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FACTORS FOR ONE-SIDED TOLERANCE LIMITS AND FOR VARIABLES SAMPLING PLANS

by

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ABSTRACT

Tables are given of a quantity k which is used to define single-sample variables sampling plans and one-sided tolerance limits for a normal distribution. The probability is γ that at least a proportion P of a normal population is below $\bar{x} + ks$, where $\bar{x}$ has a normal distribution with mean μ and variance σ^2/n and fs^2/σ^2 has a chi-square distribution with f degrees of freedom. The quantity k just described corresponds to a percentage point of the noncentral t-distribution and is extensively tabulated. Tabulations of other functions computed from the noncentral t-distribution and various expected values are also given. Many other applications are discussed and various approximations compared. One section gives the mathematical derivations and there is an extensive bibliography which has been cross referenced to several indices of mathematical and statistical literature.

The variables sampling plans given are to be preferred to most other such variables plans (including the MIL STD plans) in cases where the protection of the consumer is of primary interest and the costs of items are high. These plans may also be preferred in other circumstances but an analysis of costs of the alternative plans should precede any decision on which plan to use.

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FACTORS FOR ONE-SIDED TOLERANCE LIMITS AND FOR VARIABLES SAMPLING PLANS

1. INTRODUCTION.

1.1 One-sided tolerance limits for a normal distribution.

For a normal random variable X with known mean μ and known standard deviation σ, it is possible to say that exactly a proportion P of the normal population is below $\mu + K_p\sigma$, where K_p is read from a table of the inverse normal probability distribution (e.g., see Reference [52], p. 12). For example, one can say that exactly 95% of the population is below $\mu + 1.64485\sigma$. The quantity $\mu + K_p\sigma$ is an upper tolerance limit.

In most cases, however, μ and σ are unknown and it is necessary to estimate both of them from a sample. Then a tolerance limit of the form $\bar{x} + ks$ may be used where $\bar{x}$ is an estimate of μ and s is an estimate of σ. Since $\bar{x}$ and s will be random variables, however, the tolerance limit statement can only be made with a given probability attached.

The problem then reduces to finding k such that the probability is γ that at least a proportion P of the population is below $\bar{x} + ks$. Tables of factors for one-sided tolerance limits for a normal distribution have been given in References [29], [37], [50], and [52] for the case where a sample $x_1, x_2, \ldots, x_n$ is taken and the sample mean,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

and the sample standard deviation,

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

are computed.

A value of k is given in the tables of Section 2 such that "at least a proportion P of the normal population is less than $\bar{x} + ks$ with probability equal to γ." The value $\bar{x} + ks$ is called an upper tolerance limit. For a lower tolerance limit $\bar{x} - ks$ is used and the statement is "at least a proportion P of the population is greater than $\bar{x} - ks$ with confidence γ." If a two-sided limit is desired the reader is referred to References [12], [35], [52], and [76].

If the normal distribution has mean μ and standard deviation σ and either of these are known, there are entries in the tables of Sections 3 and 4 which will give the required tolerance limit. When the mean is known, k may be read from the tables of Section 4 with $n = \infty$, i.e., the tables of Sections 4.1.15, 4.2.15, and 4.3.15. Similarly, if the standard deviation is known, k may be read from the tables of Section 4 with $f = \infty$, i.e., as the last entry for each table. The tables of Sections 3.1, 3.2 and 3.3 may be useful if $n = 1$ or ∞ or if $f = 1$ or ∞.

It is convenient to define the term degrees of freedom for $\bar{x}$ as that value of n which occurs in the statement $\bar{x}$ has mean μ and standard deviation $\sigma/\sqrt{n}$. Similarly, the degrees of freedom for s is that value of f which occurs in the statement fs^2/σ^2 has a chi-square distribution with f degrees of freedom.

In addition to giving more extensive tables of k than [29], [37], and [50], this report extends the tables of k to the cases where the degrees of freedom for s are not necessarily one less than the degrees of freedom for $\bar{x}$. The degrees of freedom for s will be designated by f, and the degrees of freedom for $\bar{x}$ will be designated by n. Values for $n = 1, 2, 3$ and 4 only are given in [52] for this case where $f \neq n - 1$. The present report can also be considered an extension of the work in References [35] and [76] which cover the two-sided tolerance limit problem with $\bar{x}$ based on n degrees of freedom and s based on f degrees of freedom, where again f is not necessarily equal to $n - 1$. The extension given here, of course, is from the two-sided case to the one-sided case.

The values of k given in Sections 2, 3.1, 3.2, 3.3, and 4 correspond to percentage points (divided by the square root of n) of the noncentral t-distribution. Specifically,

$$\Pr\left\{\text{noncentral } t \leq k\sqrt{n} \mid \delta = K_p\sqrt{n}\right\} = \gamma,$$

where the noncentral t has f degrees of freedom and K_p is such that $\Pr\left\{\text{a standardized normal variable} \leq K_p\right\} = P$.

1.2 Johnson and Welch type tables for computing k.

A discussion of the tables of Section 5 follows. Among other things these tables may be used whenever there is a combination of values of f, n, and P for which there is not an entry in the tables of Sections 2, 3 or 4 and for which interpolation in Sections 2, 3 or 4 would not be satisfactory. Note also that the values of γ which are available in Section 5 include $(1 - \gamma)$ for each γ listed since $\Pr\left\{\text{noncentral } t \leq t_o \mid \delta\right\} = 1 - \Pr\left\{\text{noncentral } t \leq -t_o \mid -\delta\right\}$ and both positive and negative values of t and δ appear in the tables.

Section 5 follows a procedure used by Johnson and Welch [32] and contains values of γ such that if

$$\eta = \frac{\delta}{\sqrt{2f}}\left(1 + \frac{\delta^2}{2f}\right)^{-\frac{1}{2}},$$

and

$$t_o = \frac{\delta + \lambda\left(1 + \frac{\delta^2}{2f} - \frac{\lambda^2}{2f}\right)^{\frac{1}{2}}}{\left(1 - \frac{\lambda^2}{2f}\right)},$$

These values of k and the standard deviations of $\bar{x} + ks$ are given in Section 3.6. The tables of Sections 3.7 and 3.8 give the mean and standard deviation of the noncentral $(t/\sqrt{n})$ -distribution and the noncentral t-distribution.

The reader wishing a broader mathematical coverage of the noncentral t-distribution and the methods of computing and checking the accompanying tables is referred to Section 8 for this information. Some special cases where n and f are not necessarily tied together as $f = n - 1$ are considered in Sections 9, 10 and 11. In Section 12 a comparison of four approximations to the exact values of k are given and at the end there is a bibliography and list of references to the noncentral t-distribution and its various applications.

