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NASA Technical Memorandum 107703

# ICASE INTERIM REPORT 24

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(NASA-TM-107703) ELECTROMAGNETIC POTENTIAL VECTORS AND THE LAGRANGIAN OF A CHARGED PARTICLE (ICASE) 17 p

N93-16555

Unclas

G3/70 0137576

485127

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Operated by the Universities Space Research Association

NASA
National Aeronautics and Space Administration

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Hampton, Virginia 23665-5225

NOV 1992

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# Electromagnetic Potential Vectors and the Lagrangian of a Charged Particle

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**Abstract**

Maxwell's equations can be shown to imply the existence of two independent three-dimensional potential vectors. A comparison between the potential vectors and the electric and magnetic field vectors, using a spatial Fourier transformation, reveals six independent potential components but only four independent electromagnetic field components for each mode. Although the electromagnetic fields determined by Maxwell's equations give a complete description of all possible classical electromagnetic phenomena, potential vectors contain more information and allow for a description of such quantum mechanical phenomena as the Aharonov-Bohm effect. A new result is that a charged particle Lagrangian written in terms of potential vectors automatically contains a 'spontaneous symmetry breaking' potential.

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## Introduction

It is well known that the electric field **E** and the magnetic field **H** can be determined through the four-potential ($\varphi$,**A**). However, the scalar potential $\varphi$ is merely an 'auxiliary' variable since, as will be shown here, it is only the three-vector **A** which is dynamically important. In transforming to a potential formulation, the first-order Maxwell's equations for **E** and **H** are replaced by second-order equations for the vector potential **A**. These second-order equations for **A** can be expressed as first-order equations for **A** and **C**, where $\mathbf{C} \equiv \partial_t \mathbf{A}$ (**C** and **A** must be treated as independent vectors, unless **A** is known for all time *a priori*). Thus, a formulation of classical electrodynamics in terms **E** and **H** (the electromagnetic fields) is embedded in a formulation in terms of **A** and **C** (the 'potential vectors'), which serves to describe electrodynamics in both classical and quantum mechanics.

Maxwell's equations contain dynamic (*i.e.*, time-evolution) equations only for the curls of **E** and **H**, while their divergences are prescribed. The equations for the potential vectors **A** and **C**, on the other hand, dynamically determine both their curls and divergences. If these systems of vector fields and partial differential equations are spatially Fourier transformed, then these preceding statements can be given in terms of the Fourier coefficients associated with the various spatial modes (which are identified by their wave vector **k**). Maxwell's equations dynamically determine only the transverse parts of the electromagnetic fields (*i.e.*, $\mathbf{k}\times\mathbf{E}(\mathbf{k},t)$ and $\mathbf{k}\times\mathbf{H}(\mathbf{k},t)$), while the longitudinal parts (*i.e.*, $\mathbf{k}\cdot\mathbf{E}(\mathbf{k},t)$ and $\mathbf{k}\cdot\mathbf{H}(\mathbf{k},t)$) are prescribed. The potential vector equations, on the other hand, dynamically determine all the components associated with a given mode: $\mathbf{k}\times\mathbf{A}(\mathbf{k},t)$, $\mathbf{k}\times\mathbf{C}(\mathbf{k},t)$, $\mathbf{k}\cdot\mathbf{A}(\mathbf{k},t)$, and $\mathbf{k}\cdot\mathbf{C}(\mathbf{k},t)$. Maxwell's equations thus dynamically determine only four components for each mode (the transverse ones) which is sufficient classically, while the potential vector equations determine six components (both longitudinal and transverse) which is needed quantum mechanically.

