sigma_1/n'\sigma_2)e^{iP(n')x_2}$,

$$p(n,n') = |(n/S/n')|^2$$

since they are independent of σ_1 and σ_2, provided the current vanishes on these surfaces. The following substitution, representing a transformation to a common reference surface,

$$a^{(-)'}_{\lambda k} e^{i k x_1} \rightarrow a^{(-)'}_{\lambda k}, \qquad a^{(+)'}_{\lambda k} e^{-i k x_2} \rightarrow a^{(+)'}_{\lambda k}$$

yields the generating function

$$\exp(i\mathcal{W}_0) \prod_{\lambda k} \exp\left[a^{(-)'}a^{(+)'} - i a^{(-)'} J - i a^{(+)'} J^*\right] \tag{34}$$

$$= \sum_{n,n'} (F^{(-)'}/n)(n/S/n')(n'/F^{(+)'}).$$

On picking out the coefficient of a particular $(F^{(-)'}/n)$, we obtain the partial generating function

$$\exp(i\mathcal{W}_0) \prod_{\lambda k} \left[\frac{(a^{(+)'} - iJ)^n}{(n!)^{1/2}} \exp(-i a^{(+)'} J^*) \right] \tag{35}$$

$$= \sum_{n'} (n/S/n')(n'/F^{(+)'}),$$

and, similarly,

$$\exp(i W_0) \prod_{\lambda k} \left[\frac{(a^{(-)\prime} - i J^*)^{n'}}{(n'!)^{1/2}} \exp(-i a^{(-)\prime} J) \right] = \sum_n (F^{(-)\prime} | n)(n|S|n'). \tag{36}$$

Preliminary to a direct verification of the unitary property of the operator S, we evaluate the imaginary part of W_0. According to (24),

$$2 \operatorname{Im} W_0 = \int (dx)(dx') J_m(x) \operatorname{Im} (\delta_{mn} D_+(x-x')^{(T)} J_n(x'),$$

where, referring to (23),

$$\operatorname{Im} (\delta_{mn} D_+(x-x'))^{(T)}$$

$$= \operatorname{Re} \sum_{\lambda=1,2} \int \frac{(d\underline{k})}{(2\pi)^3} \frac{1}{2k_0} e_m(\lambda k) e^{ikx} e_n(\lambda k) e^{-ikx'}.$$

This form is valid without restriction on $x_0 - x_0'$. Hence,

$$2 \operatorname{Im} W_0 = \sum_{\lambda k} |J_{\lambda k}|^2.$$

Alternatively, the invariant expression (17) yields

$$2 \operatorname{Im} W_0 = \int (dx)(dx') J_\mu(x) \operatorname{Im} D_+(x-x') J_\mu(x'),$$

with

$$\operatorname{Im} D_+(x-x') = \operatorname{Re} \int \frac{(d\underline{k})}{(2\pi)^3} \frac{1}{2k_0} e^{ikx} e^{-ikx'},$$

which can be written

$$2 \operatorname{Im} W_0 = \sum_k I_k.$$

Here

$$I_k = \frac{(d\underline{k})}{(2\pi)^3} \frac{1}{2k_0} \left| \int (dx) e^{-ikx} J_\mu(x) \right|^2 = \sum_{\lambda=1,2} |J_{\lambda k}|^2$$

in which it must be understood that the complex conjugation does not extend to $J_4 = i J_0$. The necessary equivalence of the two evaluations indicates the complete cancellation of the integrals associated with $\lambda = 3$, and 4.