2.4 Values of k for f = n - 1 and γ = .95 (Continued)

$$\Pr\left\{T_f \le k\sqrt{n} \,\middle|\, K_p\sqrt{n}\right\} = \gamma$$

	P							
n	.75000	.90000	.95000	.97500	.99000	.99900	.99990	.99999
46	.974	1.664	2.086	2.457	2.890	3.801	4.555	5.211
47	.971	1.659	2.081	2.450	2.883	3.792	4.544	5.199
48	.967	1.654	2.075	2.444	2.876	3.783	4.533	5.187
49	.964	1.650	2.070	2.438	2.869	3.774	4.523	5.175
50	.960	1.646	2.065	2.432	2.862	3.766	4.513	5.164
51	.957	1.641	2.060	2.427	2.856	3.758	4.504	5.153
52	.954	1.637	2.055	2.421	2.850	3.750	4.494	5.142
53	.951	1.633	2.051	2.416	2.844	3.742	4.485	5.132
54	.948	1.630	2.046	2.411	2.838	3.735	4.477	5.123
55	.945	1.626	2.042	2.406	2.833	3.728	4.468	5.113
56	.943	1.622	2.038	2.401	2.827	3.721	4.460	5.104
57	.940	1.619	2.034	2.397	2.822	3.714	4.452	5.095
58	.938	1.615	2.030	2.392	2.817	3.708	4.445	5.086
59	.935	1.612	2.026	2.388	2.812	3.701	4.437	5.078
60	.933	1.609	2.022	2.384	2.807	3.695	4.430	5.070
61	.930	1.606	2.019	2.380	2.802	3.689	4.423	5.062
62	.928	1.603	2.015	2.376	2.798	3.684	4.416	5.054
63	.926	1.600	2.012	2.372	2.793	3.678	4.410	5.047
64	.924	1.597	2.008	2.368	2.789	3.673	4.403	5.039
65	.921	1.594	2.005	2.364	2.785	3.667	4.397	5.032
66	.919	1.591	2.002	2.361	2.781	3.662	4.391	5.025
67	.917	1.589	1.999	2.357	2.777	3.657	4.385	5.018
68	.915	1.586	1.996	2.354	2.773	3.652	4.379	5.012
69	.913	1.584	1.993	2.351	2.769	3.647	4.373	5.005
70	.911	1.581	1.990	2.347	2.765	3.643	4.368	4.999
71	.910	1.579	1.987	2.344	2.762	3.638	4.362	4.993
72	.908	1.576	1.984	2.341	2.758	3.633	4.357	4.987
73	.906	1.574	1.982	2.338	2.755	3.629	4.352	4.981
74	.904	1.572	1.979	2.335	2.751	3.625	4.347	4.975
75	.903	1.570	1.976	2.332	2.748	3.621	4.342	4.970
76	.901	1.568	1.974	2.329	2.745	3.617	4.337	4.964
77	.899	1.565	1.971	2.327	2.742	3.613	4.333	4.959
78	.898	1.563	1.969	2.324	2.739	3.609	4.328	4.954
79	.896	1.561	1.967	2.321	2.736	3.605	4.323	4.949
80	.895	1.559	1.964	2.319	2.733	3.601	4.319	4.944
81	.893	1.557	1.962	2.316	2.730	3.597	4.315	4.939
82	.892	1.556	1.960	2.314	2.727	3.594	4.310	4.934
83	.890	1.554	1.958	2.311	2.724	3.590	4.306	4.929
84	.889	1.552	1.956	2.309	2.721	3.587	4.302	4.925
85	.888	1.550	1.954	2.306	2.719	3.583	4.298	4.920
86	.886	1.548	1.952	2.304	2.716	3.580	4.294	4.916
87	.885	1.547	1.950	2.302	2.714	3.577	4.291	4.911
88	.884	1.545	1.948	2.300	2.711	3.574	4.287	4.907
89	.882	1.543	1.946	2.297	2.709	3.571	4.283	4.903
90	.881	1.542	1.944	2.295	2.706	3.567	4.279	4.899

2.10 Values of k for f = n - 1 and γ = .05 (Continued)

$$\Pr\left\{T_f \le k\sqrt{n} \,\middle|\, K_p\sqrt{n}\right\} = \gamma$$

	P							
n	.75000	.90000	.95000	.97500	.99000	.99900	.99990	.99999
46	.423	.989	1.317	1.598	1.922	2.589	3.134	3.606
47	.426	.992	1.321	1.602	1.925	2.594	3.140	3.612
48	.428	.995	1.324	1.605	1.929	2.598	3.145	3.618
49	.431	.997	1.327	1.608	1.933	2.603	3.150	3.624
50	.433	1.000	1.329	1.611	1.936	2.607	3.155	3.629
51	.435	1.003	1.332	1.614	1.940	2.611	3.160	3.635
52	.438	1.005	1.335	1.617	1.943	2.615	3.165	3.640
53	.440	1.007	1.337	1.620	1.946	2.619	3.169	3.645
54	.442	1.010	1.340	1.623	1.949	2.623	3.174	3.650
55	.444	1.012	1.343	1.626	1.952	2.627	3.178	3.655
56	.446	1.014	1.345	1.628	1.955	2.630	3.182	3.660
57	.448	1.016	1.347	1.631	1.958	2.634	3.186	3.664
58	.450	1.018	1.350	1.634	1.961	2.637	3.190	3.669
59	.451	1.020	1.352	1.636	1.964	2.641	3.194	3.673
60	.453	1.022	1.354	1.638	1.966	2.644	3.198	3.678
61	.455	1.024	1.356	1.641	1.969	2.647	3.202	3.682
62	.457	1.026	1.358	1.643	1.972	2.650	3.206	3.686
63	.458	1.028	1.360	1.645	1.974	2.653	3.209	3.690
64	.460	1.030	1.362	1.648	1.976	2.657	3.213	3.694
65	.461	1.032	1.364	1.650	1.979	2.659	3.216	3.698
66	.463	1.033	1.366	1.652	1.981	2.662	3.220	3.702
67	.465	1.035	1.368	1.654	1.984	2.665	3.223	3.706
68	.466	1.037	1.370	1.656	1.986	2.668	3.226	3.709
69	.467	1.038	1.372	1.658	1.988	2.671	3.229	3.713
70	.469	1.040	1.374	1.660	1.990	2.673	3.232	3.716
71	.470	1.042	1.375	1.662	1.992	2.676	3.235	3.720
72	.472	1.043	1.377	1.664	1.994	2.679	3.238	3.723
73	.473	1.045	1.379	1.665	1.996	2.681	3.241	3.726
74	.474	1.046	1.380	1.667	1.998	2.684	3.244	3.730
75	.476	1.048	1.382	1.669	2.000	2.686	3.247	3.733
76	.477	1.049	1.384	1.671	2.002	2.688	3.250	3.736
77	.478	1.050	1.385	1.672	2.004	2.691	3.253	3.739
78	.479	1.052	1.387	1.674	2.006	2.693	3.255	3.742
79	.480	1.053	1.388	1.676	2.008	2.695	3.258	3.745
80	.482	1.054	1.390	1.677	2.010	2.697	3.260	3.748
81	.483	1.056	1.391	1.679	2.011	2.700	3.263	3.751
82	.484	1.057	1.392	1.681	2.013	2.702	3.265	3.753
83	.485	1.058	1.394	1.682	2.015	2.704	3.268	3.756
84	.486	1.059	1.395	1.684	2.017	2.706	3.270	3.759
85	.487	1.061	1.397	1.685	2.018	2.708	3.273	3.762
86	.488	1.062	1.398	1.686	2.020	2.710	3.275	3.764
87	.489	1.063	1.399	1.688	2.021	2.712	3.277	3.767
88	.490	1.064	1.400	1.689	2.023	2.714	3.279	3.769
89	.491	1.065	1.402	1.691	2.025	2.716	3.282	3.772
90	.492	1.066	1.403	1.692	2.026	2.718	3.284	3.774

8. DERIVATION OF THE MATHEMATICAL RELATIONSHIPS.

8.1 The noncentral t-distribution.

Let X have a normal distribution with mean zero and variance one and let Y be independent of X and have a chi-square distribution with f degrees of freedom and let δ be any constant. Then $T_f(\delta) = (X + \delta)/\sqrt{Y/f}$ has a noncentral t-distribution with noncentrality parameter δ and f degrees of freedom. Note that if $\delta = 0$ then $T_f(\delta)$ has a (central) Student t-distribution with f degrees of freedom.

The joint density of X and Y is then given by

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\Gamma(f/2)2^{f/2}} y^{(f-2)/2} e^{-y/2}$$

If the change in variables given by $X = (ZU/\sqrt{f}) - \delta$ and $y = U^2$ is made in this last expression and U is integrated out, then the density of $Z = T_f$ is obtained. That result is then integrated with respect to Z from minus infinity to t to give the cumulative distribution function,

$$\Pr\left\{T_f \le t\right\} = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{f}{2}\right) 2^{(f-2)/2}} \int_0^{\infty} G\left(\frac{tU}{\sqrt{f}} - \delta\right) U^{f-1} G'(U)\, dU,$$

where

$$G'(X) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and

$$G(X) = \int_{-\infty}^{X} G'(t)\, dt.$$

If this cumulative distribution function is integrated by parts repeatedly, one obtains for odd values of f

$$\Pr\left\{T_f \le t\right\} = G(-\delta\sqrt{B}) + 2T(\delta\sqrt{B}, A) + 2\left[M_1 + M_3 + \dots + M_{f-2}\right]$$

where

$$A = \frac{t}{\sqrt{f}} \text{ and } B = \frac{f}{f + t^2}$$

and

$$T(h, a) = \frac{1}{2\pi} \int_0^a \frac{\exp\left[-\frac{h^2}{2}(1 + X^2)\right]}{1 + X^2}\, dX$$

is a function discussed and tabulated in [48] and [49]. The M's are defined below.

