Point Processes of Seasonal Thunderstorm Rainfall
3. Relation of Point Rainfall to Storm Areal Properties

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Following a two-part study of the stochastic nature of thunderstorm occurrence and the probability of daily rainfall depths, the importance of point rainfall information to areal rain distribution is indicated. A probabilistic expression is developed for the relationship between the point depth of rainfall, the local probability distribution of storm cell maximum depth, and the dimensionless expression of storm depth-area pattern. In a sample test the expression is successfully used to reproduce point rainfall depth probability from storm maximum depth distribution and depth area data from Tombstone, Arizona.

In two previous papers [*Smith and Schreiber*, 1973, 1974], statistical properties of daily thunderstorm rainfall in southern Arizona were studied in detail. The statistical processes of daily rainfall occurrence and daily depth distribution were explicitly separated for analysis, and the joint use of these processes allowed simulation of seasonal and other period depth distributions.

The scattered nature of thunderstorm 'cells' in the summer rainfall season implies that point statistics alone are inadequate to describe rainfall input within even a small area. This inadequacy is more apparent when the hydrologic response of an area is the subject of interest. The statistics of point rainfall in a thunderstorm season, however, provide important information in the investigation of areal thunderstorm properties.

Investigation of areal properties requires a dense rain gage network covering sufficient area to enclose the rather isolated storms. Only a few areas in the United States are so instrumented. Two examples familiar to the author have produced probabilistic models based on a 10- to 20-year watershed history. Other examples of thunderstorm research sites include work by the Illinois State Water Survey [*Stout and Huff*, 1962] and the National Severe Storms Laboratory in Oklahoma [*Sanders*, 1965]. A preliminary model for areal storm properties (size and temporal and areal distribution) has been proposed by *Osborn et al.* [1972] from data at the Agricultural Research Service (ARS) Walnut Gulch Experimental Watershed. Spatial properties of rainfall at the University of Arizona Atterbury Watershed were discussed by *Woolhiser and Schwalen* [1960], who presented an expression for point depth probability based on discretized rainfall rates and circular storm isohyetal shapes. Such a discretized relation was also presented by *Fogel and Duckstein* [1969].

The purpose of this paper is to derive the mathematical relation between the probability of storm center depth, a dimensionless description of storm shape, and the probability of point rainfall depth within the storm area. As a sample test, the consistency of published relationships for storms on Walnut Gulch is evaluated with this relationship.

PROBABILISTIC DEVELOPMENT

Define a thunderstorm event as a short continuous period of rainfall over a limited area. The cases familiar to the author involve durations of less than 4 hours and areas generally less than 60 mi². Storms are often described as being composed of cells moving, overlapping, or combining sequentially in time. This concept comes from meteorological analysis. Here the emphasis is on the resultant pattern of rainfall depths measured on the ground from an individual event.

The following assumptions establish the somewhat idealized condition of the derivation to follow:

1. With reference to Figure 1 assume that thunderstorms produce depth isohyets that may be described by a monotonic dimensionless depth-area relation as follows:

$$d_* = d_*(a_*) \tag{1}$$

in which

$d_* = d_p/D_c$;
d_p depth at any point p within the storm;
D_c maximum point depth;
$a_* = a_p/A_t$;
a_p area within which $d > d_p$;
A_t total area of the storm.

2. Assume that all storms are closely similar, as described in (1). Obviously, $d_*(0) = 1$, and $d_*(1) = 0$.

3. Finally, assume that storm cells occur randomly uniformly distributed in space.

On these assumptions we consider the population composed of all storms from which rainfall occurred at a point p. By assumption 3, rainfall at this point may have occurred with equal likelihood from any point within a storm. Rewriting (1) to describe the area within the locus of points having a given depth less than or equal to storm maximum depth, we have

$$a_* = a_*[(d_p)/D_c] = a_*(d_*) \tag{2}$$

Let the random variable x_p ($x_p > 0$) represent rainfall depth at point p within the storm and the random variable Z ($Z > 0$) represent the random value of center or maximum depth D_c. For any x, Z is always greater than or equal to x, and the event $\{x_p \leq d\}$ includes all storms for which $Z \geq d$ but $x < d$. For p within a storm, this event may be expressed as

