Probability Bounds Analysis Is a Global Sensitivity Analysis

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Abstract: Probability bounds analysis provides analysts a convenient means to characterize the neighborhood of possible results that would be obtained in probabilistic calculations of plausible alternative inputs. For this reason, it constitutes a method of global sensitivity analysis that does not require any notion of decomposing or partitioning total uncertainty. We show the relationship between probability bounds analysis and the methods of interval analysis and probabilistic sensitivity analysis from which it is jointly derived, and indicate how the method can be used to assess the quality of probabilistic models such as those developed in Monte Carlo simulations for risk analyses. We also illustrate how a meta-level sensitivity analysis can be conducted within a probability bounds analysis by pinching inputs to precise distributions or real values.

Keywords: probability bounds analysis, interval analysis, second-order probability, sensitivity analysis, convolution, robust Bayes, meta-level sensitivity analysis

1. INTRODUCTION

Sensitivity analysis is the general term for quantitative study of how the inputs to a model influence the results of the model. Sensitivity analysis has many manifestations in probabilistic risk analyses and there are many disparate approaches based on various measures of influence and response. Sensitivity analyses are conducted for fundamentally two reasons: to understand the reliability of the conclusions and inferences drawn from an assessment, and to focus future empirical studies so that effort might be expended to improve estimates of inputs that would lead to the most improvement in the estimates of the outputs. Because of the obvious and fundamental importance of sensitivity analyses in calculations, there has been a confluence of ideas to this issue from disparate analytical disciplines.

Leamer [1] defined global sensitivity analysis as a systematic study in which "a neighborhood of alternative assumptions is selected and the corresponding interval of inferences is identified". There are two disparate ways to effect such a study. One natural way is to bound the neighborhood with interval ranges. Another natural way is to ascribe a probability distribution to the elements in the neighborhood. Consider, for example, the context of a deterministic calculation. When the model involves uncertainty about the real-valued quantities used in the calculation, the definition of global sensitivity analysis is equivalent to that of interval analysis [2,3,4,5]. Probability theory, implemented perhaps by Monte Carlo simulation, can also be viewed as a global sensitivity analysis of a deterministic calculation in that it yields a distribution describing the probability of alternative possible values about a point estimate [6,7,8,9]. In the figure below these two possible paths are shown as right and left downward arrows respectively.

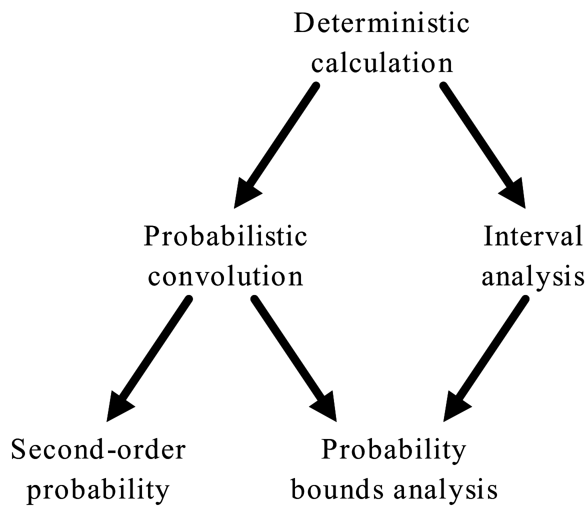

Figure 1. Relationships among different calculation strategies. Arrows represent generalizations.

Of course, the calculations on which it might be desirable to conduct sensitivity analyses are not all deterministic. In fact, many of them are already probabilistic, as is the case in most modern risk analyses and safety assessments. One could construct a probabilistic sensitivity analysis of a probabilistic calculation. The resulting analysis would be a second-order probabilistic assessment. However, such studies are often difficult to conduct because of the large number of calculations that are required. It is also sometimes difficult to visualize the results in a way that is easily comprehensible. Alternatively, one could apply bounding arguments to the probabilistic calculation and arrive at interval versions of probability distributions. We call such calculations "probability bounds analysis" (PBA) [10,11,12]. This approach represents the uncertainty about a probability distribution by the set of cumulative distribution functions lying entirely within a pair of bounding distributions called a "probability box" or a "p-box". Probability bounds analysis is a global sensitivity analysis of a probabilistic calculation because it defines neighborhoods of probability distributions (i.e., the p-boxes) that represent the uncertainty about imperfectly known input distributions and projects this uncertainty through the model to identify a neighborhood of answers (also characterized by a p-box) in a way that guarantees the resulting bounds will entirely enclose the cumulative distribution function of the output. A probability distribution is to a p-box the same way a real scalar number is to an interval. The bounding distributions of the p-box enclose all possible distributions in the same way that the endpoints of the interval circumscribe the possible real values.