A direct, algebraic transformation from the set of modal vectors $\mathbf{A}(\mathbf{k},t)$ and $\mathbf{C}(\mathbf{k},t)$ to the set $\mathbf{E}(\mathbf{k},t)$ and $\mathbf{H}(\mathbf{k},t)$ will be given, and will show that the transformation is, in fact, a projection from (for each mode) a six dimensional space onto a four dimensional space. Again, a knowledge of

$\mathbf{A}(\mathbf{k},t)$ and $\mathbf{C}(\mathbf{k},t)$ for all modes gives a description of electromagnetic phenomena which encompasses both classical and quantum mechanics, while a knowledge of $\mathbf{E}(\mathbf{k},t)$ and $\mathbf{H}(\mathbf{k},t)$ for all modes gives a description which is restricted to classical mechanics alone. For example, an understanding of the Aharonov-Bohm effect requires that we recognize that $\mathbf{A}$ has a direct influence on physical processes; in classical electrodynamics, $\mathbf{A}$ is a convenient mathematical artifice, a potential which enters into physics only through its derivatives.

In this paper, the preceding concepts will be presented in detail, utilizing spatial Fourier expansions. (Temporal Fourier expansions are not utilized because the coupling between the electromagnetic field and matter is, in general, non-linear, which precludes assigning a specific frequency to each spatial mode.) In the end, one important result will become clear: the Lagrangian of any system containing electromagnetically interacting matter must contain a term linera and a term quadratic in the particle. Since these charge density coefficients are themselves quadratic functions of quantum mechanical wave functions, *the Lagrangian of any electrically charged particle must contain a term which is quadratic and a term which is quartic in the wave function associated with that particle.* These naturally occurring terms are similar to the *ad hoc* 'spontaneous symmetry breaking' potentials which have been introduced into current theories of elementary particles.

## Potential Vectors

To begin the discussion here, consider Maxwell's equations:

$$\begin{array}{ll} \text{a)}\ \nabla\cdot\mathbf{D} = \rho & \text{b)}\ \nabla\times\mathbf{E} = -\partial_t\mathbf{B} \\ \text{c)}\ \nabla\cdot\mathbf{B} = 0 & \text{d)}\ \nabla\times\mathbf{H} = \partial_t\mathbf{D} + \mathbf{j} \\ \multicolumn{2}{c}{\mathbf{D} = \varepsilon_o\mathbf{E}, \quad \mathbf{B} = \mu_o\mathbf{H}} \end{array} \tag{1}$$

Here, $\rho$ is the total charge density and $\mathbf{j}$ is the total electric current density. Although (1) is written in SI units, it will be more expedient to shift to 'natural' units where the electric permittivity is unity: $\varepsilon_o=1$, as is the magnetic permeability: $\mu_o=1$. Maxwell's equations then become:

$$\text{a) } \nabla\cdot\mathbf{E} = \rho \qquad \text{b) } \nabla\times\mathbf{E} = -\partial_t\mathbf{H}$$
$$\text{c) } \nabla\cdot\mathbf{H} = 0 \qquad \text{d) } \nabla\times\mathbf{H} = \partial_t\mathbf{E} + \mathbf{j} \qquad (2)$$

Equation (2b) states that $\nabla\cdot\mathbf{H}$= constant and (2c) states that this constant is zero. Equation (2a) can be viewed as equating charge density $\rho$ to $\nabla\cdot\mathbf{E}$. The rest of $\mathbf{E}$ is dynamically determined by (2d); taking the divergence of (2d) and using (2a) we obtain the equation of continuity (or time-evolution equation for $\nabla\cdot\mathbf{E}$):

$$\partial_t\rho + \nabla\cdot\mathbf{j} = 0 \qquad (3)$$

The equations (2a) and (2d) ensure the existence of the equation of continuity, which has an obvious physical meaning; alternatively, (3) can be viewed as merely mathematically defining future values of $\nabla\cdot\mathbf{E}$.

In a similar manner, taking the curls of (2b) and (2d) can produce time-evolution equations for $\nabla\times\mathbf{H}$ and $\nabla\times\mathbf{E}$, respectively. Presuming the values of $\mathbf{E}$ and $\mathbf{H}$ at infinity are known, then the Helmholz theorem [1] states that the six component equations found in (2b) and (2d) serve as independent, first-order, time-evolution equations which exactly determine the six components of $\mathbf{E}$ and $\mathbf{H}$, as is mathematically required [2].