$$\{x_p \leq d\} = \{Z < d\} \cup [\{Z \geq d\} \cap \{d_p \leq d\}] \tag{3}$$

The event $\{d_p \leq d\}$ represents the event that p falls in a region where d may from (1) be expressed as

$$\{d_p \leq d\} = \{a_*(d_{*p}) \geq a_*(d_*)\} \tag{4}$$

by virtue of assumption 1. In probability terms we replace (4) by an expression of the proportion of area in which $x \leq d$:

$$P[d_p \leq d] = 1 - a_*(d_*) \tag{5}$$

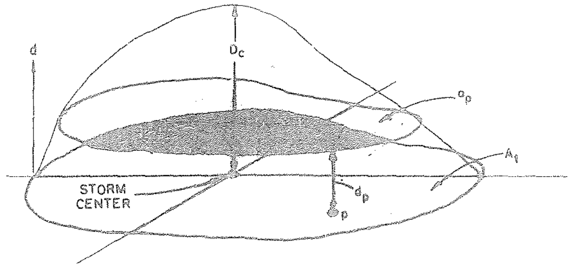

Fig. 1. Definition sketch of storm depth-area pattern.

By combining (3) and (5) we may deduce the distribution function for point depth distribution as

$$P[x_p \leq d] = P[Z \leq d] + \int_{y=d}^{\infty} \left[1 - a_*\left(\frac{d}{y}\right)\right] dP[Z \leq y] \qquad (6)$$

Letting the $p_z(d)$ represent the probability density function corresponding to the distribution function $P_z(d) = P[Z \leq d]$, we may simplify and write (6) as

$$P[x_p \leq d] = 1 - \int_{y=d}^{\infty} a_*\left(\frac{d}{y}\right) p_z(y)\, dy \qquad (7)$$

This equation establishes a general relation between normalized storm isohyetal pattern, center depth probability, and point rainfall probability.

Independence of $a_*(d/y)$ and y is not necessary; that is, the normalized isohyetal shape may change with cell center depth, and the variation can be treated explicitly in (7).

APPLICATION

Equation 7 provides a means to test the consistency of published depth-area relationships and storm cell center depth frequency. Figure 2 shows the dimensionless reductions of several depth-area relations, including those of *Woolhiser and Schwalen* [1960], *Fogel and Duckstein* [1969], and *Osborn and Lane* [1972]. The relation of Fogel and Duckstein provides no explicit limit on area, and any reasonable value assumed for A_t allows a significant discontinuity at $a_* = 1$. Their general empirical formula relates storm area to center depth, but all relations are reducible to the same normalized exponential shape.

Somewhat less information is published for storm center depth distribution. Both *Woolhiser and Schwalen* [1960] and *Fogel and Duckstein* [1969] used log normal distributions for storm center depth, but in each case a complex truncation of data was involved. This precluded their direct use in (7).

The model of *Osborn et al.* [1972] includes a probabilistic model for storm center depth as well as a depth-area relation (derived from studies of larger storms) and was therefore chosen for the sample application of relationship 7 derived above to investigate the consistency of their proposed model. The resulting simulated point depth distribution from the Osborn model is shown in Figure 3 along with measured point depth distribution at Tombstone. These results suggested that the large storms on which Osborn and Lane based their depth-area relation were not representative of the total spectrum of storms.

In order to better represent mean storm shape the depth-area relation was modified to

$$d_* = 1 - a_* \qquad (8)$$

As Figure 3 indicates, the simulation with this assumption was remarkably good. The transient nature of the curve near $d = 0$ was well reproduced. The fit may in part be fortuitous in that possible natural violations of the assumptions behind (7) and relations for $P_z(d)$ and $d_*(a_*)$ can both have counteracting biases.

It is also reasonable to assume that $d_*(a_*)$ is not independent of Z. A simple test was performed to determine if agreement with measured data comparable to the average condition represented by (8) could be obtained under such dependence. The relation of *Woolhiser and Schwalen* [1960] was modified so that

$$d_* = 1 - a_*^{\,b(Z)} \qquad (9)$$

In this function, $b(Z)$ acts as a shape factor having $b_{\min} < b < b_{\max}$; as $P_z(d) \to 1$ (largest storms), $b \to b_{\min}$.

