Modelling Geomorphological Systems
Edited by M. G. Anderson

Chapter 10

Modelling erosion on hillslopes

L. J. LANE and E. D. SHIRLEY
Southwest Rangeland Watershed Research Center, Agricultural Research Service, Tucson
and
V. P. SINGH
Department of Civil Engineering, Louisiana State University

10.1 INTRODUCTION

Surface runoff on upland areas such as hillslopes is often accompanied by soil erosion. Soil particles may be detached when the impact of raindrops exceeds the soil's ability to withstand the impulse at the soil surface. Detachment may also occur when shear stresses caused by flowing water exceed the soil's ability to resist these erosive forces. Vegetation as canopy and ground cover, and other surface cover such as gravel and rock fragments, protect the soil surface from direct raindrop impact, and also provide hydraulic resistance, reducing the shear stresses acting on the soil. Plant roots, incorporated plant residue, and minerals increasing cohesion tend to protect the soil by reducing the rate of soil particle detachment by flowing water and raindrop impact.

Once detachment has occurred, sediment particles are transported by raindrop splash and by overland flow. Conditions which limit raindrop detachment limit the sediment supply available for transport by splash and flow mechanisms. Vegetative canopies intercept splashed sediment particles and limit sediment transport by splash. The rate of sediment transport by overland flow is influenced by the factors controlling the amount of sediment available for transport, the sediment supply, and by hydraulic processes occurring in overland flow such as raindrop impacts, depth of flow, velocity, and accelerations due to microtopographic flow patterns. Obviously, the steepness, shape, and length of slopes affect both flow patterns and the resulting sediment transport capacity of the flowing water.

After sediment particles are detached from soil areas above, between, and near locations of small flow concentrations, they may enter the flow concentration areas for subsequent

Table 10.1. Some examples of empirical and conceptual models

Model Type	Model	Author
Empirical	Musgrave Equation	Musgrave (1947)
	Universal Soil Loss Equation (USLE)	Wischmeier and Smith (1978)
	Modified Universal Soil Loss Equation (MUSLE)	Williams (1975)
	Sediment Delivery Ratio Method	Renfro (1975)
	Dendy–Boltan Method	Dendy and Boltan (1976)
	Flaxman Method	Flaxman (1972)
	Pacific Southwest Interagency Committee (PSIAC) Method	Pacific Southwest Interagency Interagency Committee (1968)
	Sediment Rating Curve	Campbell and Bauder (1940)
	Runoff–Sediment Yield Relation	Rendon-Herrero (1974), Singh, Baniukiwicz and Chen (1982)
Conceptual	Sediment Concentration Graph	Johnson (1943)
	Unit Sediment Graph	Rendon-Herrero (1978)
	Instantaneous Unit Sediment Graph	Williams (1978)
	Discrete Dynamic Models	Sharma and Dickinson (1979)
	Renard–Laursen Model	Renard and Laursen (1975)
	Sediment Routing Model	Williams and Hann (1978)
	Muskingum Sediment Routing Model	Singh and Quiroga (1986)
Physically based	Quasi-Steady State Erosion	Foster, Meyer and Onstad (1977)
	Kinematic Wave Models	Hjelmfelt, Piest and Saxton (1975), Shirley and Lane (1978), Singh and Regl (1983)
	Continuum Mechanics Model	Prasad and Singh (1982)

During the period 1940–1954, work in the Corn Belt of the United States resulted in a soil loss estimation procedure incorporating the influence of slope length and steepness (Zingg, 1940), conservation practices (Smith, 1941; Smith and Whitt, 1947), and soil and management factors (Browning *et al.*, 1947). In 1946, a national committee reappraised the Corn Belt factor values, included a rainfall factor, and produced the resulting Musgrave equation (Musgrave, 1947).

During the period 1954–1965, the USLE was developed by the United States Department of Agriculture (USDA), Agricultural Research Service in cooperation with the USDA–Soil Conservation Service and state agricultural experiment stations. Plot data from natural storms and from rainfall simulator studies formed the USLE data base. During the 1965–1978 period, additional data and experimental results were incorporated, resulting in the current USLE (Wischmeier and Smith, 1978).