Probability bounds analysis is related to other forms of uncertainty analysis. It is a marriage of probability theory and interval analysis that generalizes and is faithful to both traditions. As depicted in Figure 1, PBA can arise either by bounding probability distributions (the left path down to PBA) or by forming probability distributions of intervals (the right path). The advantage of this marriage is that variability (aleatory uncertainty) and incertitude (epistemic uncertainty) are treated separately and propagated differently so each maintains its own character. PBA is a comprehensive global sensitivity analysis that is an alternative to complicated second-order or nested Monte Carlo methods. PBA is very similar in spirit to Bayesian sensitivity analysis (which is also known as robust Bayes [13]), although the former concerns arithmetic and convolutions, and the latter addresses the issues of updating and aggregation. Unlike Bayesian sensitivity analysis, probability bounds analysis is always easy to employ because it does not depend on the use of conjugate pairs to make calculations simple. PBA is a practical approach to computing with imprecise probabilities [14]. Like a Bayesian sensitivity analysis, imprecise probabilities are represented by a class of distribution functions. PBA is simpler because it defines the class solely by reference to two bounding distributions. (It therefore cannot fully represent a situation in which there are intermediate distributions lying within the bounds that are excluded from the class. In the context of risk and safety assessments, however, this is rarely a significant drawback.)

2. PBA CIRCUMSCRIBES POSSIBLE DISTRIBUTIONS GIVEN UNCERTAINTY

PBA can produce rigorous bounds around the output distribution from an assessment. These bounds enclose all the possible distributions that could actually arise given what is known and what is not known about the model and its inputs. Because it is based on the idea of bounding rather than approximation, it provides an estimate of its own reliability [15,16, cf. 17]. Probability bounds analysis can comprehensively account for possible deviations in assessment results arising from uncertainty about

- distribution parameters,
- distribution shape or family,
- intervariable dependence, and even
- model structure.

Moreover, it can handle all of these kinds of uncertainties in a single calculation that gives a simple and rigorous characterization of how different the result could be given all of the professed uncertainty. The requisite computations used in PBA are actually quite simple and have been implemented in straightforward algorithms [18,19,15,16,11,20]. The computations are generally much faster than even simple Monte Carlo convolution and vastly faster than a numerically intensive sensitivity analysis with traditional methods [21,22,23,24,6,7,25,8].

Probability bounds analysis is useful whenever the uncertainty about the marginal distributions can be characterized by interval bounds about their cumulative distribution functions. These bounds can be specified using empirical information available about each distribution. For instance, if the parameters of a normal distribution can be given within interval ranges, best-possible bounds on the distribution are easy to construct. If the shape of the underlying distribution is not known, but some statistics such as the mean, mode, variance, etc. can be specified (or given as intervals), rigorous bounds can generally be constructed that are guaranteed to enclose the true distribution subject to the given constraints. Often these bounds will be optimally narrow given the stated information. The resulting p-boxes are distribution-free in the sense that they make no assumptions whatever about the distribution family (whether it is normal, lognormal, Weibull, etc.). Such bounds on distributions can then be combined according to the calculations in the assessment. Currently, software is available to handle (i) arithmetic convolutions (addition, multiplication, minimum, etc.), (ii) magnitude comparisons (greater than, less than), (iii) logical operations (conjunction, disjunction, etc.), and (iv) transformations (logarithm, exponentiation, roots, etc.).

It is also possible to handle uncertainty about the dependencies among variables in a model. Recent algorithmic developments permit uncertainty about the dependencies among variables to be propagated through the calculations of a probabilistic assessment [26]. A pair-wise dependency may be modeled with any of the following six assumptions:

(i) independence,
(ii) comonotonicity (maximal correlation),
(iii) countermonotonicity (minimal correlation),
(iv) linear relationship and correlation within a specified interval,
(v) linear relationship with unknown correlation,
(vi) signed (positive or negative) but otherwise unknown dependency, and
(vii) unknown dependency (including any nonlinear relationship).