By inspection of an array of depth-area relations published by *Osborn and Lane* [1972] a simple empirical function $b(Z)$ was estimated:

$$b(Z) = C_1 \exp(-C_2 Z) \qquad (10)$$

in which C_1 and C_2 are parameters. The solution of (7), incorporating (9) and (10), was performed numerically by using a few trial parameter values in (10). The result of one such simulation is shown in Figure 3, indicating at least that the accurate simulation incorporating (8) should not imply that a dependence of storm maximum depth and storm isohyetal shape does not exist. Evaluation of the existence and nature of such dependence depends on the continuing collection of data.

SUMMARY AND CONCLUSIONS

By development of a general relationship between storm depth-area relations and distributions for point and storm center depths (equation 7) it has been shown how the point rainfall probability distribution may act as an important constraint in the determination of consistent relations for storm depth-area and storm center-depth distribution for air mass thunderstorms. Use of (7) has demonstrated, for example, that the mean depth-area relation for the southeastern Arizona area is apparently a simple linear relation and is quite different from published depth-area relations based on large storms alone. The relation of *Woolhiser and Schwalen* [1960] (equation 9) was treated as a general dimensionless depth-area relation for summer thunderstorms, an exponent parameter being used as a cell center depth dependent shape

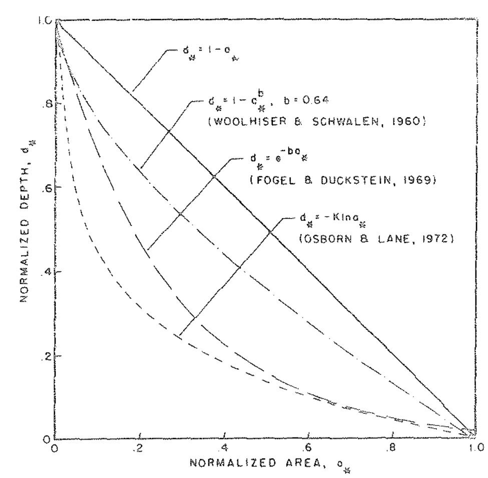

Fig. 2. Normalized depth-area relations for air mass thunderstorms proposed by three investigators.

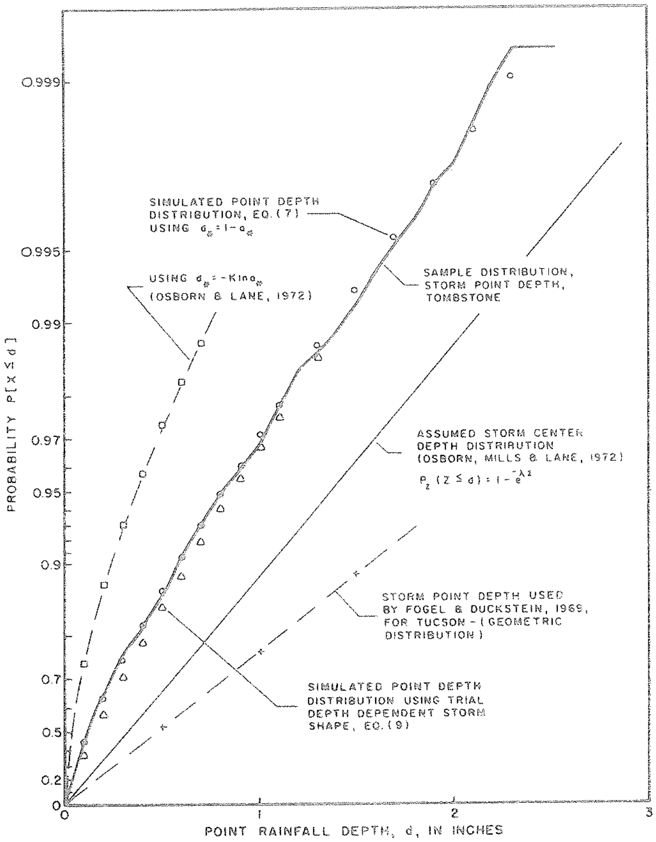

Fig. 3. Storm center-depth distribution and several point depth distributions, including measured data for Tombstone, Arizona, and simulated distributions from equations 8 and 9.

factor. This relation was used to show that such dependency between shape and storm cell depth may be treated in the general relation of (7) and satisfactory point depth distributions obtained. More extensive data on dimensionless storm depth-area relations and center-depth distributions are needed before the importance of incorporating such dependencies into (7) can be evaluated.

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