The USLE in equation form is:

$$A = RKLSCP \tag{1}$$

where:

2. Separate the baseflow Q_b from the runoff hydrograph Q_T using a standard hydrograph separation technique to obtain the direct runoff hydrograph Q,

$$Q(t) = Q_T(t) - Q_b(t) \tag{2}$$

3. Using the same baseflow separation technique, separate out the sediment concentration due to baseflow. It should be noted that Rendon-Herrero assumed that the maxima of runoff and sediment concentration occurred at the same time.
4. Compute sediment discharge Q_s due to direct runoff by noting that sediment discharge is the product of water discharge and sediment concentration,

$$Q_s = Q_T C_T - Q_b C_b \tag{3}$$

5. Compute the volume of direct runoff, which is the area under the direct runoff hydrograph.

$$V_Q = \int_0^{\infty} Q(t)\mathrm{d}t \tag{4}$$

6. Compute the sediment yield, which is the area under the sediment graph due to direct runoff.

$$V_s = \int_0^{\infty} Q_s \, \mathrm{d}t \tag{5}$$

7. Divide the ordinates of the sediment graph by the sediment yield to obtain ordinates of the USG, H_s,

$$H_s = \frac{Q_s}{V_s} \tag{6}$$

The USG varies somewhat with the intensity of the effective rainfall. It can be used to generate a sediment graph for a given storm if the wash load produced by that storm is known. A relationship between V_s and V_Q was proposed. Using this relation, V_s can be determined. Therefore, Q_s can be determined by multiplying H_s with V_s. It must be noted that the duration of the USG chosen to determine Q_s must be the same as that of the effective rainfall generating V_Q. This USG method was tested on a small wash load-producing watershed, Bixler Run Watershed, near Loysville, Pennsylvania.

Rendon-Herrero (1974) proposed the use of the so-called 'series' graph to determine the sediment hydrograph. This method has the advantage that the duration of the effective rainfall is neglected altogether, but requires construction of the series graphs beforehand. Thus, this method cannot be extended to ungauged basins. Williams (1978) and Singh *et al.* (1982), among others, have used the USG to model watershed sediment yield.

10.1.3 Development of Physically Based Erosion Models

Fundamental erosion mechanics were of interest to scientists and engineers as early as 1936 (Cook, 1936), and were described in terms of subprocesses by Ellison (1947). Negev (1967)

10.1.5 Purpose

The first purpose of this chapter is to describe the evolution and status of erosion models for hillslopes based upon the kinematic wave equations for overland flow, and on the interrill and rill terms for erosion. The second purpose is to examine a particular erosion model for which analytic solutions can be obtained, and then to discuss the mathematical properties and implications of the solutions as they relate to experimental design and interpretation of experimental data.

10.2 OVERLAND FLOW AND EROSION EQUATIONS

The development of improved erosion equations for overland flow is based upon prior development of improved flow equations. That is, the development of methodology for simulation of unsteady and spatially varying overland flow made the subsequent simulation of interrill and rill erosion possible.

10.2.1 The Shallow Water Equations

Unsteady and spatially varying and one-dimensional flow per unit width on a plane was described by Kibler and Woolhiser (1970) using the following equations:

$$\frac{\partial h}{\partial t} + \frac{u\partial h}{\partial x} + \frac{h\partial u}{\partial x} = R \tag{7}$$

and

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{g\partial h}{\partial x} = g(S_o - S_f) - (R/h)(u - v) \tag{8}$$

where

h = local depth of flow (dimension of length, L),
u = local mean velocity (L/T),
t = time (T),
x = distance in the direction of flow (L),
R = lateral inflow rate per unit area (L/T),
g = acceleration of gravity (L/T^2),
S_o = slope of the plane,
S_f = friction slope, and
v = velocity component of lateral inflow in the direction of flow (L/T).