Let us multiply (35) with the complex conjugate equation,

$$\exp(-i W_0^*) \prod_{\lambda k} \left[\frac{(a^{(-)\prime} + i J^*)^n}{(n!)^{1/2}} \exp(i a^{(-)\prime} J) \right]$$

$$= \sum_{n'} (F^{(-)\prime} | n')(n'|S^\dagger | n), \tag{37}$$

and perform the summation with respect to n,

$$(F^{(-)'}|S^{\dagger}S|F^{(+)'})$$

$$= \prod_{\lambda k} \exp\left[-|J|^2 + (a^{(-)} + i J^*)(a^{(+)} - iJ) + i a^{(-)} J - i a^{(+)} J^*\right]$$

$$= \prod_{\lambda k} \exp(a^{(-)'} a^{(+)'}) = (F^{(-)'}|F^{(+)'}),$$

whence

$$S^{\dagger}S = 1$$

A symmetry property of S may be noted here. The invariance of the generating function (34), under the substitution

$$a^{(+)'}_{\lambda k} \to a^{(-)'}_{\lambda k}(J_{\lambda k}/J^*_{\lambda k}), \quad a^{(-)'}_{\lambda k} \to a^{(+)'}_{\lambda k}(J^*_{\lambda k}/J_{\lambda k}),$$

shows that

$$(n|S|n') = (n'|S|n) \prod_{\lambda k} (J/J^*)^{n-n'},$$

which has the consequence

$$p(n, n') = p(n', n)$$

As an elementary application of the generating function, we place $n'_{\lambda k} = 0$ in (36), which yields

$$(n|S|0) = \exp(i W_0) \prod_{\lambda k} \frac{(-iJ)^n}{(n!)^{1/2}},$$

and

$$p(n, 0) = \prod_{\lambda k} \left[\frac{(|J|^2)^n}{n!} \exp(-|J|^2)\right] \tag{38}$$

for the situation in which no quanta are present initially. If we are not concerned with the polarization of the emitted quanta, we can employ the binomial theorem to replace (38) with

$$p(n, 0) = \prod_{k} \left[\frac{(I_k)^n}{n!} \exp(-I_k)\right].$$

The general matrix element of S is obtained as

$$(n|S|n') = \exp(iW_0) \prod_{\lambda k} \left[\frac{i^{n+n'}}{(n!\,n'!)^{1/2}} J^n J^{*\,n'} f_{n,n'}(|J|^2) \right]$$

where the function $f_{n,n'}(x)$, which is symmetrical in n and n', is given by

$$f_{n,n'}(x) = x^{-n'} e^{x} (d/dx)^{n} (-x)^{n'} e^{-x} = x^{-n} e^{x} (d/dx)^{n'} (-x)^{n} e^{-x}$$
$$= (-1)^{n_>} n_<!\, x^{-n_<} L_{n_<}^{(n_> - n_<)}(x) .$$

In the indicated relation to the Laguerre polynomials[5], $n_>$ and $n_<$ repre-

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[5] We employ the definition of W. Magnus and F. Oberhettinger, Special Functions of Mathematical Physics (Chelsea Publishing Company, New York, 1949) p. 84)

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sent the greater and lesser of the integers n and n'. The general transition probability is thus obtained as

$$p(n,n') = \prod_{\lambda k} \left[\frac{n_<!}{n_>!} \left(|J|^2\right)^{(n_> - n_<)} \left(L_{n_<}^{(n_> - n_<)}(|J|^2) \right)^2 \cdot \exp(-|J|^2) \right]. \qquad (39)$$

In particular, the probability that there be no change in the numbers of quanta is

$$p(n,n) = \prod_{\lambda k} \left[\left(L_n^{(0)}(|J|^2) \right)^2 \exp(-|J|^2) \right]$$

