# MENSURATION TAUGHT OBJECTIVELY

WITH

# LESSONS ON FORM.

BY

SUP'T W. W. ROSS, A. M.,

OF FREMONT, OHIO.

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MANUAL FOR THE USE OF THE AUTHOR'S

# DISSECTED SURFACE FORMS

AND

# GEOMETRICAL SOLIDS.

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PATENT TO BE APPLIED FOR ON

DISSECTED CUBE, THE CUBE-INSCRIBED-SPHERE,

THE

CYLINDER-INSCRIBED-SPHERE,

AND

THE GENERAL COMBINATION.

# MENSURATION.

- 1 Of Surfaces.
  - 1 Plane.
    - 1 Rectilinear.
      - 1 Quadrilaterals.
        - 1 Parallelograms.
          - 1 Rectangles.
            - 1 Oblong.
            - 2 Square.
          - 2 Rhombold.
        - 2 Trapezoids.
        - 2 Trapeziums.
      - 2 Triangles.
      - 3 Pentagons.
      - 4 Hexagons, &c.
    - 2 Circular.
      - 1 Circles.
      - 2 Ellipses.
  - 2 Curved.
    - 1 Of Cylinder.
    - 2 Of Cone.
    - 3 Of Sphere.
- 2 Of Volumes.
  - 1 Plain Solids.
    - 1 Prisms.
      - 1 Quadrangular.
        - 1 Rectangular.
          - 1 Oblong.
        - 2 Rhomboidal.
          - 1 Cube.
      - 2 Triangular.
      - 3 Pentagonal.
      - 4 Hexagonal, &c.
    - 2 Pyramids.
      - 1 Triangular.
      - 2 Quadrangular.
      - 3 Pentagonal.
      - 4 Hexagonal, &c.
    - 3 Frustum of P'd
      - 1 Triangular.
      - 2 Quadrangular.
      - 3 Pentagonal, &c.
    - 4 Regular Solids.
      - 1 Tetrahedon, equal faced triangular pyramid, 4 equal triangular facees.
      - 2 Hexahedron, cube, 6 equal square faces.
      - 3 Octahedron, 8 equal triangular faces.
      - 4 Dodecahedron, 12 equal pentagons, for faces.
      - 5 Icosahedron, 20 equal triangles, for faces.
  - 2 Round Solids.
    - 1 Cylinder.
    - 2 Cone.
    - 3 Frustum Cone.
    - 4 Sphere.

# LESSONS ON FORM AND LANGUAGE EXERCISES.

In geometry we are accustomed to begin with the generic terms on the left of the outline and proceed to the specific forms on the right.

In lessons on Form and Mensuration we should begin with the specified Forms on the right and proceed to the generic terms on the left.

### THE OBLONG.

The oblong has four sides. It has four corners or angles. It has two long sides and two short sides. The long sides are equal and the short sides are equal.

The opposite sides are parallel, that is equally distant at opposite points, and will not meet how far so ever they be produced.

### CONDENSED STATEMENT FOR COMPOSITION.

The oblong has two equal long sides and two equal short sides and four right angles.

### DRAWING.

Make a drawing of an oblong 6 in. by 4 in.

### THE SQUARE.

The square has four equal sides and four right angles. Draw squares of different dimensions and divide into square inches. Oblongs and and squares are rectangles. Each have four sides and four right angles. The square is an equal sided rectangle.

### RHOMBOID.

The rhomboid has two equal long sides and two equal short sides, two sharp corners, or acute angles, and two blunt corners, or obtuse angles.

Draw.

An equal sided rhomboid is a rhomb.

Oblongs, squares and rhomboids have their opposite sides parallel and are therefore parallelograms.

### TRAPEZOIDS.

Trapezoids are four sided figures only two of whose sides are parallel. Draw them of different shapes.

### TRAPEZIUMS.

A trapezium is an irregular four sided figure with no sides parallel.

Oblongs, squares, rhomboids, trapezoids, and trapeziums have each four sides and are therefore quadrilaterals.

### LESSONS ON THE TRIANGLE.

Triangles
- 1 As to Sides.
  - 1 Equal Sided, or Equilateral.
  - 2 Isosceles, Two Sides Equal.
  - 3 Scalene, No Sides Equal.
- 1 As to Angles
  - 1 Acute.
  - 2 Obtuse.
  - 3 Right.
    - 1 Hypothenuse.
    - 2 Base.
    - 3 Perpendicular.

Illustrate with the wooden forms, with tablets or by drawing the forms. Show by the forms that a triangle is either half of a square, half of an oblong or half of a rhomboid.

### PENTAGON, HEXAGON, &C., AND CIRCLE.

Show by the forms and other wise that the pentagon, hexagon, heptagon, octagon, &c., are made up respectively of five, six, seven, eight, &c., triangles; each having for a base one of the sides of the corresponding polygon, and by the dissected circle, that the circle is made up of an infinite number of such triangles.

### THE CIRCLE.

Develop the following parts by forms and drawings.

A circle is a figure bounded by a curved line every point of which is equally distant from a point within called the center.

The boundary line is the circumference, or the circumference is a curved line every point of which is equally distant from a point within called the center. The diameter is a straight line passing through the center and terminating in the circumference. A radius is half the diameter or the distance from center to the circumference.

An arc is part of a circle or rather circumference. A quadrant is the fourth part of a circle. A chord joins the extremities of an arc.

A diameter is the chord of a semi-circle. A segment is the space enclosed between an arc and its chord.

A sector is the space enclosed between an arc and radi drawn to the extremities of the arc.

### TH ELLIPSE.

With a slack string fastened to two tacks driven into a board or the floor, draw with pencil or crayon an ellipse following the tightened cord.

An ellipse is a curve such that the sum of the distances from any point of the curve to two fixed points within called foci is invariable.

The sum of the distances is the entire length of the cord and the tacks are the fixed points or foci.

A point midway between the two foci is the center. A line passing through the center and terminating in the curve is a diameter. The longest diameter, which passes through the foci, is called the major axis. The shortest diameter, which is perpendicular to the major axis, is the minor axis.

A line drawn from the curve to either focus is the radius vector.

Show that when the tacks are placed together at the center and the distance between the foci and the center, in other words the eccentricity of the ellipse, is zero the ellipse becomes a circle.

Show that when the tacks are the cords distance apart, the ellipse becomes a straight line, the eccentricity being considered equal to one, and therefore that an ellipse may vary all the way from a circle to a straight tine.

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## SOLIDS.

### PRISM, CYLINDER, CUBE AND SPHERE.

Show by the solids that the cube has six faces and eight corners; that the faces are square and equal; that the oblong or square prism has four oblongs for its convex surface, that both are rectangular prisms, that the rhomboidal or oblique prism has two rhomboidal surfaces; that all thus far are quadrangular prisms; that the triangular prism has triangles for its ends and three oblong surfaces; that the pentangular or pentagonal prism has five oblong surfaces with pentagons for the ends; that the hexagonal prism has six oblong surfaces and hexagons for its ends; that the cylinder is a prism with an infinite number of oblongs in its curved surface and circles for its ends; that as the oblong prism by an increase in the number of its convex surfaces terminates in the oblong cylinder, and the cube in a cylinder whose altitude and diameter are the same, so the cube by an increase in all its surfaces may be made to pass through a variety of solids terminating in the sphere

### ILLUSTRATIVE LESSON ON THE CUBE.

How many faces has the cube? Ans. The cube has six faces.

What kind of faces? Ans. The faces are square. Put that in your sentence. The cube has six square faces.

Measure them and what do you observe? The faces are equal. Put that in your sentence. The cube has equal square faces.

How many corners? Ans. Eight. Place that in your sentence.

The cube has six equal square faces and eight corners.