The fields $\mathbf{E}$ and $\mathbf{H}$ can be represented in terms of a four-potential ($\varphi$, $\mathbf{A}$):

$$\text{a) } \mathbf{E} = -\nabla\varphi - \partial_t\mathbf{A} \qquad \text{b) } \mathbf{H} = \nabla\times\mathbf{A} \qquad \text{c) } \partial_t\varphi + \nabla\cdot\mathbf{A} = 0 \qquad (4)$$

These equations are well known: (4a) and (4b) satisfy (2b) and (2c) identically, while (4c) is the Lorentz condition, which arises because the divergence of $\mathbf{A}$ needs to be defined (the curl of $\mathbf{A}$ is defined by (4b)). However, the need for relativistic invariance requires the left-hand-side of (4c) be defined, rather than just $\nabla\cdot\mathbf{A}$. The right-hand-side of (4c) can initially be an arbitrary (well-behaved) scalar function $\sigma$, rather than zero; however, a gauge transformation of ($\varphi$, $\mathbf{A}$)$\rightarrow$($\varphi$+

$\partial_t\psi$, $\mathbf{A}$-$\nabla\psi$) can return the divergence condition to the form it has in (4c), as long as $\psi$ satisfies the *inhomogeneous* wave equation $\partial_t^2\psi - \nabla^2\psi=\sigma$. Thus, the Lorentz condition can always be regained from an arbitrary choice of gauge.

Using (3), and (4), Maxwell's equations (2a) and (2d) become:

$$\text{a)}\ \nabla^2\varphi - \partial_t^2\varphi = -\rho \qquad \text{b)}\ \nabla^2\mathbf{A} - \partial_t^2\mathbf{A} = -\mathbf{j} \tag{5}$$

Here it would appear that the six (five non-trivial) independent evolution equations of (2) have been replaced by at most four equations; at least this is what is implied by stating that **E** and **H** are determined by $\varphi$ and **A**.

The source of the apparent over-determination is, of course, that Maxwell's equations (2) are first-order equations, while the equations (5) are second-order. Before changing (5) to a set of first-order equations, let us use (4c) and (5a) to obtain the following:

$$\nabla^2\varphi = -\rho - \nabla\cdot\partial_t\mathbf{A} \tag{6}$$

Now, defining $\mathbf{C} \equiv \partial_t\mathbf{A}$, equations (5b) and (6) can be written as:

$$\text{a)}\ \nabla^2\varphi = -\rho - \nabla\cdot\mathbf{C} \qquad \text{b)}\ \partial_t\mathbf{C} = \nabla^2\mathbf{A} + \mathbf{j} \qquad \text{c)}\ \partial_t\mathbf{A} = \mathbf{C} \tag{7}$$

Thus we arrive at the realization that **E** and **H** can be found by determining **A** and **C**, and not just **A** (and $\varphi$) alone. The six independent evolution equations in (7b) and (7c) contain (2b) and (2d), and (7a) simply determines $\varphi$ at each instant that $\rho$ and **C** are known ($\rho$ is presumably determined by using the equations of motion of whatever charged matter is present). If **A** is a *known* vector function, then **C** is determined through (7c) and only in that special case can **E** and **H** be found using **A** alone; otherwise, both **A** and **C** are needed for completeness. To continue this discussion, let us bring in Fourier representations at this point.

## Fourier Representation

The electromagnetic field and vector potential formulations of electromagnetism can easily be compared in terms of spatial Fourier expansions (here, the physical domain is assumed to be a finite-sized box):

$$\mathbf{E}(\mathbf{x}) = (2\pi)^{-3/2} \sum_{|\mathbf{k}| \le k_{max}} \mathbf{E}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \text{etc.} \tag{8}$$

In (8), the number of modes for $|\mathbf{k}| \le k_{max}$ is N, and the argument of **E** denotes whether we are in physical space or in Fourier space; in either case, time is omitted from the argument for brevity. It is presumed that the limit $k_{max} \to \infty$ may be taken and also that summation over discrete **k** can be transformed into a continuous integral over d**k** by expanding the size of the physical domain out to infinity. (Fourier expansions in time are not utilized since the system in which the electromagnetic fields arise must, in general be assumed to be non-linear, which does not allow for simple dispersion relations, *i.e.*, a single frequency to be assigned to each mode **k**.)