Should the quantum numbers n and n' be large in comparison with unity and $\Delta n = n - n' \ll n, n'$, for a particular mode of the radiation field, we can replace the factor in the transition probability referring to that made with the Bessel function asymptotic form

$$\left[J_{\Delta n}(2 n^{1/2} |J|) \right]^2 .$$

One can devise another generating function for the transition probabilities which has the further advantage of yielding the expectation values of powers of the final occupation numbers. We first perform the substitution

$$a^{(+)'}_{\lambda k} \rightarrow a^{(+)'}_{\lambda k} e^{-i\gamma_{\lambda k}}$$

in (35), where the $\gamma_{\lambda k}$ are arbitrary constants. The result,

$$\exp(iW_0) \prod_{\lambda k} \left[\frac{(a^{(+)'} - i e^{i\gamma} J)^n}{(n!)^{1/2}} \exp(-i a^{(+)'} e^{-i\gamma} J^*) \right]$$

$$= \sum_{n'} \prod_{\lambda k} \left[\exp(i\gamma(n-n')) \right] (n|S|n')(n'|F^{(+)'}),$$

is then multiplied by (37) and the summation with respect to n performed. This gives

$$\sum_{n,n',n''} (F^{(-)'}|n')(n'|S^\dagger|n) \prod_{\lambda k} \left[\exp(i\gamma(n-n'')) \right] (n|S|n'')(n''|F^{(+)'})$$

$$= \prod_{\lambda k} \exp\left[a^{(-)'} a^{(+)'} - i a^{(-)'}(e^{i\gamma}-1)J - i a^{(+)'}(e^{-i\gamma}-1)J^* + (e^{i\gamma}-1)|J|^2 \right].$$

which exhibits the same structure as (34). On confining our attention to the diagonal matrix elements, $n'_{\lambda k} = n''_{\lambda k}$, we find

$$\sum_n \left[\prod_{\lambda k} \exp(i\gamma(n-n')) \right] p(n,n') = \left\langle \prod_{\lambda k} \exp(i\gamma(n-n')) \right\rangle_{n'}$$

$$= \prod_{\lambda k} \left[L^{(0)}_{n'}((e^{i\gamma}-1)(e^{-i\gamma}-1)|J|^2) \exp((e^{i\gamma}-1)|J|^2) \right]. \qquad (40)$$

The right side thus serves as a generating function for the transition probabilities if developed in positive and negative powers of the $e^{i\gamma_{\lambda k}}$. The expansion in power of the $\gamma_{\lambda k}$ exhibits it as the generator of expectation values of all powers of the quantities $n_{\lambda k} - n'_{\lambda k}$.

The alternative presentation of this result,

$$\left\langle \prod_{\lambda k} (1 + x_{\lambda k})^{n-n'} \right\rangle_{n'} = \prod_{\lambda k} \left[L^{(0)}_{n'}\left(-\frac{x^2}{1+x}|J|^2\right) \exp(x|J|^2) \right],$$

supplies the expectation values of products successively decreasing by unity. Thus, in the special example referring to the vacuum as the initial state,

where

$$\left\langle \prod_{\lambda k} (1+x)^{n} \right\rangle_0 = \prod_{\lambda k} \exp(x|J|^2) ,$$

we find, for a particular mode,

$$\left\langle \frac{n!}{(n-k)!} \right\rangle_0 = (|J|^2)^k ,$$

which is characteristic of the Poisson distribution. The first two expectation values, derived from the general generating function are

$$\langle (n_{\lambda k} - n'_{\lambda k}) \rangle_{n'} = |J_{\lambda k}|^2 \tag{41}$$

and

$$\langle (n_{\lambda k} - n'_{\lambda k})^2 \rangle_{n'} = \langle (n_{\lambda k} - n'_{\lambda k}) \rangle^2_{n'} + (2n'_{\lambda k}+1)|J_{\lambda k}|^2 .$$

We need hardly remark on the statistical independence of different modes.

