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COMPUTING MULTIATTRIBUTE VALUE MEASUREMENT RANGES UNDER A HIERARCHY OF THE CRITERIA

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Abstract: A method to quickly compute the range of values from the most optimistic and the most pessimistic viewpoint (best to worst) for a hierarchically arranged multiattribute problem under the assumption of an additive value function is presented. The method is an analysis tool to be applied after commensurate attribute values have been determined for each alternative but without the need to specify or determine weights on the attributes explicitly. The decision maker can change the priority order of the criteria or attributes at any tier in the hierarchy and quickly recompute the range from best to worst of the overall value measurement. A simple algorithmic method is presented that requires no linear programming solver. This solution method makes it easy to determine the result of modifying priorities in portions of the hierarchical architecture without recalculating the contributions of unaffected branches. The simple calculations are illustrated and generalization to any hierarchical structure is readily apparent.

Introduction

This work is prompted by the need for tools that can be quickly understood and applied as encouraged in Dyer et al. (1992) to multi-attribute decision making situations. In particular we consider a problem that has been formulated as a hierarchical multi-attribute decision problem. The development here is an extension of the work in Salo and Hämäläinen (1992) and Yakowitz, Lane and Szidarovszky (1993) that considers the impact on an additive value function caused by allowing the weights to vary subject to a priority ordering of the criteria. These works fall under the category of partial information in multi-attribute utility theory (Fishburn, 1965; Kirkwood and Sarin, 1985; or see Hazen, 1986 for a discussion and numerous references). Most techniques proposed in the literature for assessing weights, either solicit weights directly, or seek to discern them indirectly, from the decision maker (DM) (Goicoechea, Hansen and Duckstein, 1982; Keeney and Raiffa, 1976; Saaty, 1980 are examples). The resultant ranking of the alternatives can be very sensitive to the importance order of the attributes, and therefore the weights. The calculations for examining this sensitivity are very straightforward in the case of a simple ordinal priority ranking of the attributes (Salo and Hämäläinen, 1992; Yakowitz, Lane and Szidarovszky , 1993). However, the complexity imposed by a hierarchical architecture on examining the range of values of an additive value function, at first appears to be difficult to overcome.

Under a hierarchy, the structure of the decision priorities does not necessarily imply an ordinal ranking of each individual attribute. Therefore, examining the solutions of the linear programs in Salo and Hämäläinen (1992) and Yakowitz, Lane and Szidarovszky (1993), given an ordinal ranking of the attributes, without considering the added freedom (or relaxation of the weights) possible due to the hierarchical structure, does not provide a complete picture of weight vector sensitivity or the range of the additive value function. A method to quickly compute the range of values from best to worst for a hierarchically arranged multiattribute problem under the assumption of an additive value function is presented here. The method allows the decision maker to quickly assess the most optimistic and most pessimistic decision maker viewpoint given alternative preference orders of the attributes at any tier in the hierarchy. It is assumed that attribute values for each alternative have been determined by some method such as the Analytic Hierarchy Process (Saaty, 1980).

Computing the range of value of an additive value function makes it possible for various stakeholders to determine the sensitivity of the ranking of alternatives to the hierarchical order. Often this order is a compromise between various objectives. Quickly examining the range of the value function under other hierarchical scenarios may reveal an alternative choice that is favored by each and eliminates the need to discern weights explicitly.

We begin by defining the hierarchy followed by a brief descriptions of the algorithm for computing the range of the value function. Algorithmic details are then developed followed by detailed description of the calculations needed for a specific hierarchical structure. The generalization is readily apparent.

Defining the Hierarchy

Figure 1 illustrates a generic hierarchical architecture for a multi-attribute decision problem. Note that branches or subgraphs of the hierarchy may terminate at different tiers or levels. Dummy elements may of course be added at intermediate levels if equal depth branches are desired. At the highest level (Tier 1) is the *Major Goal*. This could be *Sustainable Agricultural System* in the case of a problem to determine the best farm management system from a finite number of alternatives for a given farm or region, or *Traffic Plan* for a problem to define a traffic policy from among several alternative plans. The subsequent levels (Tier 2 through N) contain sub-elements of the parent or previous levels. Thus, for example in the sustainable agriculture problem, Tier 2 could include environmental, economic, and social sub-goals. Subsequent levels of the environmental branch could then include surface water, sub-surface water, and soil, followed by criteria including fertilizer and pesticide impacts, and erosion under the proper parent category. Of interest in this work is the effect of changing the priority (importance or preference) order of the elements of the hierarchy. In all figures, it is assumed that the priority order is from left to right. That is, for elements emanating from a common branch, an element to the left of another element in the same tier has a higher priority and therefore more "weight" in the decision making process at that level. No assumption is made regarding the priority relationship between elements on different branches.