The various differential **x**-space relations can easily be transformed to the Fourier domain by the substitution $\nabla \to i\mathbf{k}$. Thus, the fields **E** and **H**, as given in (4), and **A**, **C**, and $\varphi$, as given in (7), can be algebraically related through their Fourier coefficients:

$$\text{a) } \mathbf{E}(\mathbf{k}) = -i\mathbf{k}\,\varphi(\mathbf{k}) - \mathbf{C}(\mathbf{k}) \qquad \text{b) } \mathbf{H}(\mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\mathbf{k})$$
$$\text{c) } \varphi(\mathbf{k}) = k^{-2}\left[\rho(\mathbf{k}) + i\mathbf{k}\cdot\mathbf{C}(\mathbf{k})\right] \tag{9}$$

In particular,each Fourier coefficient of the magnetic field **H**(**k**) is clearly due only to the transverse part of **A**(**k**); also, it is clear that $\mathbf{k}\cdot\mathbf{A}(\mathbf{k})$ does not play any role in determining the electromagnetic field components **E**(**k**) and **H**(**k**).

The scalar function $\mathbf{k}\cdot\mathbf{C}(\mathbf{k})$, on the other hand, helps determine $\varphi(\mathbf{k})$, as (9c) shows. Placing (9c) into (9a) yields:

$$\text{a)}\ \mathbf{E}(\mathbf{k}) = -i\mathbf{k}k^{-2}\rho(\mathbf{k}) - \left(\mathbf{I} - k^{-2}\mathbf{k}\mathbf{k}\right)\cdot\mathbf{C}(\mathbf{k}) \qquad \text{b)}\ \mathbf{H}(\mathbf{k}) = i\mathbf{k}\times\mathbf{A}(\mathbf{k}) \tag{10}$$

where **I** is the unit dyadic. In this set of equations, it is clear that the transverse parts of **E**(**k**) and **H**(**k**) are determined only by the transverse parts of **C**(**k**) and **A**(**k**), respectively, and that $i\mathbf{k}\cdot\mathbf{E}(\mathbf{k}) = \rho(\mathbf{k})$ and $i\mathbf{k}\cdot\mathbf{H}(\mathbf{k}) = 0$, as required. Apparently, the six components of **A**(**k**) and **C**(**k**) project onto only four components of **E**(**k**) and **H**(**k**).

The dynamic Maxwell's equations (2d) and (2b), and vector potential equations (7b) and (7c), are, respectively:

$$\text{a)}\ d\mathbf{E}(\mathbf{k})/dt = i\mathbf{k}\times\mathbf{H}(\mathbf{k}) - \mathbf{j}(\mathbf{k}) \qquad \text{b)}\ d\mathbf{H}(\mathbf{k})/dt = -i\mathbf{k}\times\mathbf{E}(\mathbf{k}) \tag{11}$$

$$\text{a)}\ d\mathbf{C}(\mathbf{k})/dt = -k^2\mathbf{A}(\mathbf{k}) + \mathbf{j}(\mathbf{k}) \qquad \text{b)}\ d\mathbf{A}(\mathbf{k})/dt = \mathbf{C}(\mathbf{k}) \tag{12}$$

Taking a dot product of (10a) and (11a) with $i\mathbf{k}$ and combining the results yields the modal form of the continuity equation (3):

$$d\rho(\mathbf{k})/dt + i\mathbf{k}\cdot\mathbf{j}(\mathbf{k}) = 0 \tag{13}$$

This is not so much an evolution equation as an identity which must be satisfied by the dynamic equations of both the electromagnetic fields and charged matter.