How many edges? Ans. Twelve. Add that.

The cube has six equal square faces, eight corners, and twelve edges.

How many angles formed by the edges? Count them. Ans. Twenty-four square corners or right angles.

Show that two surfaces or faces form face or diedral angles, acute, obtuse or right.

How many face angles in the cube? Count the face angles formed by the walls, ceiling and floor of a room. Ans. Twelve face anges, or twelve diedral right angles.

How many triedral or solid angles formed by three faces in a cube? Count those in a room. Ans. There are eight solid angles in the cube.

DESCRIPTION OF THE CUBE.

The cube has six equal sqqare faces, eight corners and twelve edges. The edges of the cube form twenty-four right angles.

The surfaces taken two and two form twelve face angles, and taken three and three. eight solid angles at the respective corners.

PYRAMIDS, CONES, FRUSTUMS, SPHERE.

Show by the Forms that the convex surface of pyramids is always made up of triangles and that the base may be a triangle, square, or any other polygon, and that the pyramid when the number of its trianglar faces is infinite becom s a cone with an infinite sided polygon or circle for its base; and that the convex surface of frustums is made up of trapezoids.

By means of the dissect d cube show that a cube is made up of six pyramids, each having for its base a face of the cube and for its altitude the radius or semi-diameter of the cube; that in like manner the octahedron is made up of eight equal triangular pyramids, the dodecahedron of twelve equal pentagonal pyramids; the icosahedron of twenty equal triangular pyramids and the sphere itself of an infinite number of pyramids whose bases make up the surface of the sphere and whose altitudes are the radius of the sphere.

These lessons on form are valuable as language exercises, for their bearing on drawing, and as a preparation for Mensuration and may be given with profit in the primary and lower grammar grades and the entire graded school as special lessons, or to advanced grammar grade and High School pupils in the actual work of Mensuration as will be seen by what follows.

# MENSURATION OF SURFACES.

## OUR OBJECT NOW IS TO SHOW HOW TO TEACH OBJECTIVELY THE MENSURATION OF THESE FORMS.

### OBLONG AND SQUARE.

Mensuration should begin with the oblong or square. Pupils should be taught to see that as there is no natural unit for the measurement of surfaces it is necessary to fix upon an artificial unit and that, that artificial unit is a square, a square inch, square foot, square yard, etc., and that to find the area or superficies of a surface is to find how many times this measuring unit can be applied to, or is contained times in the given surface.

First show by the actual application of your measuring unit, a small form an inch square, that a form six inches long and one inch wide contains six square inches, and if four inches wide, show by a similar application of the 6×1 oblong, or by proper ruling, that it contains four times six square inches, or twenty-four square inches. The analysis for this first and fundamental step in mensuration being as follows:

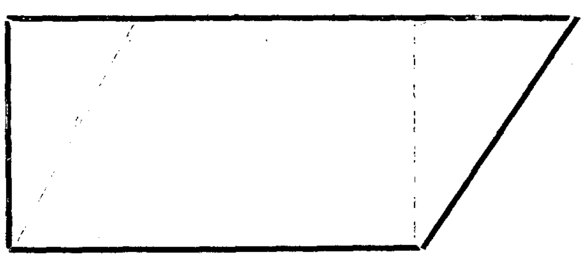
A surface six inches long and one inch wide contains six square inches; if four inches wide it contains four times six square inches, or twenty-four square inches.

### RULE.

Multiply the length by the breadth, or more accurately, multiply the number of square units corresponding to the length by the breadth. This disposes of the oblong and the square.

### RHOMBOID.

To show that a rhomboid is equivalent to an oblong with the same base and perpendicular breadth or altitude, cut your paper, heavy drafting paper is the best, in the shape of a trapezoid as indicated by the continuous lines in the following figure:



By folding back the left hand upper corner you have the desired rhomboid.

By unfolding this left hand upper corner, and folding back the equivalent right hand upper corner, the rhomboid is converted into an equivalent oblong with the same base and altitude.

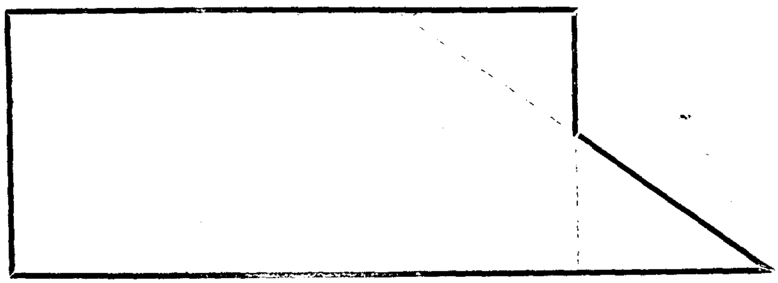
DISSECTED RHOMBOID.

Detatch the piece from the right of the rhomboid and attach it to left, or from one side to the other. It will be seen that the rhomboid is equivalent to an oblong with the same base and altitude.

Having perceived the equivalency of the rhomboid and the oblong, the pupil comprehends that the area of the rhomboid like that of the oblong is found by multiplying its length by the perpendicular breadth or by the altitude.

THE TRAPEZOID.

A trapezoid may be shown to be equivalent to an oblong with the same altitude, the sum of whose two opposite sides is equal to the sum of the parallel side of the trapezoid by cutting the paper in accordance with the model described by the continuous lines in the following fiigure:



By folding back the right hand upper corner, you have the desired trapezoid. By unfolding this upper corner and folding back the equivalent lower corner, the trapyzoid is converted into an equivalent oblong. It is a very easy matter to show by measurement, that the sum of the long sides of the oblong is equal to the sum of the parallel sides of the trapezoid, and that as the oblong is equal to its base, *half the sum of its long sides*, multiplied by its altitude, so is the surface of the trapezoid equal to half the sum of its parallel sides multiplied by the perpendicular distance between them, the common altitude of the oblong and the trapezoid.

DISSECTED TRAPEZOIDAL FORMS.

Take the dissected trapezoidal form, detatch the dowelled piece and attach to the dowels above. It is immediately converted into an equivalent oblong. Reasoning as before, as the area of the oblong is equal to half the sum of its long sides by its altitude so the area of the trapezoid is equal to half the sum of its parallel sides by its altitude or perpendicular breadth.

RULE FOR THE TRAPEZOID.

Multiply half the sum of its parallel sides by the perpendicular breadth between them.

## EXAMPLES.

1 What is the cost of a rectangular piece of land 240 yards long and 110 yards wide at $36.50 per acre? Ans. $200.75.

2 What is the area of an oil cloth 42 ft. by 5 ft. 8 in.? Ans. 26 4-9 sq yds.

3 What is the number of sq. yds. in a rhomboid whose base is 37 feet and altitude 5 feet 3 inches? Ans. 21 7-12.

4 How many acres in a street 2 miles and 240 rods long and 66 feet wide? Ans. 22.

5 How many acres in a field 20 rods square?

6 A floor 16 feet long contains 24 square yards of carpeting; how wide is the room? Ans. 13½ feet.

7 What is the area of a rhombus (equilateral rhomboid) whose perimeter or boundary is 48 yards and altitude 8 yards? Ans. 96 yards.

8 Find the area of a trapezoid, the longer of whose parallel sides is 175 feet, the shorter 135 feet, and whose altitud is 80 feet.

Solution—$(175+135)\div2=155$; $155\times80=12400$ sq. yds. area.

9 How many square feet in a tapering plank whose length is 18 feet, and whose width at one end is 8 inches and at the other end 6 inches? Ans. 10½ square feet.

10 The parallel sides of a field are 70 and 90 rods, and its breadth 209 yards. What is the value of the field at $75 per acre? Ans. $1425.