As is the case in most multi-attribute solution methods, the goal of the methodology is to determine the value of an additive value function that can be used to rank the set of alternatives. An additive value function in the following form is assumed:

$$V(w,v) = \sum_i w_i v_i$$

Where i ranges over the terminal elements of each branch and the weights, w, are consistent with the hierarchy and normalized so that they sum to 1. We emphasize that V is a function of both the individual criterion values determined for each alternative and the weights determined for each attribute or criteria. Since we are primarily concerned with the effects on the above function caused by changing hierarchical element priorities, we will assume that for each alternative, v_i is fixed for all i. We refer to the above as V, or subscripted, V_j, when wishing to distinguish between alternatives.

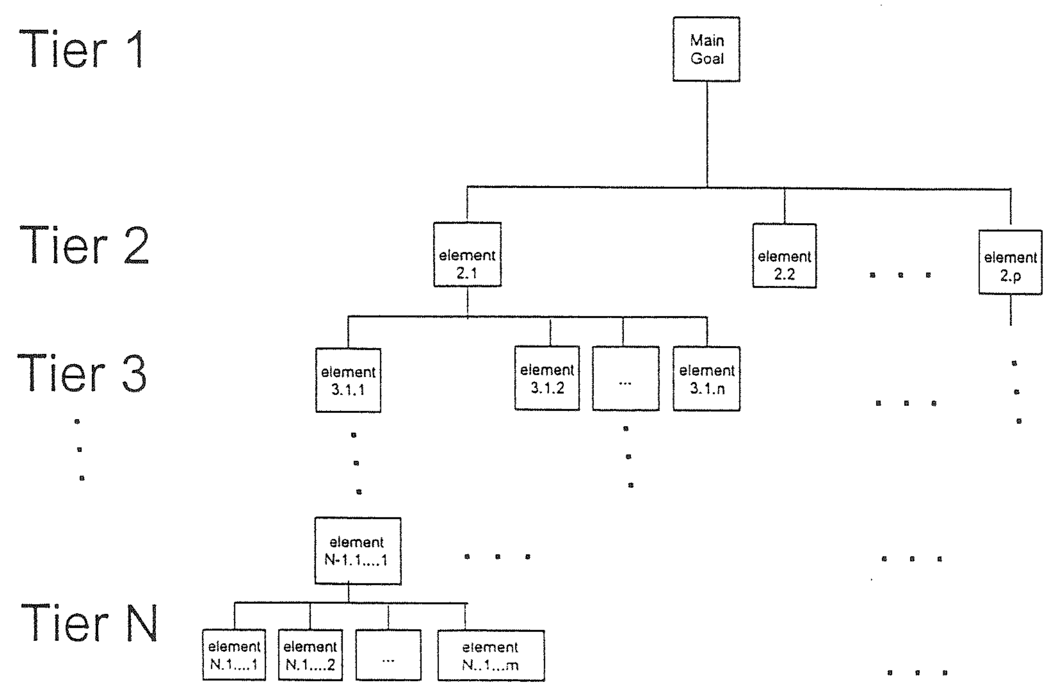

Figure 1. Generic Decision Hierarchy

Algorithm Outline for Computing the Range of Values Under a Hierarchy

The algorithm for assessing the full range from best to worst under our assumptions begins at the lowest tier of each branch of the hierarchy. Best and worst additive values are computed for each element using closed form solutions to two simple linear programs that maximize and minimize V at the parent element over all weights consistent with the priority order of the decision elements. These same programs are used at intermediate elements, substituting the maximum (or minimum) values previously computed as the values for those elements that have descendent elements, until the main or first tier is reached. Altering the priority at any level requires redoing only those calculations that occur after that point to the main or first tier. This fact makes it easy to examine the effects of changing priorities or decision maker preferences, which may be especially useful if there are more than one decision maker or affected parties involved.

Algorithmic Details:

Details of the algorithm will now be described. Please refer to the annotated portion of Figure 1.

Computing Best and Worst Subvalues for the Lowest Tier of each Subgraph

Given the importance order of the criteria at the lowest tier of each subgraph, best and worst additive values can be found without requiring the decision maker to set specific weights for each of the criteria (Salo and Hämäläinen,1992; Yakowitz, Lane and Szidarovszky,1993).