For even values of f,

$$\Pr\left\{T_f \le t\right\} = G(-\delta) + \sqrt{2\pi}\left[M_o + M_2 + \ldots + M_{f-2}\right]$$

where

$$M_{-1} = 0$$

$$M_o = A\sqrt{B}\, G'(\delta\sqrt{B})\, G(\delta A\sqrt{B})$$

$$M_1 = B\left[\delta A M_o + \frac{A}{\sqrt{2\pi}} G'(\delta)\right]$$

$$M_2 = \frac{1}{2} B\left[\delta A M_1 + M_o\right]$$

$$M_3 = \frac{2}{3} B\left[\delta A M_2 + M_1\right]$$

$$M_4 = \frac{3}{4} B\left[\frac{1}{2}\delta A M_3 + M_2\right]$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$M_k = \frac{k-1}{k} B\left[a_k \delta A m_{k-1} + M_{k-2}\right]$$

and where

$$a_k = \frac{1}{(k-2)a_{k-1}} \qquad \text{for } k \ge 3 \qquad \text{and} \qquad a_2 = 1.$$

Some special properties of the noncentral t-distribution are now immediately obvious from the above formulas:

$$\Pr\left\{T_f \le t \mid \delta\right\} = 1 - \Pr\left\{T_f \le -t \mid -\delta\right.$$

$$\Pr\left\{T_f \le 0 \mid \delta\right\} = G(-\delta)$$

If $f = 1$, $t = 1$ and $G\left(\frac{\delta}{\sqrt{2}}\right) = P$, then $\Pr\left\{T_1 \le 1\right\} = 1 - P$.

Also if $\delta = 0$, the above noncentral t-distribution reduces to the Student t-distribution. Note that $T(0, A) = (\arctan A)/(2\pi)$. Note also that in this case the M's with odd subscripts can be computed independently of the M's with even subscripts and vice versa.

For odd values of f, therefore, the Student t-cumulative distribution function reduces to

$$\Pr\left\{\text{Student-t} \leq t\right\} = \frac{1}{2} + (\arctan A)/(\pi) + (AB)/(\pi)\left[b_o + b_1 B + b_2 B^2 + \ldots + b_{(f-3)/2} B^{(f-3)/2}\right]$$

where, $b_o = 1$ and $b_r = \frac{2r}{2r+1} b_{r-1}$; and for even values of f

$$\Pr\left\{\text{Student-t} \leq t\right\} = \frac{1}{2} + \frac{A\sqrt{B}}{2}\left[c_o + c_1 B + c_2 B^2 + \ldots + c_{(f-2)/2} B^{(f-2)/2}\right]$$

where $c_o = 1$ and $c_r = \frac{2r-1}{2r} c_{r-1}$ and, as before,

$$A = t/(\sqrt{f}) \text{ and } B = f/(f+t^2).$$

8.2 <u>One-sided tolerance limits.</u>

A method for finding a one-sided tolerance limit will now be given such that at least a proportion P of a normal population will be above (or below) the tolerance limit with probability γ (i.e., if the experimental procedure as described below were repeated an infinite number of times and all of the hypotheses were met exactly, then $100\gamma\%$ of the tolerance limits would cover at least a proportion P of the population).

A sample of n observations is taken at random from a normal distribution with unknown mean μ and unknown standard deviation σ. The sample values are denoted by $x_1, x_2, \ldots, x_n$ and

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

are computed from these sample values.

In more complicated problems the estimate of σ may be obtained from an analysis of variance computation and be based on f degrees of freedom. The quantity s as defined above has $f = n - 1$ degrees of freedom. In what follows, s will be used for either the analysis of variance estimate or the estimate above. How s or $\bar{x}$ are computed is not important to the problem here so long as fs^2/σ^2 has a chi-square distribution with f degrees of freedom and $\bar{x}$ has a normal distribution with mean μ and standard deviation equal to $\sigma/\sqrt{n}$ (μ and σ are unknown). The problem then becomes a problem in finding k so that either $\bar{x} + ks$ or $\bar{x} - ks$ is the required tolerance limit.

Mathematically, the problem is to find k such that

$$\Pr\left\{\Pr(X \leq \bar{x} + ks) \geq P\right\} = \gamma$$

where X has a normal distribution with mean μ and standard deviation σ, and P and γ are specified probabilities. Define K_p by:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_p} e^{-t^2/2}\, dt = P,$$

and then,

$$\Pr\left\{\Pr(X \le \bar{x} + ks) \ge P\right\} = \Pr\left\{\frac{\bar{x} + ks - \mu}{\sigma} \ge K_p\right\}.$$

Rewriting once more this becomes

$$\Pr\left\{\frac{\frac{\bar{X} - \mu}{\sigma}\sqrt{n} - K_p\sqrt{n}}{\frac{s}{\sigma}} \ge -k\sqrt{n}\right\} = \gamma.$$

This is now in the form of the noncentral t-distribution with f degrees of freedom and with $\delta = -K_p\sqrt{n}$ and $t = -k\sqrt{n}$.
Or equivalently, this may be written

$$\Pr\left\{T_f \le k\sqrt{n} \mid \delta = K_p\sqrt{n}\right\} = \gamma,$$

where T_f is a noncentral t random variable. Hence the quantity k which is desired may be computed from the percentage points of the noncentral t-distribution.

8.3 The sampling plan procedure.

The problem is to find a value of k such that the $\Pr\{\bar{x} + ks \le U\} = 1 - \gamma$, i.e., the probability of accepting a lot is $1 - \gamma$. (The quantity γ is usually equal to 0.90.) This is equivalent to

$$\Pr\left\{\left(\frac{\bar{x} - \mu}{\sigma}\right)\sqrt{n} - \left(\frac{U - \mu}{\sigma}\right)\sqrt{n} \le -\frac{ks}{\sigma}\sqrt{n}\right\} = 1 - \gamma.$$

Next we divide both sides of the inequality by s/σ and the quantity on the left is of the form of the noncentral t. Hence, we have

$$\Pr\left\{T_f \le -k\sqrt{n} \mid \delta = -\left(\frac{U - \mu}{\sigma}\right)\sqrt{n}\right\} = 1 - \gamma,$$

where T_f is a noncentral t variable with f degrees of freedom (the number of degrees of freedom of s^2). In most applications $f = n - 1$. We want to accept a lot with no more than $100(1 - P)\%$ of the population above U only $100(1 - \gamma)\%$ of the time and hence $K_p = \frac{U - \mu}{\sigma}$. Then we have

$$\Pr\left\{T_f \le k\sqrt{n} \mid \delta = K_p\sqrt{n}\right\} = \gamma.$$

This equation is identical with the one given for the tolerance limit problem.

8.4 The density of noncentral t in terms of the cumulative.

The value of k, as defined in Section 8.2, may now be obtained by using Newton's method for interpolating for a root of an equation. To implement this method, the derivative of the distribution

These last two formulas were often used to start the iterations described in Sections 8.4 and 8.5. Other start values were obtained by use of the approximations given in Section 12.