If we are not interested in the polarizations of the emitted quanta, it suffices to identify the parameters distinguishing the different polarizations,

$$\gamma_{\lambda k} = \gamma_k .$$

To obtain statements referring also to unpolarized incident quanta, we must average, with equal weight, over the various polarized photon numbers that are consistent with a given number of photons in a certain propagation mode,

$$n'_{1k} + n'_{2k} = n'_k$$

This can be accomplished with the aid of the addition theorem for the Laguerre polynomials. The resulting generating function, without reference to polarization, is

$$\sum_n \left[\prod_k \exp(i\gamma(n-n')) \right] p(n,n') = \left\langle \prod_k \exp(i\gamma(n-n')) \right\rangle_{n'}$$

$$= \prod_k \left[(n'+1)^{-1} L^{(1)}_{n'} \left((e^{i\gamma}-1)(e^{-i\gamma}-1) I \right) \exp\left((e^{i\gamma}-1) I \right) \right] .$$

Some expectation values are

$$\langle (n_k - n'_k) \rangle_{n'} = I_k$$

and

$$\langle (n_k - n_k')^2 \rangle_{n'} = \langle (n_k - n_k') \rangle_{n'}^2 + (n_k' + 1) I_k$$

For the second example that is concerned with the evaluation of transition probabilities, we suppose that the current is time-independent in the vicinity of σ_2, varies in an arbitrary manner in the region between the parallel surfaces σ_1 and σ_2, but again becomes time-independent in the neighborhood of σ_1. These limiting forms, $J_\mu(x,1)$ and $J_\mu(x,2)$, need not be the same.

On each surface, we use the description appropriate to the current on that surface,

$$(\bar{F}^{(-)\prime}\sigma_1 | \bar{F}^{(+)\prime}\sigma_2) = \exp\left[-i\int d\sigma\, J_k(1)\, \omega^{-2} F_{0k}^{(-)\prime} + \tfrac{1}{4}\int d\sigma\, J_k(1)\omega^{-3} J_k(1)\right]\cdot \tag{42}$$

$$\cdot (F^{(-)\prime}\sigma_1 | F^{(+)\prime}\sigma_2) \exp\left[i\int d\sigma\, J_k(2)\, \omega^{-2} F_{0k}^{(+)\prime} + \tfrac{1}{4}\int d\sigma\, J_k(2)\,\omega^{-3} J_k(2)\right],$$

where

$$\bar{F}_{0k}^{(-)} = F_{0k}^{(-)} + \tfrac{1}{2} i\,\omega^{-1} J_k(1),$$

$$\bar{F}_{0k}^{(+)} = F_{0k}^{(+)} - \tfrac{1}{2} i\,\omega^{-1} J_k(2).$$

Were the current constant, this transformation function would possess the form (32). Accordingly, it must be possible to express all additional contributions in terms of the time derivative of the current. The manner in which this occurs can be illustrated with the evaluation of

$$\int_{\sigma_1} d\sigma\, F_{0k}^{(-)\prime}(x) \langle A_k(x) \rangle = -\int_{\sigma_1} d\sigma\, F_{0k}^{(-)\prime}(x) \int_{t_2}^{t_1} dx_0' \int (dx') \partial_0' D_+(x-x')\, i\,\omega^{-1} J_k(x')$$

$$= \int d\sigma\, F_{0k}^{(-)\prime} \tfrac{1}{2}\omega^{-2} J_k(1) + \int_{\sigma_1} d\sigma \int_{\sigma_2} d\sigma'\, F_{0k}^{(-)\prime}(x) D_+(x-x')\, i\,\omega^{-1} J_k(x',2)$$

$$+ \int_{\sigma_1} d\sigma\, F_{0k}^{(-)\prime}(x) \int_{\sigma_2}^{\sigma_1} (dx') D_+(x-x')\, i\,\omega^{-1} \partial_0' J_k(x').$$

The terms containing $J_k(1)$ and $J_k(2)$ are cancelled on expressing (42) as a function of the variables $\bar{F}_{0k}^{(-)\prime}$ and $\bar{F}_{0k}^{(+)\prime}$.