Equation 7 is the continuity of mass equation, and equation 8 is the one-dimensional momentum equation. In general, equations 7 and 8 must be solved numerically. Modelling real overland flow with one-dimensional equations represents significant abstractions and simplifications. Real overland flow occurs in complex mixes of sheet flow and small concentrated flow areas. The routes of concentrated flow are often determined by irregular microtopographic features which vary in the downstream direction (x) and in the lateral direction (y). Definitive analyses of the influences of such simplications upon hydraulic and erosion paramaters are nonexistent.

are a good approximation to the solutions to the shallow water equations, provided the kinematic flow number is larger than about 20. It is important to note that this refers to the accuracy with which the kinematic wave solutions approximate solutions to the shallow water equations for sheet flow on a plane. The kinematic flow number says nothing about how well the shallow water equations, with one-dimensional flow and spatially uniform parameters, approximate overland flow on natural surfaces.

10.2.3 Equations for Erosion by Overland Flow

The sediment continuity equation, with the kinematic assumptions, is quite similar to the water continuity equation on the left hand side. The right hand side of the sediment continuity equation is commonly separated into an interrill erosion term, E_I, and the rill erosion term, E_R. With these assumptions, the continuity equation for sediment is:

$$\frac{\partial(ch)}{\partial t}+\frac{\partial(cq)}{\partial x}=E_I+E_R \tag{12}$$

where:

c = sediment concentration (M L^{-3}),
E_I = interrill erosion rate per unit area per unit time (M L^{-2} T^{-1}), and
E_R = net rill erosion (or deposition) rate (M L^{-2} T^{-1}),

and the other variables are as described earlier. The procedure is to solve the flow equations first, and then solve equation 12 for sediment concentration. Total sediment yield for a storm, V_s, is then found by integrating the product cq over the period of runoff.

The interrill term, E_I

The rate of interrill erosion is a function of the rate of detachment by raindrop impact and the rate of transport from the point of detachment to a rill.

As discussed in the introduction, interrill erosion is, by definition, caused by raindrop detachment and the rate of transport in the shallow interrill flow. On steep slopes, the rate of detachment by raindrop impact limits interrill erosion, whereas transport capacity in interrill flow limits the rate of delivery on flat slopes (Foster, Meyer, and Onstad, 1977). These authors, and others, document the dependence of interrill erosion on soil characteristics, slope steepness, and canopy and ground cover. In equation form, this can be expressed as

$$E_I=f(I,S,C,Soil) \tag{13}$$

where I, S, and C are rainfall intensity, slope of the land surface, and cover effects, respectively. *Soil* refers to the soil characteristics, primary particle-size distribution, type and amount of clay and crusting, and land use influencing soil properties, such as density and aggregation, which affect raindrop detachment and shallow flow. Following are some selected interrill erosion terms.

A simple functional form incorporating rainfall intensity, I, as a measure of the erosivity of raindrop impact is

$$E_I=aI^2 \tag{14}$$

equation 18 can describe the rate of deposition if the coefficient f is a deposition coefficient. The deposition coefficient is primarily a function of particle characteristics, and is often calculated as a function of the particle fall velocity and the steady-state discharge rate (Foster, 1982).

10.2.4 Numerical Solutions

As stated earlier, equations 7 and 8 are solved numerically. Finite difference techniques are usually used (i.e. see Kibler and Woolhizer, 1970). If R, in equation 9, varies in space and time, then equations 9 and 10 must be solved numerically. If R in equation 10 varies, or if E_I and E_R in equation 12 are complex functions, then equation 12 must be solved numerically. The advantage of numerical techniques in solving the above equations is that one need not make as many assumptions as is required for analytic solutions, and the rainfall excess term can vary in time and space.

The disadvantages of numerical techniques, compared with analytic solutions, is that the former usually require much more computer time, the solutions are approximations of the real solutions, and the mathematics required for sensitivity analysis, limits, and other manipulations may be unavailable or very complex and difficult.

10.2.5 Analytic Solutions

Equations 9 and 10 can be solved analytically (by the method of characteristics) if R is uniform over the plane, and the temporal variation in R is described by a series of step functions. However, to obtain an analytic solution for equation 12, R in equation 9 must be uniform and constant for a finite or infinite duration. Equations 9 and 10 must be solved first to substitute into equation 12. Also, the form of T_c, in equation 18, should be simple, for example, a linear function of q, to obtain an analytic solution.