11 A lot is bounded by four streets; two of these are parallel, and measure 70 feet and 90 feet along the lot and are 135 feet apart; what is the surface of the lot? Ans. 10800 square feet.

12 A man has a lot 54 yards long and 24 yards wide, which he exchanged for a square lot of equal surface; how much more fence would the first lot require than the last? Ans. 12 yards.

TRAPEZIUM.

The trapezium is measured by dividing it up into triangles. This disposes of all the quadrilaterals.

TRIANGLES.

It is a very easy matter to show objectively that triangles are halves of either squares, oblongs, or rhomboids with the same base and altitude.

By means of the bisected square, or by folding a square piece of paper diagonally, or black board drawings, it will be seen that half of a square is a right angled isosceles triangle, or that a right triangle whose sides about the right angle are equal, is equal to half a square.

By means of the bisected oblong, &c., it may be shown that a right angled triangle with unequal sides about the right angle is half an oblong.

By the two bisected rhomboids, &c., it may be shown objectively that if a rhomboid is cut diagonally through the obtuse angles two equal acute angled triangles will result; if cut diagonally through the acute angles two equal obtuse angled triangles will result.

When it is seen that a triangle is in all cases either half of a square, an oblong or a rhomboid, with the same base and altitude, it is readily understood that the area is equal to the product of its base by half the altitude.

RULE FOR THE TRIANGLE.

Multiply the base by half the altitude or take half the product of the base and the altitude.

Formula. $\text{Surface} = \text{Base} \times \frac{A}{2}$

CIRCLE.

The mastery of the triangle prepares the way for the mastery of the pentagon, and all other polygons, and through these, the circle.

It were best to begin with the square. Inscribe in the square a circle. Draw the diagonals of the square, connect the center of the circle with the four points of tangency.

It will be seen that the square is divided into four equal triangles each having for its base a side of the square and for its altitude the radius of the inscribed circle and, therefore, that the area of the square is equal to the sum of the areas of the four triangles, or the sum of its sides, in other words its perimeter multiplied by half the radius of the inscribed circle.

DISSECTED SQUARE.

Or take the dissected six inch square. It is divided into four triangles each of which has a side of the square for its base and the radius of a circle inscribed in the square for its altitude.

OPERATIONS FOR FINDING AREAS OF THE SQUARE.

First operation;—$6 \times 6 = 36$=area.

Second operation;—6=Base of each triangle.

3=Altitude of each triangle=radius of inscribed circle.

$\frac{6 \times 3}{2} = 9$=Area of each triangle.

$4 \times 9 = 36$ the area of the four triangles or square.

Third operation;—6+6+6+6=24=Primeter or boundary of square= the sum of the basis of the 4 triangles.

$$\frac{24\times 3}{2}=36=\text{Area of the four triangles or square.}$$

RULE FOR LAST OPERATION.

Multiply the perimeter or boundary of the square by one half the radius of the inscribed circle.

DISSECTED HEXAGON.

The hexagon will be seen to be made up of six triangles whose bases make up the boundary of the hexagon and whose common altitude is the radius of the inscribed circle. Its area is therefore equal to the perimeter of the hexagon multiplied by one-half the radius of the inscribed circle.

A decagon is composed of ten triangles, a hundred sided figure of one hundred triangles and a circle of an infinite number of triangles the sum of whose bases is the circumference of the circle and whose common altitude is the radius of the circle. Show by the dissected circle that it is composed of triangles.

The area of a circle is therefore equal to its circumference, the sum of the bases of all the triangles which compose the circle, multiplied by one half its radius, the common altitude of the triangles.

RULE FOR THE AREA OF THE CIRCLE.

Multiply the circumference of the circle by one-half its radius.

MODIFICATIONS OF THIS RULE THUS TAUGHT OBJECTIVELY.

Multiply the circumference by one-fourth the diameter. Multiply one-half the circumference by one-half the diameter, &c., &c.

But how from these objective rules develop the ordinary rules, viz:

Multiply the square of the diameter by .7854 or multiply the square of the radius by 3.1416.

How develop the ordinary formulas:

Area of Circle=$D^2\times.7854$ and Area of Circle=$R^2\times3.1416$.

The ratio of the circumference to the diameter or the number 3.1416 cannot of course be taught or reached objectively. It can be done approximately by laying off on the black-board the circumference of a cylinder as a straight line, and then by laying off on this the diameter of the cylinder by application or division. It will be found that the circumference is about 3 1-7 times or more accurately, 3.1416 times the diameter.

Let C represent the circumference and D the diameter.

Then $C=D\times3.1416$

Circumference×by one-fourth the diameter will stand as follows: $D\times3.1416\times\frac{1}{4}D=D^2\times\frac{1}{4}$ of $3.1416=D^2\times.7854$. Hence Area$=D^2\times.7854$.

If C represents the circumference and R the radius, then;

$C=2R\times3.1416$.

Area$=2R\times3.1416\times\frac{1}{2}R=R^2\times3.1416$.

Let it be clearly seen that a circle is .7854 of a square erected on the diameter of a circle.

If the diameter of a circle is 4 the area of the square each of whose sides is 4, is 16 and the area of the circle is .7854 of 16.

This disposes of the circle and the measurement of flat and plane surfaces.

### TRIANGLES AND CIRCLES—EXAMPLES.

1 How many acres in a triangular piece of land whose base is 120 rods and altitude 132 yards?

Solution. 132 yards=24 rods, 120×24÷2=1440, 1440÷160=9 Acres.

2 What is the cost of a triangular piece of ground whose base is 110 yards and altitude 82 yards, 1 foot, 6 inches, at $64 per acre? Ans. $30.

3 Diagonal of a trapezium 100 rods; perpendiculars on it from opposite corners 50 rods and 70 rods; find the value of the trapezium at $960 per acre. Ans. $36000.

4 What is the altitude of a triangular piece of ground whose area is 3 acres and base 220 yards? Ans. 132 yards.

5 I Paid $637.50 for a triangular piece of ground, bought at $75 per acre. If its base is 340 yards, what is its altitude? Ans. 242 yards.

### RIGHT ANGLED TRIANGLES.

By means of the right angled triangular form 3 inches by 4 inches by 5 inches and the squares erected on their sides show

1 The square of the hypothenuse equals the square of the base plus the square of the perpendicular.

2 The square of the base equals the square of the hypothenuse minus the square of the perpendicular.

3 The square of the perpendicular equals the square of the hypothenue minus the square of the base.

### EXAMPLES.

6 A rectangular field is 84 rods long and 80 rods wide; what is the distance between its opposite corners? Ans. 116 rods.

7 What is the base of a right angled triangle, if its hypothenuse is 1275 yards and its perpendicular 601 yards? Ans. 1125.

8 At what distance from a house must the foot of a ladder 78 feet long be placed, that it may reach to a window 72 feet high?

Ans. 30 ft.

9 What is the area of a circle whose diameter is 18 rods?

Solution 1 Circumference$=18\times 3.1416$,
Radius$=9$
Area$=18\times 3.1416\times 9\div 2$

Solution 2 Area$=18^2\times .7854$

Solution 3 Area$=9^2\times 3.1416$ Ans. 254.4696.

10 If a stake be placed in the center of a pasture, and a horse be fastened thereto by a halter 15 feet long, over what area will he be able to graze? Ans. 706.86.

11 A circle is 40 feet in diameter, and within it is another circle 15 feet in diameter; what area of the larger circle is outside the smaller?

THE SURFACES OF THE SOLIDS.