Carrying out a similar reduction of W_0 with the aid of (25), we obtain

$$(\bar{F}^{(-)\prime}\sigma_1|\bar{F}^{(+)\prime}\sigma_2) = \exp\Big[i\bar{W}_0 - 4i\int_{\sigma_1} d\sigma \int_{\sigma_2} d\sigma' \,\bar{F}^{(-)\prime}_{0k}(x)\, D_+(x-x')\, \bar{F}^{(+)\prime}_{0k}(x')$$

$$+ 2i\int_{\sigma_1} d\sigma\, \bar{F}^{(-)\prime}_{0k}(x)\int_{\sigma_2}^{\sigma_1}(dx')\, D_+(x-x')\, i\,\omega^{-1}\partial_0' J_k(x')$$

$$+2i\int_{\sigma_2} d\sigma\, \bar{F}^{(+)\prime}_{0k}(x)\int_{\sigma_2}^{\sigma_1}(dx')\, D_+(x-x')\, i\,\omega^{-1}\partial_0' J_k(x')\Big],$$

where

$$\bar{W}_0 = \frac{1}{2}\int_{\sigma_2}^{\sigma_1}(dx)(dx')\left[J_k^{(T)}(x)\, J_k^{(T)}(x') - J_0(x)\, J_0(x')\right]\mathcal{D}(x-x')$$

$$+\frac{1}{2}\int_{\sigma_2}^{\sigma_1}(dx)(dx')\,\partial_0 J_k^{(T)}(x)\,\omega^{-1}\, D_+(x-x')\,\omega^{-1}\partial_0'\, J_k^{(T)}(x').$$

The introduction of the variables $\bar{a}^{(-)}_{\lambda k}$ and $\bar{a}^{(+)}_{\lambda k}$, in the manner of (27) and (28), brings this transformation function into the form

$$(\bar{F}^{(-)\prime}\sigma_1|\bar{F}^{(+)\prime}\sigma_2) = \exp(i\bar{W}_0)\prod_{\lambda k}\exp\Big[\bar{a}^{(-)\prime}e^{ikx_1}\,\bar{a}^{(+)\prime}e^{-ikx_2}$$

$$- i\bar{a}^{(-)\prime}e^{ikx_1}\,\bar{J} - i\bar{a}^{(+)\prime}e^{-ikx_2}\,\bar{J}^*\Big], \tag{43}$$

where

$$\bar{J}_{\lambda k} = \left(\frac{(dk)}{(2\pi)^3}\,\frac{1}{2k_0}\right)^{1/2}\int_{-\infty}^{\infty}(dx)\, e_m(\lambda k)\, e^{-ikx}\,(i/k_0)\,\partial_0 J_m(x) \tag{44}$$

is expressed as an integral over all space-time by supposing that the current in the extended domain exhibits the time-independent value appropriate to the nearest surface bounding the region of interest. In order to present this result as a generating function for the surface independent unitary matrix

$$(n|S|n') = e^{-iP(n,1)x_1}\,(n\sigma_1|n'\sigma_2)\,e^{iP(n',2)x_2},$$

$$P(n) = E(0) + \sum_{\lambda k} n_{\lambda k}\, k_0\,,$$

a further rearrangement of $\bar{W}_0$ is required. Indeed, the first term of this quantity is

$$-\int_{t_2}^{t_1} dx_0\, E(0,x_0) = -\left[t_1 E(0,1) - t_2 E(0,2)\right] + \int_{-\infty}^{\infty} dx_0\, x_0\,\partial_0 E(0,x_0).$$

The substitution

$$\bar{a}^{(-)\prime}_{\lambda k}e^{ikx_1} \to \bar{a}^{(-)\prime}_{\lambda k}, \quad \bar{a}^{(+)\prime}_{\lambda k}e^{-ikx_2} \to \bar{a}^{(+)\prime}_{\lambda k},$$

now yields the required generating function

$$\exp(i\,w_0)\prod_{\lambda k}\left[\bar{a}^{(-)\prime}\bar{a}^{(+)\prime} - i\,\bar{a}^{(-)\prime}\bar{J} - i\,\bar{a}^{(+)\prime}\bar{J}^*\right]$$

