

TECHNICAL NOTE

**TWO-STAGE MOISTURE DIFFUSION IN WOOD  
WITH CONSTANT TRANSPORT COEFFICIENTS**

**Jen Y. Liu and William T. Simpson**  
U.S. Department of Agriculture, Forest Service  
Forest Products Laboratory<sup>1</sup>, Madison, Wisconsin 53705-2398

**Key Words:** desorption; diffusion coefficient; diffusion equation; surface emission coefficient; wood

**ABSTRACT**

To determine whether transport coefficients in desorption curves for northern red oak are constant, four sets of desorption data were compared against previously established mathematical conditions for infinite-series solution non-steady-state diffusion equation. For each data set, when moisture fraction in wood is above a certain value, designated as the first stage, these conditions are satisfied, with the diffusion and surface emission coefficients being positive and finite; below that value, designated as the second stage, these conditions are still

---

<sup>1</sup>The Forest Products Laboratory is maintained in cooperation with the University of Wisconsin. This article was written and prepared by U.S. Government employees on official time, and it is therefore in the public domain and not subject to copyright. The use of trade or firm names is for information only and does not imply endorsement by the U.S. Department of Agriculture of any products or service.

satisfied with the diffusion coefficient taking a smaller positive value but the surface emission coefficient becoming negative and finite. Mathematically, these two pairs of transport coefficients can be used to predict the whole diffusion curve that describes the variation of moisture fraction with time. However, the negative surface emission coefficient in the second stage of the desorption process implies that the moisture gradient has cut the surface at a point below the equilibrium moisture content which is physically impossible. Alternatively, the second stage can be considered as a new stage with moisture fraction values normalized with respect to the lowest value in the first stage which is also the starting point of the second stage. The transport coefficients are obtained in the same manner as in the first stage and are found to be positive and finite. The two pairs of transport coefficients can describe the diffusion curve with high accuracy. However, the assumption used in the second stage is that the initial moisture content in wood is uniform which again is physically impossible. We therefore conclude that the transport coefficients for northern red oak are not constant. Although the two-stage approaches presented in this study can predict the diffusion curves accurately, their physical interaction is difficult to justify.

## INTRODUCTION

The diffusion equation can be used to predict moisture distribution in wood in an isothermal drying process because the thermal conductivity for wood is orders of magnitude higher than the diffusion coefficient. In previous work, the authors developed the mathematical conditions required to achieve constant transport coefficients in the non-steady-state diffusion equation [7]. In this paper, those conditions are used to examine resorption data of northern red oak (*Quercus rubra*). Results are analyzed from both mathematical and physical points of view.

Observations or assumptions for moisture diffusion transport coefficients are inconsistent in the literature. Some authors have reported the diffusion coefficient as moisture dependent [e.g., refs. 5, 8, 10, 12], while others have taken it as constant [e.g., refs. 1, 2, 4, 13]. The surface emission coefficient has been reported to depend on ambient conditions either completely [e.g., refs. 1, 14] or in combination with material surface conditions [e.g., refs. 2, 6, 9, 11].

In previous work [6], we presented closed-form solutions of the non-steady-state diffusion equation with the transport coefficients assumed to be constant, and then established the mathematical conditions under which the transport coefficients can be taken as constant. We demonstrated the application of our derived equations using four sets of resorption data of northern red oak. However, only portions of the data were analyzed and our conclusions were therefore incomplete. In the study reported here, we considered the four complete sets of data to determine whether they contain additional information.

### MATHEMATICAL MODEL

In a one-dimensional formulation with moisture moving in the direction normal to a wood plate of thickness  $2a$ , the diffusion equation can be written as

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial X} \left( D \frac{\partial C}{\partial X} \right) \quad (0 < X < a, t > 0) \quad (1)$$

where  $C$  is moisture content,  $t$  is time,  $D$  is a diffusion coefficient assumed to be constant and  $X$  is a space coordinate measured from the center of the plate.