For the first three cases, a convolution between two probability distributions yields a well defined probability distribution. For the latter four cases, the results are given as bounds on a (cumulative) distribution function. For each binary operation, the bounds obtained are generally the best possible bounds, i.e., they could not be any narrower yet still contain all the possible distributions permissible under the assumption.

Unlike approaches based on conventional Monte Carlo simulation, the algorithms employed for these operations yield rigorous answers that lack sampling error. In fact, the results are exact at each point of discretization, of which there may be arbitrarily many. The results are guaranteed to enclose the true distributions. Although it is straightforward to ensure that bounds remain rigorous (sure to contain the true distributions) in sequential calculations, the best possible nature of the bounds may be lost in some complication calculations. Maintaining the optimality of the bounds is, in general, a computationally challenging task that can require other methods [14]. Nevertheless, the methods of probability bounds analysis developed over the last two decades provide risk and safety analysts a practical and convenient means to conduct comprehensive sensitivity analyses on their calculations.

3. META-LEVEL SENSITIVITY ANALYSES

As outlined above, probability bounds analysis is a kind of sensitivity analysis with considerable comprehensiveness. It is possible and sometimes of interest to perform a sensitivity analysis on the results of an assessment conducted with PBA. This would, of course, constitute a meta-level sensitivity analysis. This section explores the use of pinching studies that hypothetically assess the impact on result uncertainty of additional empirical knowledge.

One of the fundamental purposes of sensitivity studies is to learn where focusing future empirical efforts would be most productive. This purpose requires estimating the value of additional empirical information. Of course, the value of information not yet observed cannot be measured, but it can perhaps be predicted. One strategy to this end is to assess how much less uncertainty the calculations would have if extra knowledge about an input were available. This might be done by comparing the uncertainty before and after "pinching" an input, i.e., replacing it with a value without uncertainty. Of course, one does not generally know the correct value without uncertainty, so this replacement must be conjectural in nature. To pinch a parameter means to hypothetically reduce its uncertainty for the purpose of the thought-experiment. The experiment asks what would happen if there were less uncertainty about this number. Quantifying this effect amounts to measuring the contribution by the input to the overall uncertainty in a calculation.

The estimate of the value of information for a parameter will depend on how much uncertainty is present in the parameter, and how it affects the uncertainty in this final result. The sensitivity could be computed with an expression like

$$100\left(1-\frac{\text{unc}(T)}{\text{unc}(B)}\right)\%$$

where B is the base value of the risk expression, T is the value of the risk expression computed with an input pinched, and unc( ) is a measure of the uncertainty of the answer. The result is an estimate of the value of additional empirical information about the input in terms of the percent reduction in uncertainty that might be achieved in the expression when the input parameter is replaced by a better estimate obtained from future empirical study. The pinching can be applied to each input quantity in turn and the results used to rank the inputs in terms of their sensitivities. (Note that these reductions will not generally add up to 100% for all the input variables.) In principle, one could also pinch multiple inputs simultaneously to study interactions.

There are multiple possible ways to define unc( ) to measure uncertainty. In the context of probability bounds analysis, one obvious measure is the area between the upper and lower bounds of the p-box. As the p-box approaches a precise probability distribution where all epistemic incertitude has evaporated and only the natural variability remains, this area approaches zero. An analyst might also elect to define unc( ) as some measure of dispersion or perhaps the heaviness of the tails [27] of the p-box. Using different measures will obviously allow the analyst to address different questions in a sensitivity analysis. If the measure of uncertainty is a scalar quantity (i.e., a real number), then the sensitivities that come from the analysis will also be scalars and can be ordered.

There are also multiple possible ways to pinch uncertainty. Pinching in different ways can result in strongly different estimates of the overall value of information. Several strategies are possible in estimating sensitivities from comparative PBA assessments:

(i) replace an input with a point value,
(ii) replace an input with a precise distribution function, or
(iii) replace an input with a zero-variance interval.