8.9 The moments of the noncentral t-distribution.

The moments of the noncentral t-distribution have been given by Hogben, Pinkham, and Wilk [26] and by others previously, e.g., [40], and are polynomial functions of δ whose coefficients are functions of f. The mean and second, third, and fourth moments about the mean are as follows:

$$\mu = c_{11}\,\delta$$

$$\mu_2 = c_{22}\,\delta^2 + c_{20}$$

$$\mu_3 = c_{33}\,\delta^3 + c_{31}\,\delta$$

$$\mu_4 = c_{44}\,\delta^4 + c_{42}\,\delta^2 + c_{40}$$

where

$$c_{11} = \frac{\sqrt{\frac{f}{2}}\;\Gamma\left(\frac{f-1}{2}\right)}{\Gamma\left(\frac{f}{2}\right)}$$

$$c_{22} = \frac{f}{f-2} - c_{11}^2$$

$$c_{20} = \frac{f}{f-2}$$

$$c_{33} = c_{11}\left[\frac{f(7-2f)}{(f-2)(f-3)} + 2c_{11}^2\right]$$

$$c_{31} = \frac{3f}{(f-2)(f-3)}\,c_{11}$$

$$c_{44} = \frac{f^2}{(f-2)(f-4)} - \frac{2f(5-f)\,c_{11}^2}{(f-2)(f-3)} - 3c_{11}^4$$

$$c_{42} = \frac{6f}{f-2}\left[\frac{f}{f-4} - \frac{(f-1)\,c_{11}^2}{f-3}\right]$$

$$c_{40} = \frac{3f^2}{(f-2)(f-4)}\,.$$

Hogben, Pinkham, and Wilk table the exact values of the c's. See Section 12.3 for approximations to the c's. See Sections 3.7 and 3.8 for tabulations of means and standard deviations of the noncentral t- and noncentral $\frac{t}{\sqrt{n}}$ -distributions. Merrington and Pearson [40] give the following formula for finding the rth moment about the origin for the noncentral t-distribution:

$$\mu_r' = E(t^r) = \frac{f}{2}^{r/2} \frac{\Gamma[(f-r)/2]}{\Gamma(f/2)} e^{-\frac{1}{2}\delta^2} D^r e^{\frac{1}{2}\delta^2}$$

where D^r indicates the rth derivative of the function following it. Hence,

$$\mu_5' = c_{11}\left[\delta^5 + 10\delta^3 + 15\delta\right]\left[f^2/\{(f-3)(f-5)\}\right]$$

and

$$\mu_6' = \left[\delta^6 + 15\delta^4 + 45\delta^2 + 15\right]\left[f^3/\{(f-2)(f-4)(f-6)\}\right].$$

8.10 An alternative expression for the noncentral t-distribution.

On page 387 of the article by Johnson and Welch [32] a formula is given for even values of f for the cumulative distribution function of noncentral t in terms of Hh functions. This formula has f/2 terms, i.e., is a finite sum and it is easily convertible into a finite sum of confluent hypergeometric functions. Owen and Amos [53] have another expression involving the hypergeometric function which holds for odd and even values of f.

As before, let

$$H_f(t) = \Pr\left\{T_f \le t\right\} = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{f}{2}\right)2^{(f-2)/2}} \int_0^\infty G\left(\frac{tx}{\sqrt{f}} - \delta\right) x^{f-1} G'(x)\,dx,$$

and let $A = \frac{t}{\sqrt{f}}$ and $B = \frac{f}{f+t^2}$. Then Owen and Amos [53] give

$$H_f(t) = 1 - G(\delta\sqrt{B}) + \sqrt{2}\, G'(\delta\sqrt{B})\left[\frac{\Gamma\left(\frac{f-1}{2}\right)}{\Gamma\left(\frac{f-2}{2}\right)} A\sqrt{B}\, S_1 - \frac{\delta}{\sqrt{2}}\sqrt{B}\, S_2\right]$$

where

$$S_1 = \sum_{k=0}^{\infty} a_k\, \phi\left(-k, \frac{1}{2}; \frac{\delta^2 B}{2}\right)$$

and

$$S_2 = \sum_{k=0}^{\infty} b_{k+1}\, \phi\left(-k, \frac{3}{2}; \frac{\delta^2 B}{2}\right),$$

and where

$$a_0 = 1, \quad a_k = a_{k-1} A^2 B \frac{(2k-f)(2k-1)}{(2k)(2k+1)};$$

$$b_0 = 1, \quad b_k = b_{k-1} A^2 B\left(\frac{2k-1-f}{2k}\right).$$

One series will always terminate when f is an integer; S_1 when f is even and S_2 when f is odd.

12. COMPARISON OF VARIOUS APPROXIMATIONS TO k WITH EXACT VALUES.

12.1 The Jennett and Welch approximation to k.

Jennett and Welch [28] derived a formula for k based on a normal approximation to the distribution of $\bar{x} + ks$. The formula is:

$$k = \frac{K_p + \sqrt{K_p^2 - AB}}{A}, \qquad \text{(approximation J)},$$

where

$$A = 1 - \frac{K_\gamma^2}{2f},$$

and

$$B = K_p^2 - \frac{K_\gamma^2}{n}.$$

This formula is also given on page 59 of Eisenhart, Hastay, and Wallis [12].

12.2 The van Eeden approximation to k.

Constance van Eeden [72] summarizes several different approximations which may be used to obtain k. Her approximation II b which is of the Cornish-Fisher type may be rewritten

$$k = \frac{t^*(f,\gamma)}{\sqrt{n}} + K_p\left[1 + \frac{2K_\gamma^2 + 1}{4f} + \frac{4K_\gamma^4 + 12K_\gamma^2 + 1}{32f^2}\right]$$

$$+ K_p^2\sqrt{n}\left[\frac{K_\gamma}{4f} + \frac{K_\gamma^3 + 4K_\gamma}{16f^2}\right] - K_p^3\, n\left[\frac{K_\gamma^2 - 1}{24f^2}\right] - K_p^4\, n\sqrt{n}\left[\frac{K_\gamma}{32f^2}\right], \qquad \text{(approximation E)},$$

where $t^*(f,\gamma)$ is the γth percentage point of Student's t-distribution with f degrees of freedom.

12.3 The normal approximation to noncentral t.

For large values of f, the degrees of freedom, the noncentral t-statistic becomes approximately normally distributed with mean and variance as given in Section 8.9. The value of t_o in the statement

$$\Pr\left\{\text{noncentral } t \le t_o \mid \delta, f\right\} = \gamma$$

is therefore given approximately by:

$$t_o \approx \delta c_{11} + K_\gamma\sqrt{c_{22}\,\delta^2 + c_{20}},$$

or equivalently k is given approximately for large values of f by:

$$k \approx c_{11} K_p + K_\gamma \sqrt{c_{22} K_p^2 + \frac{c_{20}}{n}} .$$

Johnson and Welch [32] give the following approximations for the moments and for the c's:

$$\mu \approx \delta ; \mu_2 \approx 1 + \frac{\delta^2}{2f} ; \text{ and } \mu_3 \approx \frac{\delta}{f}\left[3 + \frac{5\delta^2}{4f}\right]; \qquad \text{or}$$

$$c_{11} \approx 1; c_{22} \approx \frac{1}{2f}; c_{20} \approx 1; c_{33} \approx \frac{5}{4f^2} \quad \text{and} \quad c_{31} \approx \frac{3}{f}.$$

Johnson and Welch note that these approximations substituted in the approximate formula for k given above do not yield very good results and recommend the normal approximation to $\bar{x} + ks$ given in Section 12.1 over the one given in this section.

However, examination of the table given by Hogben, Pinkham, and Wilk [26] indicated that perhaps a good deal of the difficulty is due to the approximate values of the c's chosen. The only c which is difficult to compute is c_{11} and so a better approximation to c_{11} was sought. Empirical examination of the Hogben, Pinkham, and Wilk table led to the following approximation:

$$c_{11} \approx 1 + \frac{3}{4(f - 1.042)} .$$

This approximation gave c_{11} accurately to five decimal places for $f \geq 9$. The value of c_{22} was computed from c_{11} and c_{20} computed exactly giving approximate values of k which were labeled Approximation D. Explicitly,

$$k \approx c_{11} K_p + K_\gamma \sqrt{c_{22} K_p^2 + \frac{c_{20}}{n}} , \qquad \text{(Approximation D)}.$$

12.4 The Student approximation to noncentral t.

Empirical examination of the preceding approximation (D) indicated that a better approximation to k could be obtained by replacing K_γ by $t^*(f, \gamma)$, the γth percentage point of Student's t-distribution with f degrees of freedom, i.e., $\Pr\left\{T_f \leq t^*(f, \gamma) \mid \delta = 0\right\} = \gamma$. The new approximation was labeled C and is given by:

$$k \approx c_{11} K_p + t^*(f, \gamma) \sqrt{c_{22} K_p^2 + \frac{c_{20}}{n}} , \qquad \text{(Approximation C)}.$$

12.5 Comparison of the approximations.

For $f = n - 1$, $\gamma = 0.90$, 0.95, and 0.99; and $P = 0.75$, 0.90, 0.95, 0.975, 0.99, 0.999, 0.9999, and 0.99999, each of the approximations C, D, J, and E were computed and compared with the exact values. The number of correct decimal places in each approximation was then recorded as 3, 2, or 1 following the letter designating the approximation.