By the application of the principles for the measurement of plane surfaces, the surfaces of all solids bounded by flat, or plane surfaces may easily be calculated. Commencing with the cube it is only necessary to show by the object itself that its surface is composed of six equal square faces; that any oblong solid, for instance a brick, or ordinary room is bounded by three pair of oblong faces equal taken two and two; that the convex or upright surface on any right prism is made up of as many oblongs as the base has sides and that this surface is equal to the sum of the surfaces of the oblongs or the primeter of the base multiplied by the altitude, and that the entire surface may be found by adding to this result the surface of the two bases; that the convex surface of a pyramid is made up of as many triangles as the base has sides, and that this surface is therefore, equal to the sum of the surfaces of the triangles, or the perimeter of the base multiplied by half the slant hight of the pyramid which is the common altitude of the triangles; that the convex surface of a frustum of a pyramid is composed of as many trapezoids as the base has sides, that this surface is therefore equal to the sum of the surfaces of the trapezoids or half the sum of the perimeters of the upper and the lower base—the parallel sides of the trapezoid—by the slant height of the frustum, the common altitude of the trapezoids.

This disposes of the solids bounded by plane surfaces.

SURFACE OF CYLINDER.

The process of finding the curved surface of a cylinder may be taught by simply winding the cylinder with paper; when this paper is removed, it will be seen to be an oblong whose length is the circumference of the cylinder and whose altitude is the altitude of the cylinder; and therefore the convex surface of a cylinder is equal to the circumference—the length of the rectangle—multiplied by its altitude—the breadth of the rectangle.

It may also be shown as follows: As the convex surface of a prism of any number of sides is equal to the perimeter of its base multiplied by its altitude so the convex surface of a cylinder which is a prism with an infinite number of sides is equal to the perimeter of its base or circumference multiplied by its hight.

If paper is made to fit closely to the convex surface of a cone and then unfolded it will be found to be the sector of a circle, whose area, of course, is equal to its arc, which in this case corresponds with the circumference of the base of the cone, multiplied by half its radius which corresponds with the slant hight of the cone.

Again it has already been shown that the convex surface of a pyramid is equal to the perimeter of its base multiplied by half its slant hight and it follows that the convex surface of a cone or a pyramid with an infinite number of faces is equal to its circumference multiplied by one-half the slant hight.

### THE SPHERE.

How to find the surface of a sphere cannot be taught objectively and it is the only operation that cannot be so taught.

It need only be remembered that the surface of a sphere is equal to the surface or area of four great circles of the sphere and that as the area of one circle is equal to its circumference multiplied by one half its radius or one-fourth of its diameter, the surface of four circles is equal to the circumference multiplied by the diameter; or as the area of one circle is= to $D^2 \times \cdot 7854$, the area of 4 circles is=to $D^2 \times 4 \times .7854 = D^2 \times 3{,}1416$.

### RULE FOR THE SURFACE OF SPHERE.

Multiply the circumference or the sphere by its diameter or multiply the square of the diameter by 3,1416.

Formula— Area of sphere$=C \times D$, or $D^2 \times 3{,}1416$.

### RULE FOR CONVEX SURFACE OF PRISM AND CYLINDER.

Multiply the perimeter or circumference of the base by the altitude.

### RULE FOR SURFACE OF PYRAMID AND CONE.

Multiply the circumference of perimeter of the base by one-half the slant hight.

### RULE FOR FINDING THE CONVEX SURFACE OF FRUSTUM OF PYRAMID AND CONE.

Multiply half the sum of the perimeter or circumference of the upper and the lower base by the slant hight.

This finishes the mensuration of surfaces.

# EXAMPLES.

1 What is the surface of an eight inch cube? Ans. 384 sq. ft.

2 What is the surface of the walls, ceiling and floor of a room 12 feet long, 12 feet wide and 12 feet high? Ans. 864 sq. ft.

3 What is the entire surface of a square or quadrangular prism whose height is 10 feet and whose base is 2 feet square? Ans. 88 sq. ft.

4 What is the convex surface of a triangular prism whose altitude is 8 feet and the sides of whose bases are 3 feet, 4 feet and 5 feet.

Ans. 96.

5 What is the convex surface of a pentagonal prism whose altitude is 9 ft. and each side of the base 5 ft. Ans. 225 sq. ft.

6 What is the entire surface of a cylindrical boiler 2 ft. 9 in. in diameter and 10 feet long?

Ans. 98,273+sq. ft.

7 Find the convex surface of a triangular pyramid the slant height 20 feet and each side of the base 3 feet?

Ans. 90 sq. ft.

8 What is the entire surface of a square pyramid whose base is 25 feet square and whose slant height is 100 feet. Ans. 5625 sq. ft.

9 What is the convex surface of a cone whose slant height is 50 ft. and the diameter of its base $8\frac{1}{2}$ ft.? Ans. 667.59 sq. ft.

10 Required the entire surface of a cone whose slant height is 36 and the diameter of its base 18 ft. Ans. 1272.348 sq. ft.

11 What is the convex surface of a frustum of a square pyramid if its slant height is 6 feet, and each side of the graeter base 5 ft. and of the less base 2 feet?

Ans. 84 sq. ft.

12 What is the entire surface of the frustum of a cone if its slant height is 16 feet, and the diameter of the bases 6 ft. and 4 feet?

Ans 292.1688 sq. ft.

13 What is the surface of a sphere whose diameter is 32 feet?

Ans. 3216.9984 sq. ft.

14 The diameter of a globe is 1 ft. 8 in. what is its surface?

Ans. 8.726+ sq. ft.

# MENSURATION OF SOLIDS.

In what follows we shall show that without a knowledge of the rigorous demonstrations of gemoetry the mensuration of all volumes or solids may be so objectively taught and developed that the reasons for the processes shall be clearly perceived and comprehended.

## RECTANGULAR SOLIDS INCLUDING THE SQUARE PRISM, OR CUBE AND THE OBLONG PRISM.

As the square is the unit for the measurement of surfaces the cube, cubic inch, cubic foot, cubic yard &c. is assumed as an artificial unit for the measurement of volumes; the volume of a body being the number of times it contains one of these measuring units.

Commencing with the rectangular solid, it may be shown by the dissected prism that a solid six inches long, one inch wide and one inch thick or high, contains six small cubes or cubic inches; two such pieces or a similar solid two inches wide, will contain two times six small cubes, or twelve cubic inches, and if two inches high it will contain two times twelve small cubes or twenty-four cubic inches.

### ANALYSIS OF THIS FUNDAMENTAL STEP.

A solid six inches long, 1 inch wide and 1 inch thick, will contain 6 cubic inches; if 4 inches wide it will contain 4 times 6 cubic inches or 24 cubic inches, and if 3 inches high or thick it will contain 3 times 24 cubic inches or 72 cubic inches.

### GENERALIZATION OR RULE.

Multiply the length by the breadth and that product by the thickness. Or more accurately: Multiply the number of cubic units corresponding to the length by the breadth and that product by the thickness.

It is also well to have clearly understood that the product of the length by the breadth, in other words the area of the base, represents the number of cubic units in a solid with such a base, whose hight is one inch or unit, and consequently that this given number multiplied by the given hight will give the number of cubic inches in the whole volume, this principle being almost of universal application in the mensuration of solids.

This may be shown by placing the four oblong prisms on end. The following rule will be developed:

Multiply the area of the base by the altitude. Or more accurately; multiply the cubic units corresponding to the area of the base by the altitude.

### RHOMBOIDAL SOLID.

A rhomboidal solid may readily be shown to be equal to a rectangular solid with the same base and altitude. Detach the dowelled piece from one end of the rhomboidal solid and attach it to the other after the manner suggested to be pursued with rhomboidal surfaces.

### PRISM AND CYLINDER.

With the quadrangular prism cut through diagonally, it may be shown objectively that as a triangle is half of a square, an oblong or a rhomboid, so is a triangular prism half of a quadrangular prism with the same altitude, the base of the former being half that of the latter; therefore the volume of a triangular prism is equal to the base—half that of the quadrangular prism—multiplied by its altitude.