Replacing a p-box with a precise probability distribution would be pinching away the incertitude about the distribution. Replacing a p-box or a distribution function with a point value would be pinching away both the incertitude and the variability of the quantity. For inputs that are known to be variable (variance greater than zero), such a pinching is counterfactual, but it may nevertheless be informative. In particular, in may be especially useful in planning remediating strategies. In some situations, it may be reasonable to replace a p-box with a p-box shaped like an interval but prescribed to have a variance of zero. The effect of this would be to pinch away the variability but leave uncertainty. Such a replacement might be reasonable for p-boxes having a core (a region along the abscissa for which the upper bound of the p-box is one and lower bound is zero).

This approach of pinching inputs and recalculating the assessment is not unfamiliar to Monte Carlo analysts. Many routinely conduct sensitivity studies of the proportional contribution of variability in each variable to the overall variability in the calculated risk distribution. To determine the effects of variability in a Monte Carlo simulation using this method, each variable containing variability (i.e., expressed as a probability distribution) is reduced in turn to its mean or other appropriate point value, and the simulation is repeated. The measure of sensitivity is often the proportional effect of variability in each variable on the model, which is computed as the variance in the risk distribution from each of the simulations divided by the variance in the risk distribution from the base model result. Although the general idea of pinching is known to Monte Carlo analysts, the notions of pinching to a precise distribution and pinching to a zero-variance interval has no analog in Monte Carlo sensitivity analyses.

Figure 2 shows a numerical example of pinching to a precise distribution. The top panel of the figure depicts of addition of two p-boxes *a* and *b* (assuming independence). This is the "base case" against which the pinchings will be compared. The area between the upper and lower bounds for the sum *a*+*b* is 2.12. The middle panel of the figure shows the first pinching. The p-box *a* is replaced with a precise probability distribution that lies entirely within the p-box. When a distribution replaces the p-box in the addition with *b* (which is still the same p-box), the result is the p-box shown at the far right on the middle panel. This p-box has an area of about 1.14. The percentage reduction in this area compared to that of the p-box for the sum shown on the top panel is 46.24%. This percent, which labels the sum on the middle panel, represents the sensitivity measure for pinching the variable *a* to a precise probability distribution. The bottom panel of Figure 2 shows the reduction of uncertainty (area) for the sum *a*+*b* from pinching the p-box for *b* to a precise distribution. Compared to the base case in the top panel, the area is reduced by 47.17%. In this case, the potential reduction in uncertainty from additional information about *a* and *b* are roughly the same.

Figure 3 shows a similar set of sensitivity analyses based on pinching p-boxes to precise distribution functions. The calculation for the base case in this figure (shown in the top panel) was made without making any assumption about the dependence between the variables *a* and *b*. For this reason, even though the p-boxes for the variables *a* and *b* are just the same as were used in Figure 2, the area of the sum grows to about 3.05. The second panel of Figure 3 depicts pinching the p-box for the variable *a* to a precise distribution and its consequence for the resulting uncertainty about the sum. The third panel likewise shows the pinching for variable *b*. Both panels are annotated with the percent reduction in the area of the p-box for the sum compared to the base case in the top panel. The reduction in uncertainty from pinching the variable *a* in this situation is perhaps surprisingly small. The sensitivity to changing *b* is more than three times greater than that of *a*. The bottom panel shows the effect of pinching the dependence from the Fréchet case of assuming nothing about dependence to assuming independence. (The pinching could have specified any particular dependence.)