The computations for each of the approximations and the exact value of k were carried in the computer to 11 decimal digits. The absolute values of the difference between each approximation and the exact value was obtained. The approximations C, D, J, and E were then ordered from best (1) to worst (4). If this difference minus 0.0005 was negative, a 3 was recorded on the second line. If the difference minus 0.005 was zero or positive, then the difference had 0.005 subtracted. If this was negative, a 2 was recorded on the second line and if it was positive or zero, the next step was taken. If the difference minus 0.050 gave a negative, then a 1 was recorded on the second line. If the difference minus 0.050 gave a zero or a positive, then a 0 was recorded on the second line. Table 12.5.1 below gives the results of these computations.

TABLE 12.5.1

Comparisons of Approximations to k for f = n -1

γ = .90

	P							
n	.75000	.90000	.95000	.97500	.99000	.99900	.99990	.99999
	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E
10	4 2 3 1	4 1 3 2	4 1 3 2	3 1 2 4	3 1 2 4	3 1 2 4	2 1 3 4	2 1 3 4
	1 1 1 2	0 1 0 1	0 1 0 1	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0
15	3 2 4 1	3 1 4 2	2 1 4 3	2 1 3 4	2 1 3 4	2 1 3 4	2 1 3 4	2 1 3 4
	1 2 1 2	1 1 1 1	1 1 1 1	1 1 1 0	1 1 0 0	1 1 0 0	1 1 0 0	1 1 0 0
20	3 2 4 1	3 1 4 2	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	1 2 1 3	1 1 1 1	1 1 1 1	1 1 1 0	1 1 1 0	1 1 1 0	1 1 0 0	1 1 0 0
25	3 2 4 1	1 3 4 2	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	1 2 1 3	2 1 1 1	2 1 1 1	2 1 1 0	2 1 1 0	2 1 1 0	2 1 1 0	2 1 1 0
30	3 2 4 1	1 3 4 2	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	2 2 1 3	2 1 1 1	2 1 1 1	3 1 1 0	3 1 1 0	3 1 1 0	2 1 1 0	2 1 1 0
40	2 3 4 1	1 3 4 2	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	2 2 1 3	2 1 1 1	2 1 1 1	2 1 1 1	2 1 1 0	2 1 1 0	2 1 1 0	1 1 1 0
50	2 3 4 1	1 3 4 2	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	3 2 2 3	2 2 1 2	2 1 1 1	2 1 1 1	2 1 1 0	2 1 1 0	1 1 1 0	1 1 1 0
60	1 3 4 2	1 3 4 2	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	3 2 2 3	2 2 1 2	2 1 1 1	2 1 1 1	2 1 1 0	2 1 1 0	1 1 1 0	1 1 1 0
80	2 3 4 1	1 2 4 3	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	3 2 2 3	2 2 2 2	2 2 1 1	2 1 1 1	2 1 1 0	2 1 1 0	1 1 1 0	1 1 1 0
100	2 3 4 1	1 2 4 3	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	2 2 2 3	2 2 2 2	2 2 2 1	2 2 2 1	2 2 1 0	2 1 1 0	1 1 1 0	1 1 1 0
120	2 3 4 1	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	2 2 2 3	2 2 2 2	2 2 2 1	2 2 2 1	2 2 2 0	2 1 1 0	2 1 1 0	1 1 1 0
150	2 3 4 1	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
	2 2 2 3	2 2 2 2	2 2 2 1	2 2 2 1	2 2 2 0	2 2 2 0	2 1 1 0	2 1 1 0
200	2 3 4 1	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 3 2 4	1 3 2 4	1 3 2 4
	2 2 2 3	2 2 2 2	2 2 2 1	2 2 2 1	2 2 2 1	2 2 2 0	2 2 2 0	2 2 2 0
1000	2 4 3 1	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	3 3 3 3	3 3 3 2	3 3 3 2	3 2 3 1	2 2 2 1	2 2 2 0	2 2 2 0	2 2 2 0

TABLE 12.5.1 (Continued)

Comparisons of Approximations to k for f = n - 1

$\gamma = .95$

n	.75000	.90000	.95000	.97500	.99000	.99900	.99990	.99999
	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E
10	2 4 3 1	1 4 3 2	1 4 2 3	1 3 2 4	1 3 2 4	1 3 2 4	1 3 2 4	1 3 2 4
	1 0 1 2	1 0 1 1	3 0 1 0	2 0 1 0	1 0 1 0	1 0 1 0	1 0 0 0	1 0 0 0
15	2 4 3 1	2 4 3 1	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 1 1 2	1 0 1 1	1 0 1 0	1 0 1 0	1 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
20	2 4 3 1	3 4 2 1	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 1 1 2	1 1 1 1	1 0 1 1	1 0 1 0	1 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
25	2 4 3 1	3 4 2 1	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 1 3	1 1 1 1	1 1 1 1	1 0 1 0	1 0 1 0	1 0 1 0	0 0 1 0	0 0 1 0
30	2 4 3 1	3 4 2 1	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 1 3	1 1 1 1	1 1 1 1	1 1 1 0	1 0 1 0	1 0 1 0	0 0 1 0	0 0 1 0
40	3 4 2 1	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 1 3	1 1 1 1	1 1 1 1	1 1 1 0	1 1 1 0	1 0 1 0	1 0 1 0	0 0 1 0
50	3 4 2 1	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 1 1	1 1 1 1	1 1 1 0	1 1 1 0	1 1 1 0	1 0 1 0	1 0 1 0
60	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 2 1	1 1 1 1	1 1 1 0	1 1 1 0	1 1 1 0	1 1 1 0	1 1 1 0
80	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 1 2 3	1 1 2 1	1 1 2 1	1 1 2 1	1 1 2 0	1 1 1 0	1 1 1 0	1 1 1 0
100	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 1 2 3	1 1 2 2	1 1 2 1	1 1 2 1	1 1 2 0	1 1 2 0	1 1 1 0	1 1 1 0
120	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 2 2 3	1 1 2 2	1 1 2 1	1 1 2 1	1 1 2 0	1 1 2 0	1 1 2 0	1 1 2 0
150	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 2 2 3	1 1 2 2	1 1 2 1	1 1 2 1	1 1 2 0	1 1 2 0	1 1 2 0	1 1 2 0
200	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 2 2 3	2 2 2 2	1 1 2 1	1 1 2 1	1 1 2 0	1 1 2 0	1 1 2 0	1 1 2 0
1000	3 4 2 1	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	3 2 3 3	2 2 3 2	2 2 3 2	2 2 3 1	2 2 3 1	2 2 3 0	2 2 3 0	2 2 2 0

TABLE 12.5.1 (Continued)