As stated earlier, the disadvantages of analytic solutions, in comparison with numerical solutions, is that they usually require much more restrictive and simplifying assumptions. The main advantages of analytic solutions include the ease with which they can be implemented on a computer, the speed with which they can be evaluated, the simplicity of sensitivity analysis, and the ease with which one can examine limits and other mathematical properties of the solutions.

10.3 SIMPLIFIED EQUATIONS WITH ANALYTIC SOLUTIONS

In this section, specific assumptions and simplifications are made to allow the derivation of analytic solutions for overland flow on a plane, and for interrill and rill erosion with overland flow. Analytic solutions to the runoff and erosion equations are used to illustrate field data needed for estimation of parameter values and for interpretation of processes controlling erosion.

10.3.1 The Basic Assumptions

In addition to the assumptions necessary for derivation of the one-dimensional shallow water equations and their approximating kinematic wave equations, specific assumptions

Finally, Shirley and Lane (1978) showed that the mean concentration, C_b, over the entire hydrograph is

$$C_b = Qs/Q = B/K + (K_t - B/K)(1 - \exp(-K_R x))/K_R x \tag{27}$$

If $B/K > K_t$, then $C_o < C_b < C_f$ and $c(t,x)$ for fixed x is a non-decreasing function of t. It can also be shown for fixed t that if $B/K > K_t$, then $c(t,x)$ is a non-decreasing function of x. These two non-decreasing functions mean (in the context of this particular model) that if $B/K > K_t$, then there is more transport capacity in the rills than is being satisfied by sediment input from the interrill areas. As a result, rill erosion occurs at all times and at all positions on the plane. In terms of sediment concentration graphs measured in the field, measured concentrations would tend to start at K_t near $t = 0$, and increase throughout the duration of runoff, assuming, of course, that the model is a good representation of reality.

If $B/K < K_t$, then the opposite is true. Under these conditions, $c(t,x)$ for fixed x would be non-increasing, or tend to decrease with increasing t. Also, $c(t,x)$ would be non-increasing with x and a fixed t. Again, if the model is correct, then measured concentrations would tend to start at K_t near $t = 0$, and decrease throughout the duration of runoff. If $B/K = K_t$, then transport capacity and existing sediment load are in equilibrium, so $C_o = C_f = C_b$, and, in fact, $c(t,x) = K_t$ for all x and t.

The implications of these results for plot and hillslope studies are that sediment concentration should be measured throughout the duration of runoff, and that analysis of data, using this model for parameter identification, should concentrate on events with nearly constant rainfall intensity and nearly saturated initial soil water content. The last two conditions will tend to make rainfall excess nearly constant, as assumed in the analysis. Fortunately, these conditions can nearly be met in rainfall simulator studies if data from runs where the initial soil water content is near saturation and the infiltration rate is nearly a constant are obtained for analysis.

Therefore, as a first approximation, one can examine the shape of the sediment concentration *vs.* time curve from a particular event on an experimental plot, and infer whether transport capacity in the rills ($B/K < K_t$) or detachment rate ($B/K > K_t$) in the rills is limiting sediment yield.

10.4 DISCUSSION

Although the Universal Soil Loss Equation remains the most often used model for predicting erosion on upland areas, more physically based models are emerging, and may become practical tools in the near future (i.e. see Rawls and Foster, 1986). As these new models emerge, they will probably be based upon unsteady and nonuniform overland flow modelled with the kinematic wave equations. Moreover, interrill and rill erosion processes will probably be explicitly represented in the partial differential equation used to describe erosion and overland flow.

The implications for plot and hillslope studies are that more, and more intensive, data need to be collected throughout the duration of runoff events, and at various positions on the slope. Only then can we begin to quantify unsteady and spatially varying overland flow and erosion processes.