Referring to the notated branch or subgraph at Tier N of Figure 1, it is assumed that if $i < j$ then criterion i has a higher priority than criterion j (i.e. criterion 1 has higher priority than criterion 2 and so forth). Since there are m criteria, the priority order suggests that we should require that the weights, w_i, $i=1,m$, have the following relation:

$$w_1 \geq w_2 \geq \ldots \geq w_m.$$

Therefore, given the priority order and the criteria values for alternative j, the best (worst) composite score that alternative j can achieve is determined by solving the following linear programs (LPs) (Yakowitz, Lane and Szidarovszky, 1993):

Best (Worst) Additive Value:

$$\max(\min)_w \quad V_j = \sum_{i=1}^{m} w_i v_{ij}$$
$$s.t. \quad \sum_{i=1}^{m} w_i = 1$$
$$w_1 \geq w_2 \geq \ldots \geq w_m \geq 0.$$

The best additive value is found by maximizing the objective function while the worst additive value is found by minimizing the objective function. The first constraint is a normalizing constraint. The second, fixes the importance order and restricts the weights to be positive. The above linear programs are solvable in closed form according to Yakowitz, Lane and Szidarovszky (1993): For $k=1,\ldots,m$, let

$$S_{kj} = 1/k \sum_{i=1,\ldots,k} v_{ij}.$$

Then, the best or maximum additive value ($Max\ V$) for alternative j is given by :

$$(Max\ V)_j = max_k \{S_{kj}\}. \qquad (1)$$

The worst or minimum additive value ($Min\ V$) for alternative j is given by :

$$(Min\ V)_j = min_k \{S_{kj}\}. \qquad (2)$$

In the case of equal importance of some criteria, there are strict equalities in the importance order constraint set, i.e. $w_j = w_{j+1}$ for j in a subset, J, of the integers 1 through m. If we define $K = \{\{1, 2, \ldots, m\} \setminus J\}$, then the above formulas for *Max* and *Min V* apply if k is restricted to the set K (ie. $k \in K$). Calculations for other variations, including cases in which it is desired to specify the level of preference between criteria, criterion 1 is to have a weight at least twice that of criterion 2, for example, are given in Yakowitz, Lane and Szidarovszky (1993)*.

For each alternative, the above solutions determine the maximum and minimum additive value possible for any combination of weights that are consistent with the hierarchial order of the criteria/attributes. Having these two objective values available immediately alerts the DM to the sensitivity of each alternative to the weights possible with the current priority order of the criteria. These values can be displayed graphically (illustrated later) in the form of side by side bar graphs with the best value for each alternative at the top of each bar and the worst value at the bottom.

An alternative that exhibits little difference between the best and worst values indicates that this alternative is relatively insensitive to any vector of weights consistent with the importance order. Additionally, if the worst value of one alternative is greater than the best value of another alternative, then that alternative strongly dominates (Yakowitz, Lane and Szidarovszky, 1993) or absolutely dominates (Salo and Hämäläinen, 1992) the other alternative.

Computing Best and Worst Values for a Multi-level Hierarchy

Additional constraints can easily be added to the LPs to account for a hierarchy of the criteria and still provide the range from best to worst composite scores. For example, suppose we have a three tier hierarchy, and each element i in Tier 2 is composed of t_i sub-criteria in Tier 3, the terminating level. Let $v_{i,k,j}$ and $w_{i,k}$, $k=1,\ldots,t_i$ indicate the values (scores) for alternative j, and sub-weights (unspecified), respectively, associated with sub-criteria k of criteria i. Then, the following two constraints for each i are added to best/worst LPs to account for this hierarchy:

$$w_i = w_{i,1} + w_{i,2} + \ldots + w_{i,t_i}$$

$$w_{i,1} \geq w_{i,2} \geq \ldots \geq w_{i,t_i} \geq 0$$

The objective functions of the best/worst LPs for alternative j are then replaced by:

$$\max(\min) \sum_{i,k} w_{i,k} v_{i,k,j} \ .$$

Again, there is no need to specify weights or sub-weights to obtain the range from maximum to minimum. Linear modifications due to more general hierarchical considerations are easily made in this manner. Therefore, an explicit linear program for computing the maximum and minimum V for any hierarchy can be formulated and solved explicitly. The notation needed to indicate each level of the hierarchy, however, becomes very cumbersome. Solving these programs explicitly is not necessary since an algorithm that considers each portion of the hierarchy in an optimal manner is much more amenable to examining the effects of changing priorities. Calculating min and max V is an intuitively simple procedure when performed from the lowest tier up. To illustrate this fact, the solution procedure will be described for the four tier decision hierarchy of Figure 2.