The Lorentz condition (4c), in terms of spatial Fourier expansions, appears as:

$$d\varphi(\mathbf{k})/dt + i\mathbf{k}\cdot\mathbf{A}(\mathbf{k}) = 0 \tag{14}$$

Using (9c) and (14) yields (13), as it should. Also, a gauge transformation has the modal form: $\{\mathbf{A}(\mathbf{k}), \mathbf{C}(\mathbf{k})\} \rightarrow \{\mathbf{A}(\mathbf{k}) - i\mathbf{k}\psi(\mathbf{k}), \mathbf{C}(\mathbf{k}) - i\mathbf{k}\,d\psi(\mathbf{k})/dt\}$, where $\psi(\mathbf{k})$ satisfies the modal wave equation $d^2\psi(\mathbf{k})/dt^2 + k^2\psi(\mathbf{k}) = 0$.

In (12), the occurrence of two additional dynamic components in $\mathbf{A}(\mathbf{k})$ and $\mathbf{C}(\mathbf{k})$, compared with $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$, indicates that a vector potential formulation may have more explicit physical information than a formulation in terms of electromagnetic fields alone. In particular, (10) shows that neither $\nabla \cdot \mathbf{A}$ or $\nabla \cdot \mathbf{C}$ determine $\mathbf{E}$ or $\mathbf{H}$. However, the longitudinal parts of $\mathbf{A}$ and $\mathbf{C}$ and not just their transverse parts, may have observable effects. This implication has, in fact, been realized for the longitudinal part of $\mathbf{A}$ in the Aharonov-Bohm effect [3]. Before discussing this and other matters, however, let us consider the completeness and incompleteness of the various dynamical representations.

## Complete and Incomplete Dynamical Representations

So far, it has been shown that underlying Maxwell's equations there is a potential vector formulation of electromagnetics. In particular, it was shown that in addition to the vector potential $\mathbf{A}$, we must have another independent vector $\mathbf{C}$ to complete the formulation. This additional vector would, in fact, be equal to the partial time derivative of the original vector potential, *if that vector were already known.* Otherwise, it must be treated as completely independent, similar to the manner in which $\mathbf{x}$ and $\mathbf{v}$ are treated in the Lagrangian formulation of dynamics, or $\mathbf{x}$ and $\mathbf{p}$ in the Hamiltonian formulation of dynamics.

The analogy to the canonically conjugate variables of particle mechanics can be made more precise in the Fourier representation. Rather than the position $\mathbf{x}_i$ and momentum $\mathbf{p}_i$ of the $i^{th}$ particle serving as canonically conjugate variables, the coefficients of the electromagnetic field $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$ of the $\mathbf{k}^{th}$ mode are the canonically conjugate variables. In the vector potential formulation the canonically conjugate variables are $\mathbf{A}(\mathbf{k})$ and $\mathbf{C}(\mathbf{k})$. (In either of these formulations, it is the source vector $\mathbf{j}(\mathbf{k})$ which ultimately give rise to the coupled vector fields, and also ties them into material motion.)

The analogy with particle mechanics can be extended further: in the same way that the set of positions and momenta $\{\mathbf{x}_i, \mathbf{p}_i \mid i=1,...,N\}$ defines a 6N-dimensional phase space, the set of conjugate variables $\{\mathbf{A}(\mathbf{k}), \mathbf{C}(\mathbf{k}) \mid |\mathbf{k}| \leq k_{max}\}$ on N modes also defines a 6N-dimensional phase space. Here $\mathbf{k}$ and $-\mathbf{k}$ are counted as separate modes even though, since $\mathbf{A}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$ are real, $\mathbf{A}(\mathbf{k})=\mathbf{A}^*(-\mathbf{k})$ and $\mathbf{C}(\mathbf{k})=\mathbf{C}^*(-\mathbf{k})$. The complex vectors $\mathbf{A}(\mathbf{k})$ and $\mathbf{C}(\mathbf{k})$ are composed of 6 components each (*i.e.*, $\mathbf{A}(\mathbf{k})= \mathbf{A}_R(\mathbf{k})+i\mathbf{A}_I(\mathbf{k})$, where $\mathbf{A}_{R,I}$ are real) and the pair ($\mathbf{A}(\mathbf{k})$, $\mathbf{C}(\mathbf{k})$) thus contain twelve 'degrees-of-freedom'; however, these are the same as are found in the pair ($\mathbf{A}(-\mathbf{k})$, $\mathbf{C}(-\mathbf{k})$). Then only half of the N modes, such that $|\mathbf{k}|\leq k_{max}$, are independent and $12\times N/2=6N$.