Let the initial condition be

$$C = C_0 \quad (0 < X < a, t = 0) \quad (2)$$

where  $C_0$  is constant moisture content in the plate, and let the boundary conditions be

$$\frac{\partial C}{\partial X} = 0 \quad (X = 0, t \geq 0) \quad (3)$$

$$D \frac{\partial C}{\partial X} = S(C_e - C) \quad (X = a, t > 0) \quad (4)$$

where  $S$  is the surface emission coefficient and  $C_e$  is equilibrium moisture content.

### RESORPTION SOLUTIONS AND ANALYSES

Infinite-series solution of Equation (1) under the conditions of Equations (2) to (4) for resorption with both  $D$  and  $S$  as constant can be found in Crank [3] as

$$E = \frac{\bar{C} - C_e}{C_0 - C_e} = 2L^2 \sum_{n=1}^{\infty} \frac{\exp(-\beta_n^2 \tau)}{\beta_n^2 (\beta_n^2 + L^2 + L)} \quad (5)$$

where  $E$  is fraction of total moisture content in the plate,  $\bar{C}$  is mean moisture content,  $L$  is transport ratio defined as

$$L = \frac{aS}{D} \quad (6)$$

$\tau$  is dimensionless time defined as

$$\tau = \frac{Dt}{a^2} \quad (7)$$

and  $\beta_n$  are the positive roots of

$$\beta_n \tan \beta_n = L \quad (8)$$

Equation (5) can be reduced to closed form according to Liu and Simpson [7] as

$$\tau = -\frac{4}{\pi^2} \ln \left( \frac{\pi^2 E}{8} \right) - \frac{1}{L} \ln E \quad (0 \leq E < 0.48) \quad (9)$$

and

$$\tau = \left[ \frac{\pi(1-E)^2}{4} \right] - \frac{1}{L} \ln E \quad (1 \geq E \geq 0.48) \quad (10)$$

Substituting Equations (6) and (7) into Equation (9) yields

$$\frac{Dt}{a^2} = -\frac{4}{\pi^2} \ln \left( \frac{\pi^2 E}{8} \right) - \frac{D}{aS} \ln E$$

or

$$\frac{t}{a^2} = -\frac{4}{\pi^2 D} \ln \left( \frac{\pi^2 E}{8} \right) - \frac{1}{aS} \ln E \quad (0 \leq E < 0.48) \quad (11)$$

Let

$$a' = \frac{t}{a^2}, \quad b' = -\frac{4}{\pi^2} \ln \left( \frac{\pi^2 E}{8} \right), \quad c' = \frac{1}{a} \ln E \quad (12)$$

in which  $t$  and  $E$  are test data and  $a$  is half-thickness of the plate. Equation (11) can be put in the form

$$a'/b' = 1/D - (1/S)(c'/b') \quad (13)$$

Likewise, for Equation (10), we obtain

$$a' = \frac{t}{a^2}, \quad b' = \frac{\pi(1-E)^2}{4}, \quad c' = \frac{1}{a} \ln E \quad (14)$$

which are to be used in Equation (13) for  $1 \geq E \geq 0.48$ .

For  $D$  and  $S$  to be constant and positive we must have

$$a'/b' > 1/D \quad (15)$$

since  $a'$  and  $b'$  are positive and  $c'$  is negative. Also, the plot of  $a'/b'$  versus  $c'/b'$  must form a straight line so that

$$\frac{d(a'/b')}{d(-c'/b')} = \frac{1}{S} \quad (16)$$

and

$$a'/b' = 1/D \quad \text{at } c'/b' = 0 \quad (17)$$

which can be obtained by extrapolating the straight line representing the test data.

Multiplying through Equation (13) by  $b'/c'$ , we obtain

$$a'/c' = (1/D)(b'/c') - 1/S \quad (18)$$

The plot of  $a'/c'$  versus  $b'/c'$  must also form a straight line so that

$$\frac{d(a'/c')}{d(b'/c')} = \frac{1}{D} \quad (19)$$

and

$$a'/c' = -1/S \quad \text{at } b'/c' = 0 \quad (20)$$

From test data expressed as  $a'$ ,  $b'$ , and  $c'$  in Equations (12) and (14), we can determine  $S$  and  $D$  from Equations (16) and (17) or (19) and (20).