2. Solutions in the Regions

a. *Domain of Flow Establishment*. In this region, the flow is unsteady but uniform:

$$h(t,x) = Rt \tag{A7}$$

b. *Domain of Established Flow*. In this region, the flow is steady but not uniform:

$$h(t,x) = (Rx/K)^{1/m} \tag{A8}$$

c. *Domain of Prerecession*. In this region, the flow is steady and uniform:

$$h(t,x) = Rt_* \tag{A9}$$

d. *Domain of Recession*. In this region, the flow is unsteady and not uniform:

$$h(t,x) = f_t^{-1}\,(Rx/K) \tag{A10}$$

where

$$f_t(u) = u^m + Rmu^{m-1}\,(t - t_*) \tag{A11}$$

The solutions described above are also shown in Figure 10.1.

Summary of Solution Regions and Solutions for the Sediment Concentration Equations in the $t-x$ Plane

Recall that the erosion equations are:

$$\frac{\partial(ch)}{\partial t} + \frac{\partial(cq)}{\partial x} = E_I + E_R \tag{A12}$$

with

$$E_I = K_I R \tag{A13}$$

and

$$E_R = K_R(Bh^n - cq) \tag{A14}$$

where the variables are as defined previously in the text.

1. **Domains in the $t-x$ Plane for Solutions of the Sediment Concentration Equations**

Solutions for the concentration equations require that the positive quadrant of the $t-x$ plane be divided into seven regions. The regions listed below are also shown in Figure 10.2.

a. *Domain 1*. This region of the plane represents time from zero until cessation of rainfall excess and distance down the plane such that concentration and flow have not reached steady state:

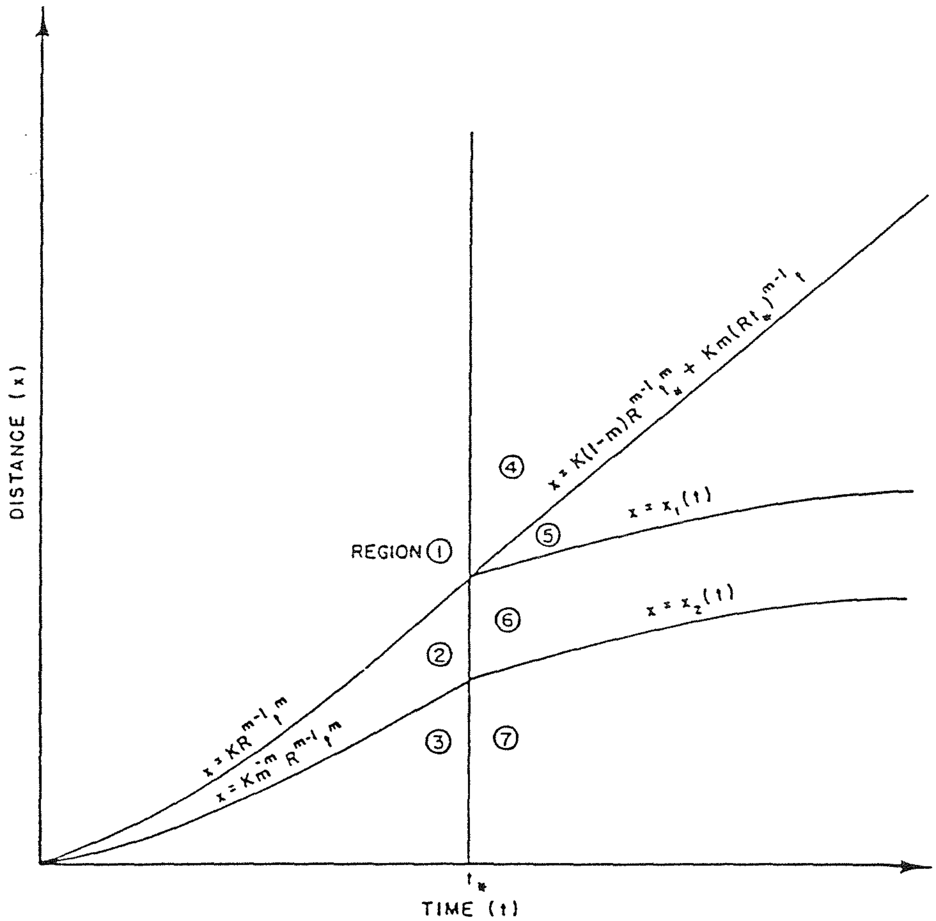

Figure 10.2. Domains in the $t-x$ plane for solutions of the overland flow erosion equations for a constant and uniform rainfall excess rate of duration t_*