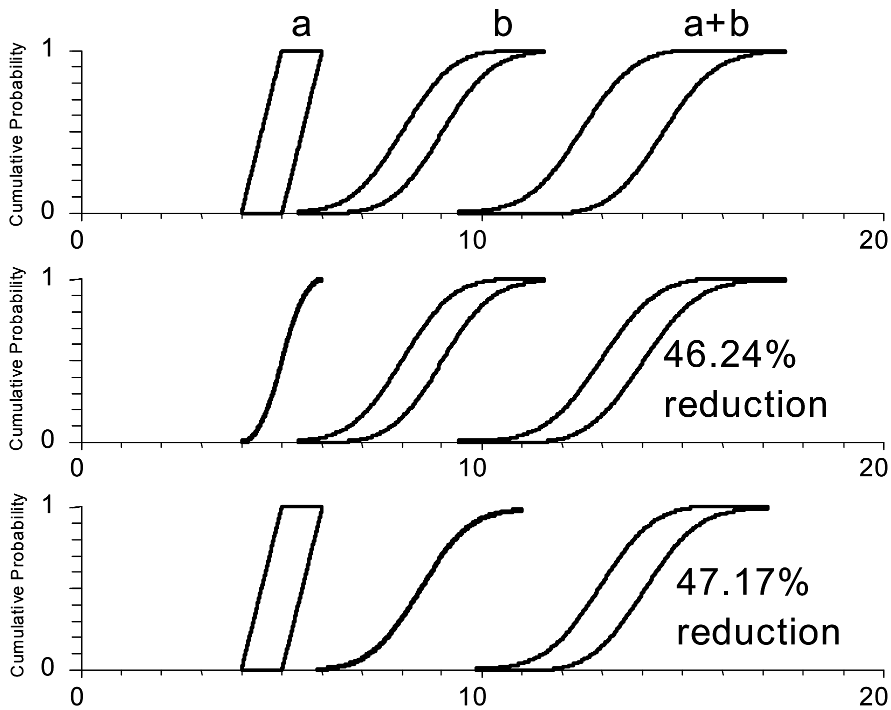

Figure 2. Meta-sensitivity analysis by pinching a p-box to a precise distribution.

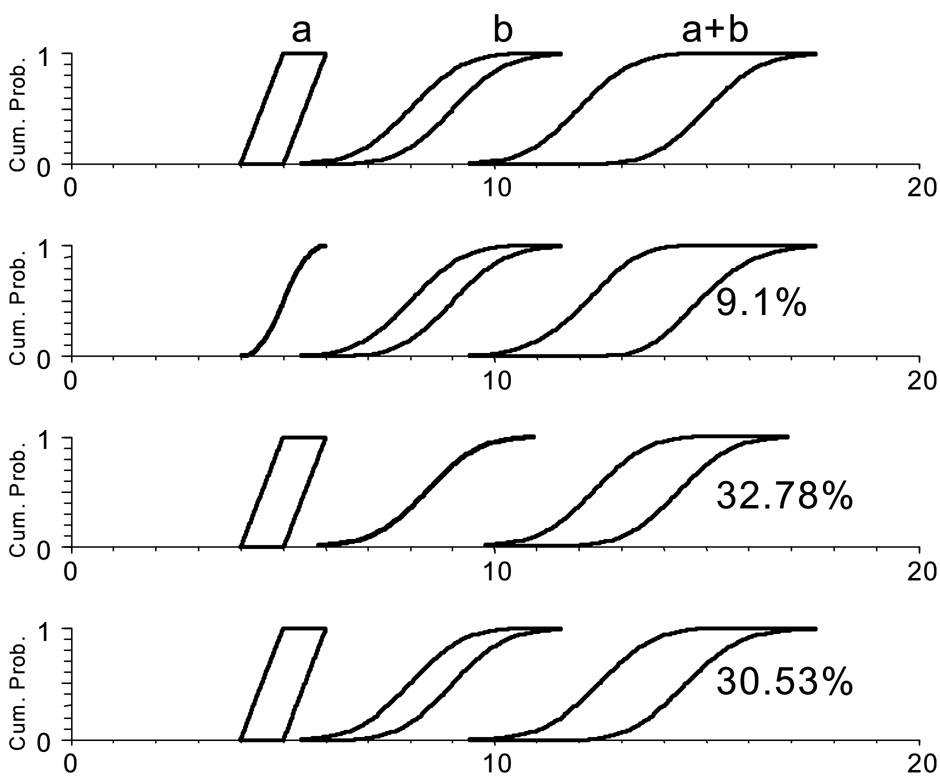

Figure 3. Meta-sensitivity analysis for the Fréchet case without dependence assumptions.