The dissected hexagonal prism will show that prisms of five, six, or a hundred faces are made up of a corresponding number of triangular prisms, and therefore, that the volumes of all prisms are equal to the areas of their bases multiplied by their altitudes and that the volume of a cylinder which is a prism of an infinite number of sides or faces, is equal to the area of its circular base multiplied by its altitude.

RULE FOR THE VOLUME OF PRISM AND CYLINDER.

Multiply the area of the base by the altitude.

## EXAMPLES.

1 Find the volume of a flag stone 10 ft. long, 3 ft. 9 in. wide and 8 in. thick.

2 How many cubic feet in a square beam 40 ft. high, each of whose ends is 9 in. square? Ans. 90 ft.

3 A sill is 40 feet long and 9 in. wide; how thick must it be to contain 25 cubic feet? Ans. 10 in.

4 What are the solid contents of a quadrangular prism whose altitude is 13 feet and ends 2 ft. 1 in. long by 1 ft. 2 in. wide? Ans. 49.652 cu. ft.

5 What is the volume of a cylinder whose altitude is 15 ft. and diameter of its ends 4 ft. Ans. 188.497 cu. ft.

6. A cylinder 10 in. in diameter contains $1963\frac{1}{2}$ cubic inches; what is its height? Ans. 25 in.

7 How many 2 in. cubes can be placed in a cubical box each of whose dimensions is 2 ft.?

8 If a piece of ice 1 ft. each way is worth 10c what is a piece of ice 2 ft. each way worth? Ans. 80 cts.

Prove it by the small cubes.

PYRAMIDS AND CONES.

The dissectial triangular prism will show that a triangular prism is composed of three equal triangular pyramids. The equality of two of these pyramids, those having for thir bases, the upper and lower base of the prism can easily be perceived by the eye; that the third pyramid is equal to these may be shown by weighing them or from the fact that they have equal bases and equal altitude.

Two of the pyramids have the basis of the original prisms, and the same altitude. Having shown that each is equal to one-third of the prism it follows that a triangular pyramid is one-third of a prism with the same base and altitude.

By means of the bisected square pyramid it may be shown that all pyramids are made up of triangular pyramids and therefore, that the volume of any pyramid is equal to one-third the product of its base and altitude.

### CONE.

As a cone is a pyramid of an infinite number of faces or sides it follows that the volume of a cone is equal to the area of its circular base multiplied by one-third of its altitude.

Again it may be shown objectively that the tin cone whose base is 4 inches in diameter and whose altitude is 4 inches is one-third of a cylinder with the same base and altitude by pouring the tin cone three times filled with water into the tin cylinder. As it exactly fills the cylinder and as the volume of the cylinder is equal to its base multiplied by the altitude it is easily seen that the volume of the cone is one-third the product of base and altitude.

Rule for the pyramid and cone; Multiply the area of the base by one-third the altitude.

### EXAMPLES.

1 What are the solid contents of a square pyramid, each side of whose base is 8 ft., and its altitude 15 ft. Ans. 320 ft.

2 What are the solid contents of a cone whose base is 3 ft. 6 in. in diameter and altitude 5 ft.? Ans. 16.035 cu. ft.

3. Find the volume of a cone whose altitude is 20 ft. and the diameter of the base 4 ft.? Ans. 83.776 cu. ft.

4 The base of a pyramid is 12 in. square and its hight 15 in. and the base of a cone is 12 in. in diameter and its height 15 in. what is the difference in solid contents? Ans. 154.512 cu. in.

5 What are the solid contents of a rectangular pyramid whose base is 40 ft. long and 30 ft. wide, and the distance from the vertex to each corner of the base 65 ft.? Ans. 24090 cu. ft.

### FRUSTUMS OF PYRARIDS AND CONES—ORDINARY RULE FOR FINDING THE VOLUME OF FRUSTUMS.

Multiply the area of the lower base by the area of the upper base and extract the square root of the product to get a mean base. Multiply the sum of the three bases,—the lower base, upper base, and mean base, by the altitude of the frustum and take one-third the product.

The reasons for this rule cannot be shown objectively. They depend upon a demonstration of geometry which shows that the frustum of a pyramid is equal to three pyramids having for their common altitude the altitude of the frustum, and for their bases the lower base of the frustum, the upper base of the frustum and a mean proportional between these bases found by extracting the square root of their product.

EXAMPLES.

What is the volume of a frustum of a square pyramid, each side of the lower base being 3 feet, and each side of the upper base 2 feet, and the altitude being 30 feet.

Area of lower base=3×3=9

Area of upper base=2×2=4

Product of the bases=36

Mean base=$\sqrt{36}$=6

Sum of the bases=9+4+6=19

Sum of bases multiplied by the altitude=19×30=570

One-third of 570=190 ft. Ans.

The foregoing is the operation under the ordinary rule, based upon a geometrical demonstration.

ANOTHER WAY WHICH CAN BE EASILY UNDERSTOOD.

If we can by a little mental arithmetic complete the pyramid of which the frustum forms a part, it is evident that the volume of the frustum is equal to the difference between the volumes of the entire pyramid and the pyramid that has been cut off.

ILLUSTRATION WITH THE EXAMPLE HERETOFORE SOLVED.

An edge of the lower base is 3 feet and the edge of the upper base 2 feet. The difference between them is one foot; that is, the pyramid has tapered off 1 foot in 30 feet and to taper off the entire amount of 3 feet it must go 3 times 30 feet or 90 feet which is the altitude of the entire pyramid.

The area of the base 9 multiplied by one-third of the altitude gives 270 feet the volume of the entire period. The square of 2 gives 4 for the area of the upper base of the frustum and the base of the pyramid cut off. The altitude is 90 feet minus 30 feet the altitude of the frustom or 60 feet. One-third of 4 times 60 feet is 80 feet the volume of the pyramid cut off. 270 feet less 80 feet is 190 feet the volume of the frustum which of course corresponds with the answer previously obtained.

FRUSTUM OF A CONE.

The volume of the frustum of a cone is found in the same manner. If the diameter of the lower base is 3 feet and the upper base 2 feet and the altitude, 30 it tapers off 1 foot in going 30 feet and to taper of 3 feet it must have an altitude of 3×30 feet or 90 feet. The cone is thus completed and has an altitude of 90 feet and the upper section of the cone has an altitude of 60 feet. The difference between these two cones gives the volume of the frustum.

The principle may be stated as follows:

The altitude of the frustum of a pyramid or cone is such part of the altitude of the entire pyramid or cone as the difference between the lower and upper edges, the lower and upper diameter, is a part of the lower edge or diameter.

EXAMPLES.

1. What is the solid contents of the frustum of a square pyramid each side of whose upper base is 5 feet, of lower base 8 feet, and whose altitude is 12 feet?

SOLUTION.

The difference between 8 feet and 5 feet is 3 feet. The pyramid tapers off 3 feet in an altitude of 12 feet. To taper off 1 foot would require an altitude of $\frac{1}{3}$ of 12 feet, or 4 feet. To taper off 8 feet would require an altitude of $8\times4$ feet or 32 feet, the altitude of the completed pyramid.

$8\times8=64$=area of base.

$(64\times32)\frac{1}{3}=682\frac{2}{3}$= vol. entire pyramid.

$5\times5=25$=area of base of upper section of pyramid.

$32-12=20$ altitude of upper section.

$(25\times20)\frac{1}{3}=166\frac{2}{3}$=vol. of upper section.

$682\frac{2}{3}-166\frac{2}{3}=516$ cu. ft. vol. of frustum. Ans.