The phase space associated with the set $\{\mathbf{E}(\mathbf{k}), \mathbf{H}(\mathbf{k}); |\mathbf{k}|\leq k_{max}\}$ of N modes has, however, a dimension of only 4N because $\mathbf{k}\cdot\mathbf{E}(\mathbf{k})$ and $\mathbf{k}\cdot\mathbf{H}(\mathbf{k})$ are prescribed. Transformation (10), which takes $\{\mathbf{A}(\mathbf{k}), \mathbf{C}(\mathbf{k}); |\mathbf{k}|\leq k_{max}\}\rightarrow\{\mathbf{E}(\mathbf{k}), \mathbf{H}(\mathbf{k}); |\mathbf{k}|\leq k_{max}\}$ is actually a projection operation and is not invertible. Although the set of vectors $\{\mathbf{E}(\mathbf{k}), \mathbf{H}(\mathbf{k}); |\mathbf{k}|\leq k_{max}\}$ may be sufficient classically, it is not, as has been mentioned, sufficient quantum mechanically. The classical phenomena described by $\mathbf{E}$ and $\mathbf{H}$ thus fall into only a subset of all possible electromagnetic phenomena.

## The Aharonov-Bohm Effect

Consider an infinitely long wire with a steady electric current 'I' coursing down it. In order to determine the magnetic field, we solve Maxwell's equation (2d) in a time-independent form, $\nabla\times\mathbf{H} = \mathbf{j}$, where $\mathbf{j}$ is zero outside the wire. The solution is elementary and well-known: in a cylindrical coordinate system $(r,\phi,z)$, whose z-axis runs down the center of the current carrying wire, outside of the wire the only non-zero component of $\mathbf{H}$ is $H_\phi=I(2\pi r)^{-1}$.

Consider, instead of a long current-carrying wire, a long, straight, and narrow solenoid. This solenoid will completely contain a magnetic field of total flux $\Phi$; the vector potential $\mathbf{A}$ will now satisfy the equation $\nabla\times\mathbf{A} = \mathbf{H}$, where $\mathbf{H}$ is zero outside the solenoid. Since this situation is mathematically the same as the current-carrying wire problem of the previous paragraph, we can immediately give the solution outside the solenoid: the only non-zero component of $\mathbf{A}$ is $A_\phi=\Phi(2\pi r)^{-1}$.

Classically, an electron beam passing by the long solenoid would feel no electromagnetic influence from the solenoid, since its motion is only affected by $\mathbf{E}$ and $\mathbf{H}$, which are zero outside the solenoid. Quantum mechanically, however, the electron's wave function is affected:

Depending on which path we trace the electron beam's motion along, the phase $\Theta$ of the wave function associated with the beam has an additional value of (here Planck's constant has a value of unity)

$$\Delta\Theta = e\int_{x_o}^{x} \mathbf{A}\cdot d\mathbf{l} \tag{15}$$

compared to what it would have if **A** (due to the flux $\Phi$) were absent [3].

In fact, a solenoid placed between the beams of a double slit electron diffraction experiment will cause a different phase shift to occur for the wave function of each beam. The difference in phase shift for the wave functions of the beams, one through the right-hand slit and the other through the left hand slit, with the solenoid between them, will be given by the application of Stokes theorem to (15) around a closed path:

$$\Delta\Theta_{right} - \Delta\Theta_{left} = e\int\int \nabla\times\mathbf{A}\cdot d\mathbf{s} = e\Phi \tag{16}$$

Thus the phase difference, which will cause the diffraction pattern to shift as the solenoid's field is varied, depends only on the total magnetic flux between the beams. This effect has been observed in a number of careful experiments [4].