Figure 4 shows a third hypothetical sensitivity study. The base case in the top panel is identical to the base case shown in Figure 2, but in this study, the p-boxes are pinched to scalar values. The second and third panels of Figure 4 depict the additions resulting from pinching one of the addends to a point value. The observed percentage reduction in the area of each resulting sum compared to the base case is shown beside its p-box. What would the reductions in uncertainty have been if the base calculation had not assumed independence? The pinchings would have yielded exactly the same results, simply because dependence assumptions have no effect when either of the addends is a point. Thus, the lower two panels of Figure 4 would look exactly the same. However, if the base calculation had not assumed independence, then the base uncertainty about the sum *a*+*b* would have been slightly greater (area = 3.05, compared to 2.12 under independence). That would make the rightmost p-box in the top panel of Figure 4 noticeably wider. Therefore the reductions in uncertainty by pinching to a point would have been somewhat greater than they were for the independent case. Instead of 50.4% and 52.36% reductions, pinching the variables *a* and *b* to points under no assumption about dependence would have respectively yielded 65.54% and 66.9% reductions in uncertainty as measured by the area within the resulting p-boxes.

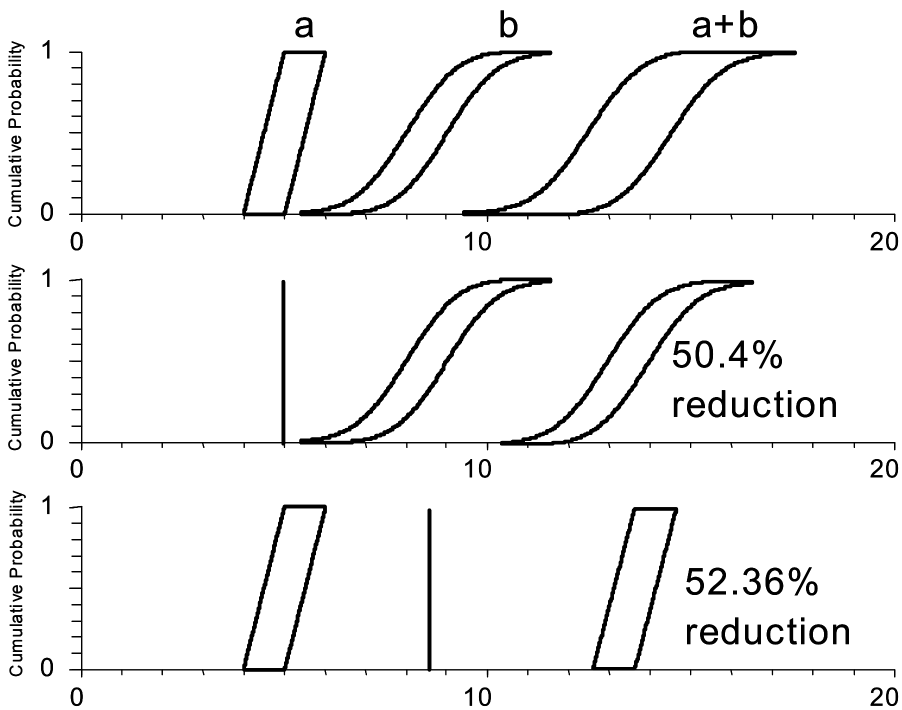

Figure 4. Meta-sensitivity analysis by pinching a p-box to a point value.

4. CONCLUSIONS

Many probabilistic assessment conducted using Monte Carlo simulations employ what-if sensitivity studies to explore the possible impact on the assessment results of varying the inputs. For instance, the effect of the truncation of some variable might be explored by re-running the model with various truncation settings, and observing the effect on the risk estimate. The effect of particular parameter and probability distribution choices, and assumptions regarding dependencies between variables can also be examined in this way. Model uncertainty can be probed by running simulations using different models. However, such studies are often very difficult to conduct because of the large number of calculations that are required. While informative, this approach is rarely comprehensive because when there are multiple uncertainties at issue (as there usually are), the shear factorial problem of computing all of the possible combinations becomes prohibitive. Usually, in practice, only a relatively tiny number of such analyses can be performed. Probability bounds analysis can be used to automate such what-if sensitivity studies and vastly increase their comprehensiveness.

Sensitivity analysis can also be conducted at a meta-level by hypothetically replacing a p-box in a probability bounds analysis with a precise distribution or perhaps a scalar number to evaluate the potential reduction of uncertainty of the result under additional knowledge.

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