Comparisons of Approximations to k for f = n - 1

$\gamma = .99$

n	P .75000	.90000	.95000	.97500	.99000	.99900	.99990	.99999
	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E	C D J E
10	3 4 2 1	3 4 2 1	3 4 2 1	2 4 1 3	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4
	0 0 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
15	3 4 1 2	3 4 2 1	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	0 0 2 2	0 0 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
20	3 4 1 2	3 4 1 2	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 0 2 2	0 0 1 1	0 0 1 0	0 0 1 0	0 0 1 0	0 0 0 0	0 0 0 0	0 0 0 0
25	3 4 2 1	3 4 1 2	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 0 2 2	0 0 1 1	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 0 0	0 0 0 0
30	3 4 2 1	3 4 1 2	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 0 2 2	0 0 2 1	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 0 0
40	3 4 2 1	3 4 1 2	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 2	0 0 2 1	0 0 1 1	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
50	3 4 2 1	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 0 2 1	0 0 2 1	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
60	3 4 2 1	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 3 1	1 0 2 1	0 0 2 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
80	3 4 2 1	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 3 1	1 1 2 1	1 0 2 0	0 0 2 0	0 0 1 0	0 0 1 0	0 0 1 0
100	3 4 2 1	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 3 1	1 1 2 1	1 1 2 0	1 1 2 0	0 0 2 0	0 0 1 0	0 0 1 0
120	3 4 2 1	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 3 1	1 1 3 1	1 1 2 0	1 1 2 0	1 0 2 0	0 0 2 0	0 0 1 0
150	3 4 2 1	3 4 1 2	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 3 1	1 1 3 1	1 1 2 1	1 1 2 0	1 1 2 0	1 1 2 0	1 0 2 0
200	3 4 2 1	3 4 1 2	2 4 1 3	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	1 1 2 3	1 1 3 2	1 1 3 1	1 1 3 1	1 1 2 0	1 1 2 0	1 1 2 0	1 1 2 0
1000	3 4 2 1	3 4 1 2	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4	2 3 1 4
	2 2 3 3	2 2 3 2	2 2 3 1	2 2 3 1	2 2 3 1	1 1 3 0	1 1 3 0	1 1 3 0

BIBLIOGRAPHY

The abbreviations and style used are those adopted by the Annals of Mathematical Statistics. See, for example, Vol 33, page 313, 1962. At the end of the references appear notations regarding reviews of the corresponding references. The following notations are used:

GTMS: Guide to Tables in Mathematical Statistics by J. Arthur Greenwood and H. O. Hartley, Princeton University Press, Princeton, New Jersey, 1962.

IJA: International Journal of Abstracts, published for the International Statistical Institute by Oliver and Boyd, Edinburgh. Vol 1 (1959-60), Vol 2 (1961), Vol 3 (1962).

IMT: An Index of Mathematical Tables, Second Edition, 1962, by A. Fletcher, J. C. P. Miller, L. Rosenhead, and L. J. Comrie, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, Vol I and Vol II.

QC&AS: Quality Control and Applied Statistics, Interscience Publishers, Inc., New York, N. Y., Vol 1 (1956) to Vol 7 (1962).

MR: Mathematical Reviews, Published by the American Mathematical Society, Providence, R. I., Vol 1 (1940) to Vol 24 (1962).

MTAC: Mathematical Tables and other Aids to Computation, published by the National Academy of Sciences--National Research Council, Baltimore, Md., Vol 1 (1947) to Vol 13 (1959) or Mathematics of Computation, published by the American Mathematical Society, Providence, R. I., Vol 14 (1960) to Vol 16 (1962).

[1] ALBERT, G. E. and JOHNSON, R. B. (1951). On the estimation of certain intervals which contain assigned proportions of a normal univariate population. Ann. Math. Statist. **22** 596-599. See MR **13** 479, GTMS 6.119/304, and MTAC **7** 23.

[2] ALTMAN, IRVING B. (1957). The new MIL-STD-414 sampling inspection by variables. Industr. Qual. Contr. **14** (4) 23-26. See QC&AS **3** 953.

[3] ANDERSON, ROBERT H. (1961). Applications of MIL-STD-414 for quality assurance of nuclear materials. Trans. 8th Ann. Western Quality Conf., Anaheim, Calif. See QC&AS **7** 9.

[4] BOWKER, A. H. and GOODE, H. P. (1952). Sampling Inspection by Variables. McGraw-Hill, New York. See GTMS 14.121/477, and MTAC **7** 88(1073).

[5] BOWKER, A. H. and LIEBERMAN, G. J. (1955). Handbook of Industrial Statistics. Prentice-Hall, New Jersey.

[6] BOWKER, A. H. and LIEBERMAN, G. J. (1959). Engineering Statistics. Prentice-Hall, New Jersey.

[7] CHERNOFF, H. and LIEBERMAN, G. J. (1957). Sampling inspection by variables with no calculations. Industr. Qual. Contr. **13** (7) 5-7. See GTMS 14.122/479, and QC&AS **2** 639.

[8] CRAIG, CECIL C. (1941). Note on the distribution of noncentral t with an application. Ann. Math. Statist. **12** 224-228. See MR **3** 9, and GTMS 4.1412/220.

[9] CRAIG, CECIL C. (1961). Performance characteristics of acceptance sampling plans for in-use testing of electric meters. Industr. Qual. Contr. **18** 18-20. See QC&AS **7** 475.

BIBLIOGRAPHY (Continued)

[10] CROARKIN, MARY C. (1962). Graphs for determining the power of Student's t-test. J. Res., Nat. Bur. Stand. **66B** 59-70. See QC&AS **7** 1087.

[11] DUNCAN, ACHESON J. (1958). Design and operations of a double-limit variables sampling plan. J. Amer. Statist. Assoc. **53** 543-550. See QC&AS **4** 439.

[12] EISENHART, C., HASTAY, M. W., and WALLIS, W. A. (1947). Techniques of Statistical Analysis. McGraw-Hill, New York (A. H. Bowker wrote Chapter 2 of this book.) See MR **9** 365, and GTMS 14.122/479, 1.19/18, and 14.32/492.

[13] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F., and TRICOMI, F. G. (1953). Higher Transcendental Functions, Bateman Manuscript Project. **1** and **2** McGraw-Hill, New York. See MR **15** 868, and MTAC **11** 114(67).

[14] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F., and TRICOMI, F. G. (1954). Tables of Integral Transforms, Bateman Manuscript Project. **1** and **2** McGraw-Hill, New York. See MR **16** 468, and MTAC **10** 252(107).

[15] FRASER, D. A. S. and GUTTMAN, I. (1956). Tolerance regions. Ann. Math. Statist. **27** 162-17? See MR **17** 871, MTAC **11** 28(16), GTMS 14.32/493, and QC&AS **2** 267.

[16] GARNER, NORMAN R. (1958). Curtailed sampling for variables. J. Amer. Statist. Assoc., **53** 862-867. See QC&AS **4** 657, and MR **20** 6173.

[17] GOLDBERG, H. and LEVINE, H. (1946). Approximate formulas for the percentage points and normalization of t and χ^2. Ann. Math. Statist. **17** 216-225. See MR **8** 42, GTMS 2.314/147, and 4.132/216.

[18] GUTTMAN, IRWIN (1957). On the power of optimum tolerance regions when sampling from normal distributions. Ann. Math. Statist. **28** 773-778. See MR **19** 895, MTAC **12** 294(149), and QC&AS **3** 401.

[19] GUTTMAN, IRWIN (1959). Optimum tolerance regions and power when sampling from some non-normal universes. Ann. Math. Statist. **30** 926-938. See IJA **1** 399(4.5), QC&AS **5** 565, MTAC **15** 307(73), and MR **22** B285.

[20] HADER, R. J. (1959). Some basic aspects of sampling. Textile Quality Control Papers, Am. Soc. Quality Control, **6** 67-77. See QC&AS **4** 857.

[21] HALD, A. (1952). Statistical Theory with Engineering Applications. John Wiley, New York. See MR **14** 188.

[22] HAMAKER, H. C. (1961). A sampling table with adjustable scales. EOQC Bull., No. 14 440-443. See QC&AS **7** 473.

[23] HARLEY, BETTY I. (1957). Relation between the distributions of noncentral t and of a transformed correlation coefficient. Biometrika **44** 219-224. See MR **19** 472.

[24] HARRIS, M., HORVITZ, D. G., and MOOD, A. M. (1948). On the determination of sample sizes in designing experiments. J. Amer. Statist. Assoc. **43** 391-402. (See also MOOD [43].) See GTMS 4.1413/222.