The vector potential outside of the solenoid due to the flux $\Phi$ satisfies $\nabla\times\mathbf{A}=\mathbf{0}$. Thus $\mathbf{k}\times\mathbf{A}(\mathbf{k}) = \mathbf{0}$ and it is only the part of $\mathbf{A}(\mathbf{k})$ along $\mathbf{k}$ which can cause the effect, *i.e.*, it is $\mathbf{k}\cdot\mathbf{A}(\mathbf{k})$ which is important here. Since this is precisely what is projected out of the electromagnetic fields, an effect such as that of Aharonov and Bohm can be overlooked if only **E** and **H** are assumed to have physically observable effects.

## Charged Particle Lagrangians

At this point, let us consider what novel effects are uncovered when the Lagrangian for a charged particle is expressed in terms of potential vectors. The Lagrangian density for any particle of classical or quantum mechanical electric charge $e$ will contain a part $\Lambda_e$ related to the electromagnetic field:

$$\Lambda_e = -e\rho\varphi + e\mathbf{j}\cdot\mathbf{A} + \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{H}^2\right) \tag{17}$$

(Note: The symbols ($\rho$, $\mathbf{j}$) will now be used to signify the particle four-current density; the electric four-current density will now have components ($e\rho$, $e\mathbf{j}$).)

The Lagrangian itself is defined as a volume integral of a Lagrangian density:

$$L(t) = \int \Lambda(\mathbf{x},t)\, d^3x \tag{18}$$

Since we have defined the system volume to be a periodic box with an associated Fourier expansion (8), the Lagrangian could equally well be written in terms of Fourier coefficients:

$$L(t) \equiv \sum_{\mathbf{k}} \Lambda(\mathbf{k},t) \tag{19}$$

where the $\mathbf{k}$-space Lagrangian density is (surpressing explicit time dependence for brevity):

$$\Lambda(\mathbf{k}) = \Lambda_m(\mathbf{k}) - e\rho^*(\mathbf{k})\varphi(\mathbf{k}) + e\mathbf{j}^*(\mathbf{k})\cdot\mathbf{A}(\mathbf{k}) + \tfrac{1}{2}\left(|\mathbf{E}(\mathbf{k})|^2 - |\mathbf{H}(\mathbf{k})|^2\right) \tag{20}$$

In (20), $\Lambda_m$ contains those terms which pertain only to the matter in the system; the specific form of $\Lambda_m$ depends on whether a classical or quantum mechanical system is under condsideration. The rest of the Lagrangian pertains equally to both classical or quantum mechanical systems. Notice that if we set ($\mathbf{E}$, $\mathbf{H}$) equal to zero in (20), then the Lagrangian density is $\Lambda(\mathbf{k}) = \Lambda_m(\mathbf{k}) - ej_\mu A^\mu$, i.e., it still depends on the four-potenial. Thus, even though $(\mathbf{E}, \mathbf{H})$ are zero, the particle is not free (for which the four-potential must also be zero), and effects such as that of Aharonov and Bohm are possible.

If the expressions for $\varphi(\mathbf{k})$, $\mathbf{E}(\mathbf{k})$, and $\mathbf{H}(\mathbf{k})$ ((9c), (10a), and (10b), respectively) are placed into (20), the result may be written as:

$$\Lambda(\mathbf{k}) = \Lambda_m(\mathbf{k}) + \Lambda_r(\mathbf{k}) - \Lambda_p(\mathbf{k}) \tag{21}$$

where 'radiation' part $\Lambda_r$ and the 'potential' part $\Lambda_p$ of $\Lambda_e$ are:

$$\begin{aligned} &\text{a) } \Lambda_r(\mathbf{k}) = \tfrac{1}{2}\left(|\mathbf{C}_\perp(\mathbf{k})|^2 - k^2|\mathbf{A}_\perp(\mathbf{k})|^2\right) \\ &\text{b) } \Lambda_p(\mathbf{k}) = \rho^*(\mathbf{k})\mu(\mathbf{k}) - e\mathbf{j}^*(\mathbf{k})\cdot\mathbf{A}(\mathbf{k}) + \tfrac{e^2}{2}k^{-2}|\rho(\mathbf{k})|^2 \end{aligned} \tag{22}$$