2. Find the solid contents of the frustum of a cone the diameters of whose bases are 10 in. and 6 in. and altitude 12 inches.

SOLUTION.

10—6=4 inches 4 in. is 4-10 or 2-5 of 10 in. the lower edge. The altitude 12 in. is therefore 2-5 of the altitude of the completed pyramid or 30 in. Ans. 615, 7536 cu. in.

3. Find the solid contents of the frustum of a pyramid whose two basis are 10 in. and 6 in. square and whose altitude is 15 inches.

$10\times6=4$; 4=4-10 or 2-5 of lower edge; 15 is 2-5 of $37\frac{1}{2}$ in. the altitude of pyramid, $37\frac{1}{2}-15=22\frac{1}{2}$=altitude of upper section of pyramid.

Ans. 980.

THE SPHERE.

As we reasoned from the square through the triangle to the circle, so we may reason from the cube by means of the pyramid to the sphere.

A cube 4 inches each way contains 64 cubic inches found by cubing the 4 or multiplying together the length, breadth and thickness.

The dissected cube will show that the cube is composed of six equal pyramids each having for its base a face of the cube and for its altitude the radius or semi-diameter of the cube or the distance from the middle of any face to the center of the cube.

$4\times4=16$=area of the base of each.

$16\times2=32$ and 32÷ by 3 gives $10\frac{2}{3}$ cubic feet, the volume of one pyramid.

$10\frac{2}{3} \times 6 = 64$ the volume of the six pyramids of the cube.

Again $4 \times 4 = 16$ the area of one face of the cube and one base of the the pyramids.

$16 \times 6 = 96$ the sum of the bases of the pyramids and the entire surface of the cube.

96 multiplied by 2 the altitude of each pyramid gives 192 and this divided by 3 gives 64 or the volume of the cube.

CONCLUSION.

The volume of a cube may be found by multiplying its surface by one-third of its radius, or by taking one-third of the product of its surface and its radius, because made up of pyramids, the sum of whose bases is the entire surface of the cube and whose altitudes are equal to the radius of the cube.

The sphere inscribed in the cubic frame will show that the radius of the cube is the same as the radius of the inscribed sphere, and, therefore, that the volume of a cube may be found my multiplying its surface by one-third of the radius of an inscribed sphere, that is, a sphere whose diameter is equal to an edge of a cube.

By cutting off the eight corners of the cube down to the serface of an inscribed sphere it will be seen that the new solid which approximates more nearly to the sphere than does the cube is made up of 14 pyramids 8 of which are triangular and 6 octagonal all of which have for their altitude the radius of the inscribed sphere.

The volume of this solid as it is composed of pyramids is equal to one-third of the product of its surface, the bases of the pyramids, and the radius of the inscribed sphere.

The eight triangular pyramids are respectively equal to each other and so are the six octagonal pyramids.

REGULAR SOLIDS.

The regular solids are only five in number. 1. The Tetrahedron or triangular pyramids all of whose faces are equal. 2. The Hexahedron or cube with six equal square faces. 3. Octahedron with 8 equal triangular faces. 4 The Dodecahedron with 12 equal pentagons for faces. 5. Icosahedron with 20 triangular faces. These last 4 solids are composed respectively of 6 equal square pyramids, 8 equal triangular pyramids, 12 equal pentagonal pyramids and 20 equal triangular pyramids.

The volume of each and all are equal of course to their surface or the bases of the pyramids multiplied by one-third of the radius of the inscribed shere.

When the number of faces of the solid become infinite the solid becomes a sphere, and a sphere is made up of pyramids. The surface of the sphere is the sum of the bases of the pyramids and the radius of the sphere is the altitude of the pyramids. Hence the volume of the sphere is equal to the surface multiplied by one-third of the radius, or one-sixth of the diameter.

Formula. Vol. of sphere=surface$\times\frac{1}{3}$ R.; or, Vol. of sphere=surface $\times$1-6 D.

But how teach illustratively the ordinary rule for finding the volume os a sphers viz:

Multiply the cube of the diameter by .5236; expressed by the formula; Vol. of sphere=$D^3\times.5236$?

FIRST PROCESS.

It has been shown objectively that the volume of a sphere is equal to its surface multiplied by 1-3 of its radius or 1-6 of its diameter because made up of pyramids.

Surface of sphere=4 circles of the sphere.

$D^2\times.7854$=one circle.

$D^2\times4\times.7854$ or 3.1416=4 circles.

Surface of sphere=$D^2\times3.1416$.

This multiplied by 1-6 D gives $D^3\times$1-6 of 3.1416 or $D^3\times.5236$.

SECOND PROCESS.

Take the metallic cylinder whose diameter and altitude are each 4 inches.

$4^2\times$ 7854=area of base. Multiply this by the altitude 4 and we have $4^2\times.7854\times4=4^3\times.7854$; that is, the volume of the cylinder whose diameter and altitude are equal, is found by multiplying the cube of its diameter by .7854.

Take next the metallic cone, fill it with water and pour its contents into the cylinder, by which the cylinder is one-third full; take then the 4 inch sphere and press it into the 4 inch cylinder. The water will rise and with the globe just fill the cylinder.

By this it is shown that a 4 inch sphere is $\frac{2}{3}$ of a 4 inch cylinder, or that any sphere is $\frac{2}{3}$ of a cylinder with an altitude and diameter equal to the diameter of the sphere.

Vol. of cylinder=$4^3\times.7854$.

Vol. of sphere=$\frac{2}{3}$ of $4^3\times.7854=4^3$ by $\frac{2}{3}$ of $.7854=4^3\times.5236$.

It is well to show by the forms the relations between the volumes of a 4 inch cube, a 4 inch cylinder and a 4 inch sphere; that the cylinder is .7854 of the cube or such part of it as its circular base is a part of the square base of the cube and that the sphere which is $\frac{2}{3}$ of the cylinder is $\frac{2}{3}$ of .7854 of the cube or .5236 of the cube.

Therefore $4^3$ expresses the volume of the cube; $4^3\times.7854$ expresses the volume of the cylinder, and $4^3\times.5236$ the volume of the sphere.

RULES FOR THE SPHERE.

1. Multiply its surface by $\frac{1}{3}$ the radius.
2. Multiply its surface by 1-6 of the diameter.
3. Multiply the cube of the diameter by .5236.

# EXAMPLES.

1. What is the volume of a 10 inch cube? A 10 inch cylinder? A 10 inch sphere? Answers, 1000; 785.4; 523.6.
2. How many cubic inches in a 9 inch globe? Ans 381.7044.
3. What is the volume of a globe whose radius is 20 inches. Ans, 18 cu. ft. 6784 cu. in.
4. Find the weight of 500 cannon balls each 2 ft. in diameter, a cubic foot weighing 430 pounds. Ans. 450 tons 592 pounds.
5. Find the solid contents of a sphere whose diameter is 8ft 6 in. Ans. 321.555 cu. ft.

RECAPITULATION OF SURFACE MENSURATION.

1. Apply the square inch to the 6×1 oblong.
2. Apply the 6×1 oblong to the 6×4 oblong and develop rule.
3. Show the equality of the rhomboid and the oblong by detaching the dowelled piece on the rhomboid and making necessary transfer.
4. Show the equality of the trapezoid and the oblong by the transfer of the dowelled piece and develop the rule.
5. Show by forms that a triangle is half a square, an oblong or a rhomboid and that its area is therefore half the product of its base and altitude.
6. Show by the dissected square that it is made up of triangles and that its area is therefore equal to its boundary multiplied by half the radius of the inscribed circle.
7. Show by the dissected circle that as it is composed of triangles, it is equal to its circumference multiplied by one-half its radius or 1-4 its diameter and develop the ordinary rule.
8. Show that the surface of prisms is made up of oblongs, pyramids of triangles, and frustums, of trapezoids, and develop the rules for the surface of prisms, cylinders, pyramids, cones and frustums.