## RESULTS AND DISCUSSION

Desorption test data of northern red oak (*Quercus rubra*) by Simpson [10] were used for numerical analysis. The relevant data are shown in Table 1.

Figure 1 presents the variation of  $a'/b'$  with  $c'/b'$  for each of the four sets of data in Table 1. For each set, the data points can be approximately represented by two straight lines, the first line inclining downward and the second line rising upward, from left to right. Clearly, the first line gives a positive value of  $S$  and the second line a negative value of  $S$  according to Equation (16). The corresponding values for  $D$  obtained from Equations (17) or (19) are different but are positive and finite. In the following text, two approaches will be used to analyze these data.

### Transport Coefficients Based on Original Data

Let the moisture fraction corresponding to the smallest value of  $a'/b'$  in each data set in Figure 1 be the point of demarcation. Above this point, the

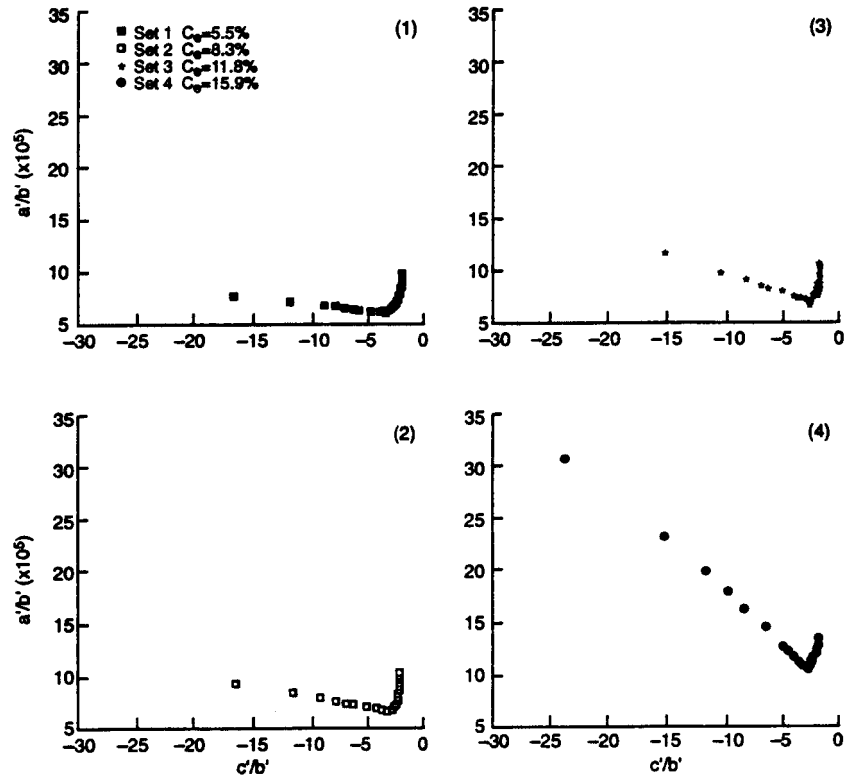


FIGURE 1. Variation of  $a'/b'$  with  $c'/b'$ .

TABLE 1. Desorption of Northern Red Oak

Set	Specimens	Specimen thickness, $2a$ (cm)	Initial moisture content, $C_0$ (%)	EMC, $C_e$ (%)	RH (%)
1	5	2.50	35.89	5.5	33
2	5	2.42	28.97	8.3	53
3	5	2.54	32.11	11.8	70
4	5	2.50	25.92	15.9	83

moisture fraction values form the first stage of resorption, below it, they form the second stage of desorption. The  $D$  and  $S$  values obtained using the method of least squares are shown in Table 2.

Because these transport coefficients are based on the test data as represented in Figure 1, they can best describe the diffusion curves shown in Figure 2. However, since  $S$  is negative in the second stage, it is equivalent to saying that in Equation (4),  $C$  and  $C_e$  on the right-hand side should be transposed for  $S$  to be positive while the left-hand side remains unchanged. Mathematically, this means that in the description process of the second stage, additional moisture should enter the material to accurately describe the test data. This can also be seen in Figure 2. When the transport coefficients of the first stage are used in the second stage, the predictions fall below the test data, implying more moisture is needed to drive them up. Such a phenomenon is physically unreasonable.