$$= \sum_{n,n'}(\bar{F}^{(-)\prime}|n)(n|S|n')(n'|\bar{F}^{(+)\prime}), \tag{45}$$

where

$$W_0 = \int_{-\infty}^{\infty} dx_0\, x_0\, \partial_0 E(0,x) + \frac{1}{2}\int_{-\infty}^{\infty}(dx)(dx')\partial_0 J_k^{(T)}(x)\, \omega^{-1} D_+(x-x')\,\omega^{-1}\partial_0' J_k^{(T)}(x') \tag{46}$$

is such that

$$2\,\mathrm{Im}\, W_0 = \sum_{\lambda k} |\bar{J}_{\lambda k}|^2 .$$

The generating function (45) is identical in structure with (34). Hence the transition probabilities and expectation values are given by (39) and (40), with $\bar{J}_{\lambda k}$ replacing $J_{\lambda k}$. If the current is zero on the boundaries of the region, an integration by parts reduces $\bar{J}_{\lambda k}$ to $J_{\lambda k}$. Notice also that if the current is time-independent, (45) asserts that

$$(n|S|n') = \delta(n,n') ,$$

attesting to the stationary character of the states labelled by the photon numbers.

<u>The Infra-Red Catastrophe</u> - According to (41), the average number of photons emitted into a particular mode is given by

$$|\bar{J}_{\lambda k}|^2 = \frac{(d\underline{k})}{(2\pi)^3}\frac{1}{2k_0^2}\left|\int_{-\infty}^{\infty}(dx)\, e_\mu(\lambda k)\, e^{-ikx}\,\partial_0 J_\mu(x)\right|^2 .$$

We shall now consider frequencies that are sufficiently low for the wave-length to be large in comparison with the linear dimensions of the spatio-temporal region in which the current changes. The average number of photons, of either polarization, that emerge in a range of such low frequencies is then

$$\int\frac{(d\underline{k})}{(2\pi)^3}\frac{1}{2k_0^3}\left|\int_{-\infty}^{\infty}(dx)\,\partial_0 J_\mu(x)\right|^2 = \frac{1}{4\pi^2}\int\frac{dk_0}{k_0}\left|\int(d\underline{x})\left(J_\mu(\underline{x},1) - J_\mu(\underline{x},2)\right)\right|^2$$

which becomes infinite as the lower frequency limit approaches zero. Any time variation of the current thus produces a logarithmically infinite number of zero frequency photons -- a fact well-known as the "infra-red catastrophe". Accordingly, we find a zero probability for the emission of a finite number of photons. To avoid this type of statement we recognize that in any experimental arrangement, there is a minimum detectable frequency, $k_{0,min}$ such that we have no knowledge of the number of photons emitted into modes with frequencies less than $k_{0,min}$. If we sum the general transition probability (39) over all final occupation numbers of referring to these unobservable modes, we are left with the same expression constructed only from the observable modes. The latter will yield non-vanishing probabilities for the emission of a finite number of photons, each transition probability being dependent upon $k_{0,min}$ through the factor

$$\exp\left(-\sum_{\lambda k}|\bar{J}_{\lambda k}|^2\right) = \exp\left[-\int_{k_{0,min}}^{\infty}\frac{(dk)}{(2\pi)^3}\frac{1}{2k_0^3}\left|\int_{-\infty}^{\infty}(dx)\, e^{-ikx}\partial_0 J_\mu(x)\right|^2\right].$$

This quantity represents the probability that no (observable) photon will be emitted, if none are present initially.