RECAPITULATION OF THE MENSURATION OF SOLIDS.

1. Show by the dissected oblong prism 6×1×1 that it contains 6 cubic inches, and 4 such pieces contain 24 cubic inches, and develop the rule, "the length, breadth and thickness multiplied together, "and the area of its base by the altitude."
2. Show by the bisected quadrangular prism, that the same principle—"the base by the altitude," will apply to tringular prisms.

3. Show by the dissected hexagonal prism that all prisms are made up of triangular prisms, and therefore, that the volumes of all prisms and the cylinder are found by multiplying the area of their bases by their altitudes.

4. Show by the trisected triangular prism that a triangular pyramid is one-third of a triangular prism of the same base and altitude.

5. Show by the bisected square pyramid that all pyramids are made up of triangular pyramids, and, therefore, that all pyramids including the cone are equal to the product of the base and one-third of the altitude.

6. Again show that a cone is one-third of a cylinder with the same base and altitude by pouring the 4 inch cone three times full of water into the 4 inch cylinder.

7. Also develop the rule for the pyramid by means of the dissected cube as follows: The area of the base of the cube or the square of 4 inches multiplied by the entire altitude 4 gives the volume of the cube. As each of the pyramids is one-sixth of the cube its volume is equal to the area of its base which corresponds to the base of the cube multiplied by one-sixth of 4, or one-third of 2, the altitude of each pyramid.

8. Show that the volumes of frustums can be best found by completing the pyramid or cone and finding the difference between the entire pyramid or cone and the upper section,

9. Show by the dissected cube that as it is composed of six pyramids, its volume is equal to its surface multiplied by one-third of its radius. Show by the cube-inscribed-sphere that the radius of the cube is the same as the radius of the inscribed sphere. Show that by cutting off the corners, a 14 faced solid is formed composed of 14 pyramids and that its volume and the volume of the variety of solids leading to the sphere, and the sphere itself which is made up of pyramids is equal to the surface of each multiplied by one-third of the radius of the sphere, or one-sixth of its diameter, and develop the ordinary rule, $D^3 \times .5236$.

10. Show by pouring the cone full of water into the 4-inch cylinder and inserting the 4-inch sphere, that the sphere is two-thirds of the cylinder and therefore, that as the volume of the 4-inch cylinder is equal to the cube of its diameter multiplied by .7854 the volume of the 4-inch sphere is equal to the cube of its diameter multiplied by two-thirds of .7854, or .5236.

11. Show by the solids the relations existing between a 4-inch cube, cylinder and sphere.

### SUMMARY OF SOLIDS.

Prism—cylinder. Base $\times$ altitude. Pyramids—cones. Base $\times$ $\frac{1}{3}$ altitude. Cube—pyramid—sphere.

### SQUARE ROOT AND CUBE ROOT FORMS.

Find the square root of 5625 feet and illustrate with forms.

```
       5625 | 75
       49      —
      ————
 145 | 725
     | 725
      ————
       000
```

To extract the square root of 5625 is to find the side of a square that contains 5625 square feet.

There are two periods and the root will consist of tens and units. The greatest square that can be constructed out of the quantity expressed by the left hand period is 7 tens or 70 feet long, the assumed dimensions of the trial block or square. The square of this, or 49 hundreds or 4900, subtracted from 5625 leaves 725 square feet of material to be added to this trial square.

To preserve the square the addition must be made on two sides or two times 7 tens or 70 feet, which is 14 tens or 140 feet in length.

By dividing the area of the materials to be added to the trial square by the length, which may be effected by dividing 72 tens by 14 tens, that is by dividing by double the first figure of the root, rejecting the right hand figure of the dividend, it is seen that an addition of 5 feet in width may be made to the two sides of the trial square.

By placing the added blocks it is seen there is a corner defficiency whose length and breadth is equal to the second figure of the root. Its surface can best be obtained along with the side additions by annexing it to the trial divisor. The product of the divisor thus completed gives 72.5 the surface of the entire addition, 145 ft. in length and 5 ft. in width which uses up the material to be added and gives a square of 75 feet.

CUBE ROOT.

Find the cube root of 13824.

```
                    13824 | 24
                     8
                    ——————
20×20×3=1200 | 5824
20× 4×3= 240 |
    4×4=  16 | 5824
        ———— | ——————
        1456 |
```

To extract the cube root of 13824 is to find the edge of a cube whose volume is 13824 cubic feet.

As there are two periods of three figures each the root will consist of tens and units.

The greatest cube that can be constructed out of 13 thousand is 8 thousand, the cube root of which is 2 tens or 20.

In other words the greatest number whose cube is contained in 8 thousand is 2 tens or 20.

The dimensions of the trial cube is 20. Its volume, 8000, being taken from 13824. usually effected by taking 8 from 13 and bringing down the next period, leaves 5824 cubic feet to be added to the trial cube.

In order to preserve the cube the addition must be made on three faces.

The square of the first figure of the root as tens multiplied by 3 gives the surface of the three faces, or 1200 square feet, and using this as a trial divisor it is found that an addition can be made upon this surface 4 feet thick.

By placing upon the three faces of the trial cube the blocks whose thickness is assumed to be 4, it is seen that there are three side deficiencies.

The surface of the three blocks necessary to fill these deficiencies is found by multiplying the first figure of the root as tens, the length of the blocks by the second figure, the width of the blocks, and by three.

By placing the three blocks in their places it will be seen that there is still a corner deficiency, the surface of which is found by squaring the second figure of the root.

The sum of the surfaces of the three faces of the trial cube, the three side and the corner deficiencies, gives 1456 the complete divisor, or entire surface of the additions, and this multiplied by 4 the second figure of the root and the thickness of the additions, gives 5824 cubic feet, the solid contents of the additions.

Subtracting this it is seen that the material to be added to the trial cube is used up and that the dimensions of the constructed cube is 24 ft.

If preferred the solid contents of each addition can be found separately and their sum substracted in accordance with the following operation.

```
                    13824 | 24
                    8
                    -----
20×20×3=1200     |  5824
                    -----
                    4800
   4×4×20×3=         960
      4×4×4=          64
                    -----
                    5824
                    -----
                    0000
```

BASE, PERPENDICULAR AND HYPOTHENUSE.

Use the right-angled triangular form with attached squares which clearly shows that the sum of the squares erected on the base and perpendicular is equal to the square erected on the hypothenuse, to develop the rules for finding the base, perpendicular and hypothenuse, of a right-anled triangle, where any two of them are given.

# TESTIMONIALS.

*From Dr. Findley, Editor of the Ohio Educational Monthly.*

AKRON, O., July 25, 1891.

Having heard Sup't Ross's admirable presentation of Form, Mensuration, etc., in Teachers' Institutes, I have no fear of commending too strongly the appliances he has devised to aid teachers in presenting these subjects. They are invaluable to teachers and pupils. Every school should have them. — SAMUEL FINDLEY.

*From Dr. W. G. Williams, of the Ohio Wesleyan University.*

DELAWARE, O., August 4, 1894.

I heartily endorse Dr. Findley's words, W. G. WILLIAMS.

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*From the Hon. Alston Ellis, Supt. of Schools, Hamilton, O., and Secretary of the Ohio State Board of School Examiners.*

I have been a worker with W. W. Ross, of Fremont, Ohio, in Teacher's Institutes, and have seen his objective presentation of the subject of "Mensuration." His work is simple and thorough, easily understood and wins enthusiastic commendation everywhere. Many of the forms used for his illustrative work are original and make "Mensuration" a delightful and intelligent study. *Anyone possessing a knowledge of the fundamental rules of arithmetic can master nearly all the processes of mensuration and the principles underlying them by an intelligent use of Mr. Ross's forms.* These forms should be a part of the outfit of every schoolroom in the country. Your work in mensuration cannot be too highly praised. — ALSTON ELLIS.