#### Transport Coefficients Based on Modified Data

To avoid the negative  $S$  values, the data in the second stage can be modified by setting

$$E' = E/E_0 \quad \text{and} \quad t' = t - t_0 \quad (21)$$

where  $E_0$  and  $t_0$  are the moisture fraction and time for the demarcation point, which now becomes the starting point of the second stage. The data thus modified are processed in the same way as those in the first stage; the  $D$  and  $S$  values obtained together with  $\bar{C}$  in Equation (5) and  $E_0$  in Equation (21) are shown in Table 3. Note that  $\bar{C}$ , the average moisture content at  $E_0$ , is only slightly larger than 20% in all cases.

The  $E'$  and  $t'$  values obtained using the  $D$  and  $S$  values in Table 3 must be converted to  $E$  and  $t$  values using Equation (21) to be combined with results of the



TABLE 2. Transport Coefficients Based on Original Data

Set	First stage		Second stage	
	$D$ (cm <sup>2</sup> /s)	$S$ (cm/s)	$D$ (cm <sup>2</sup> /s)	$S$ (cm/s)
1	1.6983E-6	8.8131E-5	6.0120E-7	-2.6990E-6
2	1.4580E-6	5.6230E-5	6.9480E-7	-4.0830E-6
3	1.5715E-6	3.0100E-5	1.0033E-6	-9.7765E-6
4	1.2613E-6	1.0254E-5	4.9593E-7	-2.8601E-6

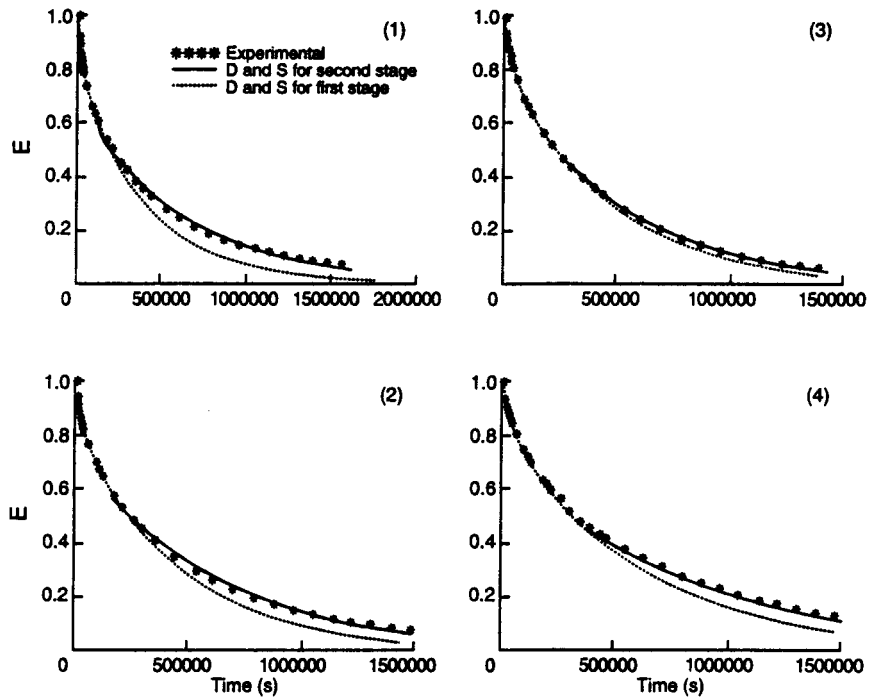


FIGURE 2. Variation of moisture fraction  $E$  with time  $t$ .