Algorithm for Computing the Range of Values Under a Given Hierarchy

The following procedure is described for solving for the range from best to worst of additive values under the hierarchy illustrated in Figure 2. The procedure for other hierarchial variations is handled in a similar manner and will become transparent.

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*The formula for S_{kj} on page 175 of Yakowitz, Lane and Szidarovszky (1993) for this case is in error, please contact the lead author for correction.

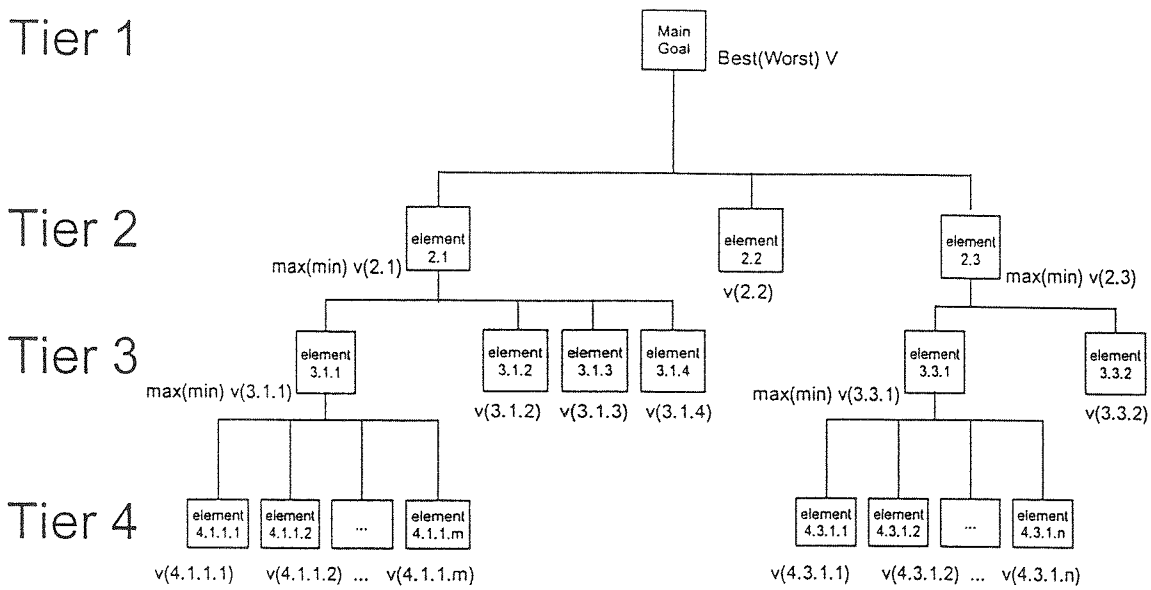


Figure 2. Decision Hierarchy for Algorithmic Explanation

For each alternative under consideration, we assume that the associated value for the terminal elements has been determined by some means. Thus, for Figure 2, the values indicated by $v(2.2)$, $v(3.1.2)$, $v(3.1.3)$, $v(3.1.4)$, $v(3.3.2)$, $v(4.1.1.1)$,..., $v(4.1.1.m)$, $v(4.3.1.1)$,..., $v(4.3.1.n)$, are known for each alternative. All indices are with respect to the hierarchy of Figure 2, which indicates the inputs and calculations required.

Calculations start at the lowest Tier in the hierarchy. In this case, Tier 4.

- Tier 4.
  - Compute for each alternative j,
    $S(4.1.1)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(4.1.1.i),\ k=1,\dots,m,$
    and $S(4.3.1)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(4.3.1.i),\ k=1,\dots,n.$
  - Then according to (1) and (2),
    $max\ (min)\ \ v_j(3.1.1) = max\ (min)_k\ \{S(4.1.1)_{kj}\}$, and
    $max\ (min)\ \ v_j(3.3.1) = max\ (min)_k\ \{S(4.3.1)_{kj}\}.$
- Tier 3.
  - Compute the following for each alternative j:
    $S_{max}(3.1)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(3.1.i),\ for\ k=1,\dots,4,$
    with $v_j(3.1.1) = max\ (v_j(3.1.1))$, and
    $S_{min}(3.1)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(3.1.i),\ for\ k=1,\dots,4,$
    with $v_j(3.1.1) = min\ (v_j(3.1.1)).$
    $S_{max}(3.3)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(3.3.i),\ k=1,2,$
    with $v_j(3.3.1) = max\ (v_j(3.3.1))$, and
    $S_{min}(3.3)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(3.3.i),\ k=1,2,$
    with $v_j(3.3.1) = min\ (v_j(3.3.1)).$