<u>The adiabatic Theorem</u> - This important statement refers to the situation in which the current changes from its initial to its final value at a rate determined by the total elapsed time $T = t_1 - t_2$. We are particularly interested in the limit in which T becomes very large compared to the periods of all observable modes,

$$k_{0,min}\, T \to \infty$$

The above description of the current time variation is expressed quantitatively by

$$\int (d\underset{\sim}{x})\, e_m(\lambda k)\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, J_m(x) = \mathcal{J}_{\lambda k}(\theta), \quad \theta = (x_0 - t_2)/T$$

Hence the integral occurring in (44) is essentially determined by

$$\int_0^1 d\theta\, e^{ik_0 T\theta}\, \mathcal{J}'(\theta)$$

Now, according to the Riemann-Lebesque lemma[6],

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[6] E. T. Whittaker and G. N. Watson, Modern Analysis (The Macmillan Company, New York, 1927), p. 172.

$$\lim_{k_0 T \to \infty} \left| \int_0^1 d\theta \; e^{i k_0 T \theta} \, \mathcal{J}'(\theta) \right| = 0 \tag{47}$$

provided only that

$$\int_0^1 d\theta \; |\mathcal{J}'(\theta)| < \infty$$

This suffices to establish that

$$\lim_{T \to \infty} |\bar{J}_{\lambda k}|^2 = 0$$

If $\mathcal{J}'_{\lambda k}(\theta)$ is of limited total fluctuation, the integral in (47) approaches zero as $(k_0 T)^{-1}$, which enables us to satisfy

$$\lim_{T \to \infty} \sum_{\lambda k} |\bar{J}_{\lambda k}|^2 = 0$$

without essential restriction on the spatial distribution of the current (it must not be as singular as the gradient of a delta function). Under these conditions, we obtain the probability zero for any change in the photon numbers, despite the alteration in the current.

This theorem can be exploited to give a uniform expression for the results of all problems involving transition probabilities. Thus, in the integration over the extended region in (44), it is supposed that the current is constant in the exterior region. If we were to replace these constant currents by currents decreasing adiabatically to zero, at infinity, the null contribution from the external region would not be affected. But we would have succeeded in substituting for the original problem one in which the current vanishes on the boundaries of the extended region. Accordingly, we can integrate by parts in (44) and regain the form (33), appropriate to null currents on the boundaries. The most general problem requiring the evaluation of transition probabilities between stationary states, involves initial and final currents that are time-independent with respect to different reference systems. When modified with the aid of the adiabatic device, this situation also falls into the class of problems covered by (34).

The adiabatic device is also applicable to eigenvalue problems. Thus, we can use the transformation function (34), appropriate to zero current on the boundary surfaces, to construct the energy eigenvalues for the situation of a time-independent current. We suppose that the current, which is zero on the surface $\sigma_{-\infty}$, grows adiabatically and maintains a constant value between surfaces σ_2 and σ_1, and reduces adiabatically to zero on σ_∞. The designations $\sigma_{\pm\infty}$ refer to the fact that the adiabatic theorem involves the limit of infinite temporal separation between σ_∞ and σ_1, and between σ_2 and $\sigma_{-\infty}$. Then

$$(n\sigma_\infty | n'\sigma_{-\infty}) = \delta(n,n') \exp[i W_0 + i P(n)(x_\infty - x_{-\infty})],$$

where (reversing the integration by parts in the first term of (46)),

$$W_0 = -\int_{-\infty}^{\infty} dx_0 \; E(0,x_0),$$

and

$$\exp(iW_0) = \exp\left(-i\int_{t_1}^{\infty} dx_0\, E(0,x_0)\right) \exp(-iE(0)(t_1-t_2)) \exp\left(-i\int_{\infty}^{t_2} dx_0\, E(0,x_0)\right).$$

On recalling the composition property of transformation functions, we recognize immediately that

$$(n\sigma_1 | n'\sigma_2) = \delta(n,n') \exp[-i E(0)(t_1 - t_2) + i P(n)(x_1 - x_2)],$$

which shows that, in the presence of a time-independent current, the energy eigenvalues of the radiation field are displaced by $E(0)$.

The methods discussed in this paper and illustrated for the electromagnetic field are equally applicable to other Bose-Einstein systems, such as the symmetrical pseudoscalar meson field.