$$\begin{aligned} &0 \leq t \leq t_* \\ &0 \leq x \leq Km^{-m} R^{m-1} t^m \end{aligned} \tag{A17}$$

d. *Domain 4.* This region, corresponding to the domain of prerecession for flow, represents time after cessation of rainfall excess before depth of flow is receding, and before the arrival of the slower travelling concentration disturbance from the interaction of the water wave with cessation of rainfall excess:

$$\begin{aligned} &t \geq t_* \\ &x \geq K(1-m)R^{m-1} t_*^m + Km(Rt_*)^{m-1} t \end{aligned} \tag{A18}$$

In Domains 5–7, let

$$a(u) = t_* + (K_0 u^m - mu^{-1}/(m+1))/R(m-1), \tag{A19}$$

2. Solutions in the Regions

a. *Domain 1.* In this region

$$c(t,x) = K_I + K_R(B/K - K_I)uF(u) \quad \text{(A28)}$$

where

$$u = KR^{m-1}\, t^m/m \quad \text{(A29)}$$

and

$$F(u) = \int_0^1 v^{1/m} \exp(K_R u(v-1))\, dv \quad \text{(A30)}$$

b. *Domain 2.* In this region

$$c(t,x) = B/K + (K_I - B/K)(1 - \exp(K_R(x_0 - x)) \\ ((K_R x_0)^2/mF(x_0/m) + 1 - K_r x_0))/(K_R x) \quad \text{(A31)}$$

where

$$x_0 = KR^{m-1}\, ((m(Rx/K)^{1/m} - Rt)/R(m-1))^m \quad \text{(A32)}$$

c. *Domain 3.* In this region

$$c(t,x) = B/K + (K_I - B/K)(1 - \exp(-K_R x))/K_R x) \quad \text{(A33)}$$

d. *Domain 4.* In this region

$$c(t,x) = B/K + (K_I - B/K)(1 - K_R x_*/mF(x_*/m))\exp((K_R x_*/t_*)(t_* - t)) \quad \text{(A34)}$$

where

$$x_* = KR^{m-1}\, t_*^{\,m} \quad \text{(A35)}$$

e. *Domain 5.* In this region

$$c(t,x) = B/K + c_0 \exp(-K_R x) \quad \text{(A36)}$$

where

$$K_0 = (R(m-1)(t - t_*) + mh(t,x)/(m+1))h^m(t,x) \quad \text{(A37)}$$

and

$$c_0 = (c(a(1/Rt_*),\, b(1/Rt_*)) - B/K)\exp(K_R b(1/Rt_*)) \quad \text{(A38)}$$

where c is computed using the formula from domain 4, equation A34.

f. *Domain 6.* In this region

$$c(t,x) = B/K + c_0 \exp(-K_R(x - x_0)) \quad \text{(A39)}$$

Johnson, Y. W. (1943). Distribution graphs of suspended-matter concentration. *Trans. of ASCE*, 108, 941–964.

Kibler, D. F. and Woolhiser, D. A. (1970). The kinematic cascade as a hydrologic model. *Colorado State Univ. Hydrology Papers, No. 39*, Colorado State Univ., Ft. Collins, Colorado.

Lane, L. J. and Woolhiser, D. A. (1977). Simplifications of watershed geometry affecting simulation of surface runoff. *J. Hydrology*, 35, 173–190.

Lane, L. J. and Shirley, E. D. (1982). Modeling erosion in overland flow, in *Estimating Erosion and Sediment Rangelands*, Proc. Workshop, Tucson, Arizona, March 7–9, 1981, US Dept. Agric., Agric. Res. Serv., Agricultural Reviews and Manuals, ARM–W–26, June, 1982, 120–128.