- Then:

$$max\ (min)\ v_j(2.1) = max\ (min)_k\ \{S_{max(min)}(3.1)_{kj}\},$$ and
$$max\ (min)\ v_j(2.3) = max\ (min)_k\ \{S(3.3)_{kj}\}.$$

- Tier 2.
  - Compute the following for each alternative j:

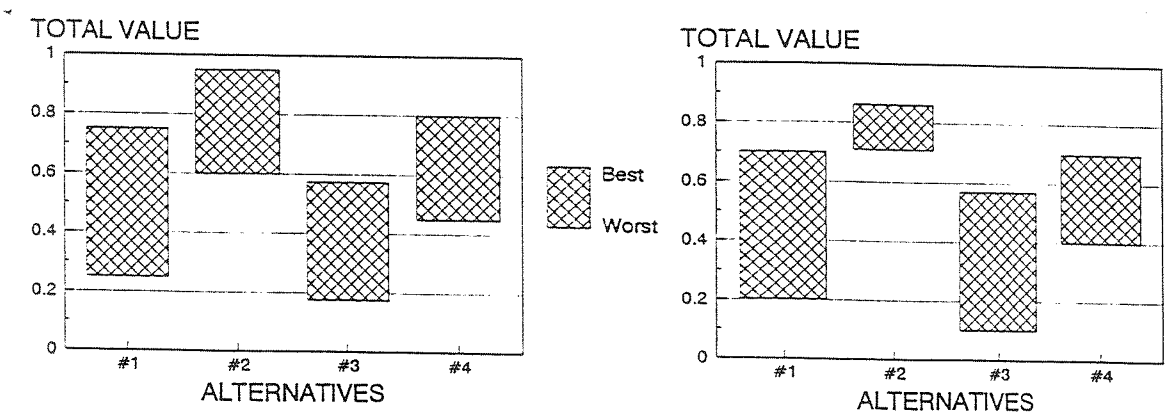
$$S_{max}(2)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(2.i),\ for\ k=1,\dots,3,$$
with $v_j(2.1) = max\ (v_j(2.1))$, and
$v_j(2.3) = max\ (v_j(2.3))$, and
$$S_{min}(2)_{kj} = 1/k \sum_{i=1,\dots,k} v_j(2.i),\ for\ k=1,\dots,3,$$
$v_j(2.1) = min\ (v_j(2.1))$, and
$v_j(2.3) = min\ (v_j(2.3))$.

  - Then:

$$Best\ (Worst)\ \ V_j = max\ (min)_k\ \{S_{max(min)}(2)_{kj}\}.$$

A bar graph with a bar for each alternative that ranges from the *Best* V_j at the top of each bar to the *Worst* V_j at the bottom of each bar would aid the decision maker by indicating domination and the sensitivity of each alternative to the priorities in the hierarchy.

Changing a priority ordering in any tier in the hierarchy, requires only recalculating appropriate max and min values in the tiers above. For example, assume Figure 3a is the bar graph obtained under the present priority order of Figure 2 for four alternatives. Clearly, Alternative 2 dominates Alternative 3 and is preferred to Alternative 1 which is very sensitive to the weights given the existing priority orders. Now, suppose one wishes to consider the scenario in which the elements previously ordered in Tier 2 are reversed. Then, only those calculations indicated under Tier 2 given above, need to be computed again. If Figure 3b is the result of the this new evaluation, then it can be argued that Alternative 2 does well with respect to both of the priority orderings and is preferred over all other Alternatives for the latter ordering. As described in Yakowitz, Lane and Szidarovszky (1993), the alternatives can be ranked based on the average of the *Best* and *Worst* V_j. In both of the example cases Alternative 1 would be ranked first. Other scenarios that reflect the differing priorities of interested parties or multiple decision makers could be quickly examined. If, as in this example, one or two Alternatives stand out as doing well under multiple decision scenarios, one would have a strong basis for supporting these alternatives and avoid unnecessary argument.



a. Priority order of Figure 2

b. Reverse priority order of Tier 2, Figure 2

Figure 3. Bar Graphs of Range of Values Under Two Priority Orderings of Tier 2 in Figure 2

Conclusions

A method to explicitly calculate the full range values possible for an additive value function subject to the priorities of a hierarchical decision structure was developed. The method involves the solutions to simple linear programs. A solution method that does not require that LPs be solved explicitly was presented. This procedure also minimizes the number of calculations needed to examine the effects of changes to the hierarchical structure. As illustrated in the example above, the method could be a valuable aid to decision makers especially in the case of multiple decision makers or stakeholders. In this case, the ability to take into account other viewpoints and examine the impact on the ranking of alternatives by the method described above could be a strong negotiation tool.

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