Here $\mathbf{C}_\perp$ and $\mathbf{A}_\perp$ are the transverse parts of $\mathbf{C}$ and $\mathbf{A}$, respectively. The coefficient $\mu(\mathbf{k})$ is:

$$\mu(\mathbf{k}) = iek^{-2}\mathbf{k}\cdot\mathbf{C}(\mathbf{k}) \tag{23}$$

Here we see that $\mathbf{k}\cdot\mathbf{C}(\mathbf{k})$, though it does not appear in the electromagnetic fields, does appear in the Lagrangian density.

In the Lagrangian density (22), the 'potential' $\Lambda_p(\mathbf{k})$ contains a term which is linear and one which is quadratic in $\rho(\mathbf{k})$: $\mu(\mathbf{k})\rho^*(\mathbf{k})+Q(\mathbf{k})$, where $Q(\mathbf{k}) = \frac{1}{2}e^2k^{-2}|\rho(\mathbf{k})|^2$. What does $Q(\mathbf{k})$ look like in $\mathbf{x}$-space? If we define $\sigma(\mathbf{k}) = \rho(\mathbf{k})/k$ for $k>0$ and $\sigma(\mathbf{k}) = 0$ for $k=0$, then we have:

$$\sum_{\mathbf{k}} \frac{e^2}{2}|\sigma(\mathbf{k})|^2 = \int d^3x\, \frac{e^2}{2}[\sigma(\mathbf{x})]^2 \tag{24}$$

Here $\sigma(\mathbf{x})$ is the inverse Fourier transform of $\sigma(\mathbf{k})$, etc. Thus $Q(\mathbf{x}) = \frac{1}{2}e^2[\sigma(\mathbf{x})]^2$.

Although the potential (22) is written in **k**-space, it has a striking resemblance to *ad hoc* 'spontaneous symmetry breaking' potentials found in current theories of particle physics [6]. (In these theories, $\rho(\mathbf{x})$ is the modulus of the quantum mechanical wave function squared.) The naturally occuring term we have uncovered here would appear to deserve further study.

## Conclusion

In this paper, four major points have been discussed. First, there is the potential vector formulation of electromagnetic fields. It was seen that the divergence of the vector potential is uniquely defined, up to a gauge transformation. The potential vector formulation contains Maxwell's equations explicitly, while it might be said that Maxwell's equations, Lorentz invariance, and gauge invariance, all taken together, implicitly contain the potential vector formulation.

The second point was that of dynamical completeness. For a given number N of Fourier modes, it was clear that the electromagnetic fields associated with these N modes span a phase space which is only 2/3rds the dimension of that spanned by the potential vectors for the same number of modes. Thus, the electromagnetic fields offer only an incomplete dynamical representation, as compared to the potential vectors.

The third point was to review how this incompleteness could miss an observable effect, albeit a quantum mechanical one. This observable effect is that of Aharonov and Bohm, where the longitudinal part of **A** proved important. This clearly demonstrates how fruitful it is to have and to use a conceptually complete dynamical representation of electromagnetic phenomena.

The fourth and final point was to recognize the existence of a naturally occuring 'spontaneous symmetry breaking' term in a charged particle Lagrangian. This term contains the longitudinal part of **C**, indicating that it is also dynamically important. The detailed effect of this term on quantum mechanical descriptions of charged particle dynamics is a separate issue from the topics covered herein, though clearly an intriguing one.

## Acknowledgement

I wish to thank Professor George Vahala for his very useful comments.

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