[25] HENDRICKS, WALTER A. (1936). An approximation to Student's distribution. Ann. Math. Statist. **7** 210-221. See GTMS 4.132/213.

[26] HOGBEN, D., PINKHAM, R. S., and WILK, M. B. (1961). The moments of the noncentral t-distribution. Biometrika **48** 465-468. See IJA **3** 157(3.2).

[27] IRESON, W., RESNIKOFF, G. J., and SMIGH, B. E. (1961). Statistical tolerance limits for determining process capability. J. Indust. Eng. **12** 126-131. See QC&AS **6** 1153.

[28] JENNETT, W. J. and WELCH, B. L. (1939). The control of proportion defective as judged by a single quality characteristic varying on a continuous scale. J. Roy. Statist. Soc. Suppl. **6** 80-88. See GTMS 2.522/171.

[29] JÍLEK, M. and LÍKAR, O. (1959). Coefficients for the determination of one-sided tolerance limits of normal distribution. Ann. Inst. Statist. Math. Tokyo, **11** 45-48. See IJA **1** 401(4.5), and MTAC **15** 307(74).

[30] JÍLEK, M. and LÍKAR, O. (1960). Tolerance limits of the normal distribution with known variance and unknown mean. Austral. J. Statist. **2** 78-83. See IJA **2** 297(4.5) and QC&AS **6** 479.

[31] JÍLEK, M. and LÍKAR, O. (1960). Tolerance regions of the normal distribution with known μ and unknown σ. Biom. Zeit. **2** 204-209. See IJA **2** 489(4.5).

[32] JOHNSON, N. L. and WELCH, B. L. (1940). Applications of the noncentral t-distribution. Biometrika **31** 362-389. See MR **1** 346, and GTMS 4.1414/224.

[33] LAUBSCHER, NICO F. (1960). Normalizing the noncentral t and F distributions. Ann. Math. Statist. **31** 1105-1112. See IJA **3** 536(3.2), and MR **22** B8589.

[34] LEVERT, C. (1959). A nomogram for confidence intervals and exceedance probabilities. Statistica Neerlandica **13** 3-14. See IJA **1** 403(4.5), and QC&AS **4** 907.

[35] LIEBERMAN, ALFRED (1957). Table for the determination of two-sided tolerance limits for the normal distribution. Report No. 373-17(55), Bureau of Ships, Navy Department, Washington.

[36] LIEBERMAN, GERALD J. and RESNIKOFF, GEORGE J. (1955). Sampling plans for inspection by variables. J. Amer. Statist. Assoc. **50** 457-516. See MR **16** 1133, GTMS 14.121/730, and MTAC **12** 69(15).

[37] LIEBERMAN, GERALD J. (1958). Tables for one-sided statistical tolerance limit. Industr. Qual. Contr. **14** (10) 7-9. See GTMS 14.32/494, MTAC **14** 388(81), and QC&AS **3** 857.

[38] LORD, E. (1950). Power of the modified t-test (u-test) based on range. Biometrika **37** 64-77. See MR **12** 117 and GTMS 4.1412/221.

[39] MAGNESS, T. A. and MCGUIRE, J. B. (1962). Comparison of least squares and minimum variance estimates of regression parameters. Ann. Math. Statist. **33** 462-470.

[40] MERRINGTON, M. and PEARSON, E. S. (1958). An approximation to the distribution of non-central t. Biometrika **45** 484-491. See IJA **1** 39(3.2).

[41] MITRA, S. K. (1957). Table for tolerance limits for a normal population based on sample mean and range or mean range. J. Amer. Statist. Assoc. **52** 88-94. See MR **19** 74, GTMS 14.32/494, MTAC **12** 288(139), and QC&AS **2** 913.

[42] MOOD, ALEXANDER M. (1950). Introduction to the Theory of Statistics. McGraw-Hill, New York. See MR **11** 445.

[43] MOOD, ALEXANDER M. (1951). Erratum: on the determination of sample sizes in designing experiments. J. Amer. Statist. Assoc. **46** 515. See GTMS 4.1413/222.

[44] NEYMAN, J., IWASZKIEWICZ, K., and KOLODZIEJCZYK, St. (1935). Statistical problems in agriculture experimentation. J. Roy. Statist. Soc. Suppl. **2** 107-180. See GTMS 4.1412/220.

[45] NEYMAN, J. and TOKARSKA, B. (1936). Errors of the second kind in testing Student's hypothesis. J. Amer. Statist. Assoc. **31** 318-326. See GTMS 4.1413/222.

[46] Office of the Assistant Secretary of Defense (Supply and Logistics) (1957). Sampling procedures and tables for inspection by variables for percent defective, MIL-STD-414. See GTMS 14.121/478.

[47] Office of the Assistant Secretary of Defense (Supply and Logistics) (1958). Mathematical and statistical principles underlying MIL-STD-414. See QC&AS **5** 459.

[48] OWEN, DONALD B. (1956). Tables for computing bivariate normal probabilities. Ann. Math. Statist. **27** 1075-1090. See MR **23** B607, QC&AS **2** 663, MTAC **12** 286(135), IMT 15.517, and GTMS 1.723/121.

[49] OWEN, DONALD B. (1957). The bivariate normal probability distribution. Report No. SC-3831(TR), Sandia Corporation, Albuquerque; available from Office of Technical Services, Department of Commerce, Washington. See IMT 15.517 and MTAC **12** 285(134).

[50] OWEN, DONALD B. (1958). Tables of factors for one-sided tolerance limits for a normal distribution. Monograph No. SCR-13, Sandia Corporation, Albuquerque; available from Office of Technical Services, Department of Commerce, Washington. See MTAC **14** 388(82).

[51] OWEN, D. B. and WIESEN, J. M. (1959). A method of computing bivariate normal probabilities with an application to handling errors in testing and measuring. Bell System Tech. J. **38** 553-572. (Bell Telephone System Monograph No. 3210 and Sandia Corporation Reprint No. SCR-61). See QC&AS **4** 1133.

[52] OWEN, DONALD B. (1962). Handbook of Statistical Tables. Addison-Wesley, Massachusetts.

[53] OWEN, D. B. and AMOS, D. E. (1963). Programs for computing percentage points of the noncentral t-distribution. Monograph No. SCR-551, Sandia Corporation, Albuquerque; available from Office of Technical Services, Department of Commerce, Washington.

[54] PATNAIK, P. B. (1955). Hypotheses concerning the means of observations in normal samples. Sankhyā **15** 343-372. See MR **17** 872 and MTAC **12** 67(10).

[55] PAULSON, EDWARD (1943). A note on tolerance limits. Ann. Math. Statist. **14** 90-93. See MR **4** 280.

[56] PEARSON, E. S. and HARTLEY, H. O. (1951). Charts of the power function for analysis of variance tests, derived from the noncentral F-distribution. Biometrika **38** 112-130. See MR **13** 52, GTMS 4.1412/221, MTAC **8** 84(1183), and IMT 14.92.

[57] PEISER, A. M. (1943). Asymptotic formulas for significance levels of certain distributions. Ann. Math. Statist. **14** 56-62 and **20** 128-129. See MR **4** 222, MR **10** 467, and GTMS 4.132/215.

[58] PROSCHAN, FRANK (1953). Confidence and tolerance intervals for the normal distribution. J. Amer. Statist. Assoc. **48** 550-564. See MR **15** 240, GTMS 14.32/493, 4.1414/225, and MTAC **8** 85(1184).

[59] RESNIKOFF, G. J. and LIEBERMAN, G. J. (1957). Tables of the Noncentral t-Distribution. Stanford University Press, Stanford. See MR **19** 187, GTMS 4.1411/219, and MTAC **12** 297(155).

[60] RESNIKOFF, GEORGE J. (1962). Tables to facilitate the computation of percentage points of the noncentral t-distribution. Ann. Math. Statist. **33** 580-586. See MR **25** 650.

[61] SATTERTHWAITE, FRANKLIN E. (1946). An approximate distribution of estimates of variance components. Biometrics **2** 110.