TABLE 3. Transport Coefficients Baaed on Modified Data<sup>a</sup>

Set	$E_0$ (%)	$\bar{C}$ (%)	$D$ (cm <sup>2</sup> /s)	$S$ (cm/s)
1	61.58	24.21	4.5455E-6	3.1177E-6
2	64.53	21.64	4.6948E-6	3.0010E-6
3	47.24	21.39	1.4991E-5	2.5566E-6
4	49.32	20.84	1.4623E-5	1.4696E-6

<sup>a</sup>For first stage  $D$  and  $S$  values, see Table 2.

that stage. Where the agreement between the predictions and the test data is not adequate, this approach can be repeated until they agree. We found that two stages are enough to cover any practical moisture ranges with high accuracy. Our results were similar to those in Figure 2 and are not plotted. The assumption in this approach is that at the demarcation point, the moisture content is uniform as described in Equation (2). Again, this is mathematically workable but physically impossible.

### CONCLUSIONS

The resorption curves for northern red oak can be divided into two stages, each characterized by a different pair of constant transport coefficients. Two approaches can be used to determine the transport coefficients for the second stage one assumes the surface emission coefficient can be negative, and the other assumes the initial moisture content is uniform. Both approaches can describe the resorption cures accurately, but they lack physical justification. This lends further proof to the contention that transport coefficients cannot be constant for northern red oak.

## NOMENCLATURE

$a$	Half-thickness of wood plate (cm)
$C$	Moisture content (%)
$C_e$	Equilibrium moisture content (%)
$C_0$	Initial moisture content (%)
$D$	Diffusion coefficient (cm <sup>2</sup> /s)
$E$	Fraction of total moisture content (Equation (5))
$L$	Transport ratio (Equation (6))
$S$	Surface emission coefficient (cm/s)
$t$	Time (s)
$X$	space coordinate (cm)
$\tau$	Dimensionless time

## REFERENCES

1. Avramidis, S. and Siau, J.F., 1987, An Investigation of the External and Internal Resistance to Moisture Diffusion in Wood, *Wood Science Technology*, 21(3) pp. 249–256.
2. Choong, E.T. and Skaar, C., 1972, Diffusivity and Surface Emissivity in Wood Drying, *Wood Fiber*, 4 (2) pp. 80–86.
3. Crank, J., 1975, *The Mathematics of Diffusion*, Chapter 4, 2d ed., Clarendon Press, Oxford.
4. Droin, A., Taverdet, J.L., and Vergnaud, J.M., 1988, Modeling the Kinetics of Moisture Adsorption by Wood, *Wood Science Technology*, 22(1) pp. 11–20.

5. Hougén, C.A., McCauley, H.J., and Marshall, W.R., Jr., 1940, Limitations of Diffusion Equations in Drying, *Transactions of the American Institute of Chemical Engineers*, 36, pp. 183–210.
6. Liu, J.Y. and Simpson, W.T., 1996, Mathematical Relationship Between Surface Emission and Diffusion Coefficients, *Drying Technology—An International Journal*, 14(3,4) pp. 677–699.
7. Liu, J.Y. and Simpson, W.T., 1997, Solutions of Diffusion Equation With Constant Diffusion and Surface Emission Coefficients, *Drying Technology—An International Journal*, 15(10) pp. 2459–2477.
8. Meroney, R.N., 1969, The State of Moisture Transport Rate Calculations in Wood Drying, *Wood Fiber*, 1(1) pp. 64–74.
9. Rosen, H.N., 1982, Functional Relations and Approximation Techniques for Characterizing Wood Drying Curves, *Wood Science*, 15(1) pp. 49–55.
10. Simpson, W.T., 1993, Determination and Use of Moisture Diffusion Coefficient to Characterize Drying of Northern Red Oak, *Wood Science Technology*, 27(6) pp. 409–420.
11. Simpson, W.T. and Liu, J.Y., 1991, Dependence of the Water Vapor Diffusion Coefficient of Aspen Wood on Moisture Content, *Wood Science Technology*, 26(1) pp. 9–22.
12. Skaar, C., 1954, Analysis of Methods for Determining the Coefficient of

Moisture Diffusion in Wood, *Journal of Forest Products Research Society*, 4  
(6) pp. 403–410.

13. Söderström, O. and Salin, J.G., 1993, On Determination of Surface Emission  
Factors in Wood Drying, *Holzforschung*, 47(5) pp. 391–397.

14. Wadsö, L., 1993, Surface Mass Transfer Coefficients for Wood, *Drying  
Technology—An International Journal*, 11(6) pp. 1227–1249.