Laws, J. O. (1940). Recent studies in raindrops and erosion. *Agric. Engr.*, 21, 431–433.

Lighthill, M. J. and Whitham, C. B. (1955). On kinematic waves: flood movement in long rivers. *Proc. Royal Society (London),. Series A*, 229, 281–316.

Meyer, L. D. (1984). Evolution of the Universal Soil Loss Equation. *J. Soil and Water Conserv.*, 39(2), 99–104.

Meyer, L. D. and Wischmeier, W. H. (1969). Mathematical simulation of the process of soil erosion by water. *Trans. American Soc. of Agric. Engr.*, 12(6), 754–758, 762.

Musgrave, G. W. (1947). The quantitative evaluation of factors in water erosion, a first approximation. *J. Soil and Water Conserv.*, 2(3), 133–138.

Negev, M. (1967). A Sediment model on a digital computer, *Tech. Report No. 76*, Dept. of Civil Engr., Stanford Univ., Stanford, California.

Nyhan, J. W. and Lane, L. J. (1986). *Erosion control technology: a user's guide to the use of the Universal Soil Loss Equation at waste burial facilities*, Manual LA–10262–M, UC–708, Los Alamos National Lab., Los Alamos, New Mexico.

Pacific Southwest Inter-Agency Committee (1968). *Factors Affecting Sediment Yield and Measures for the Reduction of Erosion and Sediment Yield*, 13 pp.

Prasad, S. N. and Singh, V. P. (1982). A hydrodynamic model of sediment transport in rill flows, *IAHS Publ. No.*, 137, 293–301.

Rawls, W. J. and Foster, G. R. (1986). USDA Water Erosion Prediction Project (WEPP), EOS. *Trans. American Geophys. Union*, 67(16), 287.

Renard, K. G. and Laursen, E. M. (1975). Dynamic behavior model of ephemeral streams. *J. Hydraul. Div., Proc., ASCE*, 101(HY5), 511–526.

Rendon-Herrero, O. (1974). Estimation of washload produced by certain small watersheds. *J. Hydraul. Div., Proc. ASCE*, 109(HY7), 835–848.

Rendon-Herrero, O. (1978). Unit sediment graph. *Water Resourc. Res.*, 14(5), 889–901.

Renfro, G. W. (1975). Use of erosion equations and sediment delivery ratios for predicting sediment yield. In *Present and Prospective Technology for Predicting Sediment Yields and Sources. Agric. Res. Serv.*, ARS–S–40, 33–45. US Dept. Agric., Washington, D.C.

Rose, C. W., Williams, J. R., Sanders, G. C., and Barry, D. A. (1983a). A mathematical model of soil erosion and deposition processes: I. Theory for a plane land element. *Soil Sci. Soc. of America J.*, 47(5), 991–995.

Rose, C. W., Williams, J. R., Sanders, G. C., and Barry, D. A. (1983b). A mathematical model of soil erosion and depositional processes: II. Application to data from an arid-zone catchment, *Soil Sci. Soc. of America J.*, 47(5), 996–1000.

Sharma, T. C. and Dickinson, W. T. (1979). Discrete dynamic model of watershed sediment yield. *J. of Hydraulic Div., Proc. ASCE*, 105(HY5), 555–571.

Shirley, E. D. and Lane, L. J. (1978). A sediment yield equation from an erosion simulation model, in *Hydrology and Water Resources in Arizona and the Southwest*, 8, 90–96. Univ. of Arizona, Tucson, Arizona.

Singh, V. P. and Prasad, S. N. (1982). Explicit solutions to kinematic equations for erosion on an infiltrating plane, in Singh, V. P. (Ed.), *Modeling Components of Hydrologic Cycle*, Water Resour. Pub., Littleton, Colorado, 515–538.

Singh, V. P., Baniukiwicz, A., and Chen, V. J. (1982). An instantaneous unit sediment graph study for small upland watersheds, in Singh, V. P. (Ed.), *Modelling Components of Hydrologic Cycle*, Water Resour. Publ., Littleton, Colorado, 539–554.