[62] SCHEUER, ERNEST M. and SPURGEON, ROBERT A. (1962). Some percentage points of the noncentral t-distribution. J. Amer. Statist. Assoc. **58** 176-182.

BIBLIOGRAPHY (Continued)

[63] SESHADRI, V. and ODELL, P. L. (1959). A method for determining a confidence bound on unreliability when time-to-failure is normally distributed. US Naval Nuclear Ordnance Evaluation Unit, Albuquerque.

[64] SRIVASTAVA, A. B. L. (1958). Effect of non-normality on the power function of t-test. Biometrika **45** 421-430. See IJA **1** 40(3.2), MR **20** 6752, and QC&AS **4** 245.

[65] SRIVASTAVA, A. B. L. (1961). Variables sampling inspection for non-normal samples. J. Sci. Engrg. Res. **5** 145-152. See MR **24** A636.

[66] STANGE, K. (1958). The graphic treatment of plans for quantitative inspection. Metrika **1** 111-129. See IJA **1** 106(8.8).

[67] STANGE, K. (1960). Variables acceptance sampling with lots of known variance. Qualitatskontrolle Operational Research **5** 2-7. See QC&AS **5** 1127.

[68] STANGE, K. (1960). Constructing range-plans for variables inspection by use of double probability paper. Metrika **3** 151-165. See IJA **2** 136(8.8).

[69] STANGE, K. (1960). Statistical test plans. Part III: plans using mean value and mean range obtained from subsamples. Qualitatskontrolle Operational Research **5** 75-80. See QC&AS **6** 361.

[70] STECK, G. P. and OWEN, D. B. (1959). Percentage points for the distribution of outgoing quality. J. Amer. Statist. Assoc. **54** 689-694. See IJA **2** 361(8.8), and QC&AS **5** 9.

[71] STORER, R. L. and DAVISON, W. R. (1955). Simplified procedures for sampling inspection by variables. Industr. Qual. Contr. **12** (1) 15-18. See GTMS 14.122/479, and MTAC **11** 34(30).

[72] van EEDEN, CONSTANCE (1961). Some approximations to the percentage points of the noncentral t-distribution. Rev. Int. Statist. Inst. **29** 4-31. See IJA **1** 383(3.2).

[73] WALD, ABRAHAM (1942). Setting of tolerance limits when the sample is large. Ann. Math. Statist. **13** 389-399. See MR **4** 165.

[74] WALD, A. and WOLFOWITZ, J. (1946). Tolerance limits for a normal distribution. Ann. Math. Statist. **17** 208-215. See MR **8** 478 and GTMS 1.19/17.

[75] WALLIS, W. ALLEN (1951). Tolerance intervals for linear regressions. Second Berkeley Symposium on Mathematical Statistics and Probability (edited by J. Neyman). University of California Press, Berkeley. 43-51. See MR **13** 481.

[76] WEISSBERG, A. and BEATTY, G. H. (1960). Tables of tolerance-limit factors for normal distributions. Technometrics **2** 483-500. See IJA **2** 671(4.5), MTAC **15** 211(47), and QC&AS **6** 97.

[77] WILKS, SAMUEL S. (1941). Determination of sample sizes for setting tolerance limits. Ann. Math. Statist. **12** 91-96. See MR **3** 9.

[78] WOLFOWITZ, JACOB (1946). Confidence limits for the fraction of a normal population which lies between two given limits. Ann. Math. Statist. **17** 483-488. See MR **8** 394.

[79] WOODS, WALTER M. (1960). Variables sampling inspection procedures which guarantee acceptance of perfectly screened lots. Applied Math. and Statist. Laboratories, Stanford University, Report No. 47. See QC&AS **6** 1253.

[80] ŽALUDOVÁ, AGNES H. (1958). Statistical quality control of production processes using individual sample values. Bull. Int. Statist. Inst. **36** III 573-578. See IJA **2** 141(8.9).

[81] ŽALUDOVÁ, AGNES H. (1960). The noncentral t-test (q-test) based on range in place of standard deviation. Acta Technica 143-185.

[82] ŽALUDOVÁ, AGNES H. (1961). Note on the noncentral t-test based on range. Second Hungarian Mathematical Congress. See IJA **2** 482(3.2).

BIBLIOGRAPHY (Continued)

[63] SESHADRI, V. and ODELL, P. L. (1959). A method for determining a confidence bound on unreliability when time-to-failure is normally distributed. US Naval Nuclear Ordnance Evaluation Unit, Albuquerque.

[64] SRIVASTAVA, A. B. L. (1958). Effect of non-normality on the power function of t-test. Biometrika **45** 421-430. See IJA **1** 40(3.2), MR **20** 6752, and QC&AS **4** 245.

[65] SRIVASTAVA, A. B. L. (1961). Variables sampling inspection for non-normal samples. J. Sci. Engrg. Res. **5** 145-152. See MR **24** A636.

[66] STANGE, K. (1958). The graphic treatment of plans for quantitative inspection. Metrika **1** 111-129. See IJA **1** 106(8.8).

[67] STANGE, K. (1960). Variables acceptance sampling with lots of known variance. Qualitatskontrolle Operational Research **5** 2-7. See QC&AS **5** 1127.

[68] STANGE, K. (1960). Constructing range-plans for variables inspection by use of double probability paper. Metrika **3** 151-165. See IJA **2** 136(8.8).

[69] STANGE, K. (1960). Statistical test plans. Part III: plans using mean value and mean range obtained from subsamples. Qualitatskontrolle Operational Research **5** 75-80. See QC&AS **6** 361.

[70] STECK, G. P. and OWEN, D. B. (1959). Percentage points for the distribution of outgoing quality. J. Amer. Statist. Assoc. **54** 689-694. See IJA **2** 361(8.8), and QC&AS **5** 9.

[71] STORER, R. L. and DAVISON, W. R. (1955). Simplified procedures for sampling inspection by variables. Industr. Qual. Contr. **12** (1) 15-18. See GTMS 14.122/479, and MTAC **11** 34(30).

[72] van EEDEN, CONSTANCE (1961). Some approximations to the percentage points of the noncentral t-distribution. Rev. Int. Statist. Inst. **29** 4-31. See IJA **1** 383(3.2).

[73] WALD, ABRAHAM (1942). Setting of tolerance limits when the sample is large. Ann. Math. Statist. **13** 389-399. See MR **4** 165.

[74] WALD, A. and WOLFOWITZ, J. (1946). Tolerance limits for a normal distribution. Ann. Math. Statist. **17** 208-215. See MR **8** 478 and GTMS 1.19/17.

[75] WALLIS, W. ALLEN (1951). Tolerance intervals for linear regressions. Second Berkeley Symposium on Mathematical Statistics and Probability (edited by J. Neyman). University of California Press, Berkeley. 43-51. See MR **13** 481.

[76] WEISSBERG, A. and BEATTY, G. H. (1960). Tables of tolerance-limit factors for normal distributions. Technometrics **2** 483-500. See IJA **2** 671(4.5), MTAC **15** 211(47), and QC&AS **6** 97.

[77] WILKS, SAMUEL S. (1941). Determination of sample sizes for setting tolerance limits. Ann. Math. Statist. **12** 91-96. See MR **3** 9.

[78] WOLFOWITZ, JACOB (1946). Confidence limits for the fraction of a normal population which lies between two given limits. Ann. Math. Statist. **17** 483-488. See MR **8** 394.

[79] WOODS, WALTER M. (1960). Variables sampling inspection procedures which guarantee acceptance of perfectly screened lots. Applied Math. and Statist. Laboratories, Stanford University, Report No. 47. See QC&AS **6** 1253.

[80] ŽALUDOVÁ, AGNES H. (1958). Statistical quality control of production processes using individual sample values. Bull. Int. Statist. Inst. **36** III 573-578. See IJA **2** 141(8.9).

[81] ŽALUDOVÁ, AGNES H. (1960). The noncentral t-test (q-test) based on range in place of standard deviation. Acta Technica 143-185.

[82] ŽALUDOVÁ, AGNES H. (1961). Note on the noncentral t-test based on range. Second Hungarian Mathematical Congress. See IJA **2** 482(3.2).