*From the Hon. Thos. W. Harvey, formerly Ohio State School Commissioner. Author of Harvey's Grammar, &c.* PAINESVILLE, O., August 15, 1891.

The appliancess invented by W. W. Ross to aid teachers in giving instruction in menuration are valuable if not indispensable aids for that purpose. II take pleasure in recommending them to students, teachers and school officers as the best appliances of their kind I have ever examined or seen used. That you will succeed in securing the general use of the "aids" in our schools, is my sincere wish and desire.

— THOS. W. HARVEY.

*From Hon. C. C. Miller, Ohio State Commissioner of Common Schools.*

To whom it may concern:— COLUMBUS, O., August 3, 1891.

I have heard Supt. Ross instruct in Teachers' Institutes and Associations. His work on Mensuration is of a very high and scholarly character. He has an excellent set of blocks and appliances for the purpose, and his use of them shows his mastery of the theme.

The apparatus that Supt. Ross offers for sale is among the most approved and thoroughly adapted of modern school aids in teaching the difficult subject of mensuration.

I commend his apparatus in strong terms for its use will greatly add to the value of the teacher's work. CHAS. C. MILLER, Commissioner.

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*From Prof. E. F. Warner, Supt. of Schools, Bellevue, Ohio.*

Dear Sir and Bro.:— MASSILLON, O., August 3, 1891.

You stated at Delta last fall that you thought some of having your mensuration blocks manufactured for school use. Has this been done? If so, I desire to procure several sets, both for my own and school use. I consider your presentation of the subject the best I have ever seen.

E. F. WARNER.

BELLEVUE, O., August 9, 1891.

To the public:—

After a careful study of various methods of presenting the subject of mensuration I take pleasure in saying that the methods and appliances devised by Supt. W. W. Ross, of Fremont, O., are by all odds the best with wihch I am acquainted. Every school and every teacher of the subject will do well to use them. E. F. WARNER.

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*From the Hon. J. J. Burns, Supt. of Schools, Canton, O., and formerly Ohio State School Commissioner.*

CANTON, O., August, 1891.

To whom it may concern:—

I have heard Supt. Ross discuss the subject of mensuration before Teachers' Institutes. His treatment of the subject is by far the best and most complete I have ever heard and seen. I cordially commend his book and appliances to the use of teachers. Very respectfully,

J. J. BURNS.

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*From Geo. H. Withey, Supt. of Schools, Oak Harbor, O.*

OAK HARBOR, O., August 8, 1891.

My dear Ross:—

I take great pleasure in calling the attention of teachers in my section of the state to the use of your Dissected Surface Forms and Geometrical Solids. The blocks are certainly the most complete now in the market. "Seeing is believing." A novice can by the aid of the blocks "see why" every operation is performed. Their novelty and simplicity aid the teacher, entertain and instruct the pupil and reduce the abstract features of mensuration to a simple, visible study. Very truly,

GEO. H. WITHEY.

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*From D. F. Mock, Supt. of Schools, of West Salem, Ohio.*

WEST SALEM, O., August 5, 1891

The Objective Forms prepared by Prof, W. W. Ross to aid in teaching Mensuration are very helpful especially to beginners. The pedagogic rules, "Proceed from the concrete to the abstract," "from the seen to the unseen" apply most happily and forcibly to this difficult and important branch of arithmetic. Every teacher should have a set in his school. Having tried them, we venture the prediction he would not teach mensuration without them.

D. F. MOCK.

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*From the Hon. J. W. Knott, of Columbus, O., a Member of the Ohio State Board of School Examiners, and Supt. of Ohio Institute for the Deaf.*

COLUMBUS, O., August 6, 1891.

I know Prof. W. W. Ross well. I have heard him give instruction on Form and Mensuration illustrated by his appliances, at Teachers' Institutes. I do not hesitate to commend his appliances for teaching the subject. In the hands of a skillful teacher they will prove highly valuable to any school and will well repay any Board of Education for the money required to buy them. Respectfully,

J. W. KNOTT.

*From the Hon. H. M. Parker, Supt. of Schools, Elyria, Ohio, and Member of the Ohio State Board of School Examiners.*

ELYRIA, O., August 6, 1891.

I have heard Supt. Ross present the subject of Mensuration at a Teachers' Institute and was highly pleased with his methods and appliances as were all who heard him. I have no hesitancy in saying that his "Dissected Surface Forms and Geometrical Solids" are the simplest and best of anything I have seen for use in the class-room.

Respectfully, H. M. PARKER

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XENIA, O., August 15, 1891

*From Hon. E. B. Cox, Member of Ohio State Board of School Examiners*

Supt. W. W. Ross, Fremont, Ohio.

Dear Sir:—Your forms for teaching Mensuration objectively are the best I have ever seen. Your exposition of their use will be an invaluable aid to any one teaching this subject. I very cordially recommend both the Forms and the Manual to boards of education and to teachers.

Yours truly, E. B. COX.

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FINDLAY, O., Sept. 23, 1891.

To Whom It May Concern:

I have seen and examined Supt. W. W. Ross's admirable Mensuration Blocks. We have used Kennedy's Blocks for years but Mr. Ross's Blocks are far superior to Kennedy's in that they illustrate not only some but all the principles of Mensuration. Very respectfully,

J. W. ZELLER.

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NORWALK, O, October 25, 1891.

*From Prof. S. F. Newman, member of the Board of Huron County School Examiners, and for many years Principal of Western Reserve Normal School, Milan, O.:*

Supt. W. W. Ross, Fremont, O., Dear Sir:

I have used many Objective Forms in teaching Mensuration, but yours are by far the best I have ever seen. They enable each pupil to make his own rules and to fully understand them. Teachers who use your Forms will no longer be compelled to give a reason for certain operations in Arithmetic, "Because the rule says so."

Boards of Education should not ask "Can we afford to buy your Blocks," but rather "Can we afford to do without them!" for, by their use a better knowledge of the subject of Mensuration can be obtained in one month than in years by the old method of instruction. They ought to be used in every school in the United States. Yours truly.

S. F. NEWMAN.

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*From Supt. C. W. Butler, Defiance, Ohio:*

DEFIANCE, OHIO, September, 22, 1891.

To Whom it May Concern:

I have carefullly examined the Mensuration Blocks invented by Supt. W. W. Ross of Fremont, Ohio, and take pleasure in recommending them. They are the best means of teaching the subject of Mensuration I have ever seen. Respectfully

C. W. BUTLER, Supt. Schools.

*From C. W. Bennett, Supt. of Schools, Piqua, O., and Ex-president of Ohio State Teachers' Association.*

PIQUA, O., Setpt. 12, 1891.

The Dissected Surface Forms and Geometrical Solids, invented by Supt. W. W. Ross are the best helps of their kind in the market and are suited to school work everywhere. Any teacher can use these forms. They are simple, easily adjusted, and an indispensable aid in teaching Arithmetic and Geometry.

Very truly,

C. W. BENNETT.

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*From the Sandusky County Board of School Examiners, consisting of Supt. F. M. Ginn, Clyde, O., G. F. Aldrich, Millersville, Ohio, and Rev. J. I. Swander, D. D., Ph. D., author of "Substantial Philosophy," "Invisible World," "History of the Reformed Church."*

We fully and heartily endorse the foregoing testimonials of Prof. Ross's appliances for teaching Form and Mensuration, and their great value in school work.

F. M. GINN, J. I. SWANDER, G. F. ALDRICH.