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STRESS DECAY IN AN ORTHOTROPIC HALF-PLANE UNDER
SELF-EQUILIBRATING SINUSOIDAL LOADING

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ABSTRACT

The problem of an elastic orthotropic half-plane subjected to sinusoidal normal loading over the entire straight boundary, apparently first analyzed by Mansfield and Best, is reexamined. Stresses are calculated for combinations of material properties which are representative of some unidirectional filamentary composites, and of $(+45^{\circ})_S$ laminates made from the same unidirectional materials. Plots of the stresses as functions of the distance from the loaded boundary show that they can differ greatly from their counterparts in the isotropic half-plane under the same loading. In addition, the stresses in some orthotropic materials are seen to exhibit oscillatory behavior. How the results bear on the question of the applicability of St. Venant's principle to orthotropic materials is briefly discussed.

INTRODUCTION

Various versions of St. Venant's principle have been invoked to justify the replacement of a given set of boundary tractions on an elastic body by another statically equivalent set, usually for the purpose of mathematical simplification. In essence, the original problem is replaced by a simpler alternative problem and, under suitable restrictions, the alternative solution is found (or assumed) to be sufficiently accurate at interior points which are sufficiently far removed from the affected part of the boundary. Typical of the restrictions involved are stipulations that the elastic body be simply connected, and that the original set (hence, the alternative set) of boundary tractions be self-equilibrating. Additional restrictions are sometimes imposed, depending on the nature of the problem.

How far an interior point must be from the affected part of the boundary in order for the alternative solution to be "accurate enough" often depends on a characteristic dimension associated with the set of surface tractions. For example, Boussinesq has shown that if normal external forces on the plane surface of an isotropic half-space are confined to a circle of radius ϵ , then the stresses at the fixed interior point at a distance $r > \epsilon$ from the center of the circle are of order of magnitude ϵ^2 , i.e., $|\sigma_{ij}| \leq C \left(\frac{\epsilon}{r}\right)^2$, when both the resultant force and moments of the applied forces are zero (see ref. 1).

With a simple boundary-value problem for a half-plane, some aspects of which were studied in reference 2, the present paper demonstrates that conventional versions of St. Venant's principle can be inadequate when the elastic body is anisotropic. Exact expressions are given for the stresses in an orthotropic half-plane subjected to sinusoidally varying normal traction on the straight boundary. Numerical results are obtained for stresses in orthotropic half-planes which are representative of some unidirectional and $(+45^\circ)_s$ composite laminates, and compared with corresponding results for an isotropic half-plane.

ANALYSIS

The problem is that of a linear elastic orthotropic half-plane $y \geq 0$ subjected to a sinusoidally varying normal "pressure" over the entire straight boundary, i.e., the line $y = 0$. For simplicity, attention is restricted to cases in which the coordinate axes are lines of material symmetry. For generalized plane stress the governing equations (see ref. 3) for the half-plane $y \geq 0$ are:

Equilibrium:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \end{aligned} \quad (1)$$

Compatibility:
$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (2)$$

where
$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E_x} - \nu_{yx} \frac{\sigma_y}{E_y}, \quad \epsilon_y = \frac{\partial v}{\partial y} = \frac{\sigma_y}{E_y} - \nu_{xy} \frac{\sigma_x}{E_x}, \quad (3)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\tau_{xy}}{G}, \quad \text{and} \quad E_x \nu_{yx} = E_y \nu_{xy}.$$

The boundary conditions are:

$$\sigma_y(x, 0) = p \cos \frac{\pi}{a} x, \quad \tau_{xy}(x, 0) = 0 \quad (4)$$

and for $y \rightarrow \infty$, $\sigma_x, \sigma_y, \tau_{xy} \rightarrow 0$.

For the boundary tractions given in (4), a separable solution is possible.

Assuming for $\sigma_y(x, y)$ the form

$$\sigma_y(x, y) = p \cos \frac{\pi}{a} x f(y) \quad (5)$$

the equations (1) - (5) require that $f(y)$ satisfy the homogeneous equation

$$\frac{d^4 f}{dy^4} - \frac{\pi^2}{a^2} \left(\frac{E_x}{G} - 2\nu_{xy} \right) \frac{d^2 f}{dy^2} + \frac{\pi^4}{a^4} \frac{E_x}{E_y} f = 0 \quad (6)$$

The function $f(y)$ satisfying (6) can have different forms, depending on the properties of the orthotropic half-plane. The solution forms which are appropriate to most engineering materials characterizable as orthotropic fall into two cases, the first of which was treated previously in reference 2:

Case I -
$$f_1(y) = \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1 e^{-\frac{\pi}{a} \lambda_2 y} - \lambda_2 e^{-\frac{\pi}{a} \lambda_1 y} \right)$$

where λ_1, λ_2 are real, $\lambda_1 = \sqrt{p_1 + p_2}$, $\lambda_2 = \sqrt{p_1 - p_2}$, (7)

$$p_1 = \frac{1}{2} \left(\frac{E_x}{G} - 2\nu_{xy} \right) > \sqrt{\frac{E_x}{E_y}}, \quad p_2 = \sqrt{p_1^2 - \frac{E_x}{E_y}}.$$

Most orthotropic materials with grossly different principal stiffnesses, e.g., unidirectional filamentary composites, fall into Case 1.

$$\text{Case 2 - } f_2(y) = \frac{1}{k_2} e^{-\frac{\pi}{a} k_1 y} \left(k_2 \cos \frac{\pi}{a} k_2 y + k_1 \sin \frac{\pi}{a} k_2 y \right)$$

$$\text{where } k_1 = k \sin \frac{\alpha}{2}, \quad k_2 = k \cos \frac{\alpha}{2},$$

$$r_1 + i r_2 = k^2 e^{i\alpha}, \quad \alpha = \tan^{-1} \frac{r_2}{r_1}, \quad (8)$$

$$0 < r_1 = -\frac{1}{2} \left(\frac{E_x}{G} - 2\nu_{xy} \right) < \sqrt{\frac{E_x}{E_y}}, \quad r_2 = \sqrt{\frac{E_x}{E_y} - r_1^2}.$$

The $(+45^\circ)_s$ laminates considered herein fall into this case.

Then, for Case 1 the stresses have the form

$$\begin{aligned} \sigma_x &= p \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left(\lambda_1 e^{-\frac{\pi}{a} \lambda_1 y} - \lambda_2 e^{-\frac{\pi}{a} \lambda_2 y} \right) \cos \frac{\pi}{a} x \\ \sigma_y &= p \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1 e^{-\frac{\pi}{a} \lambda_2 y} - \lambda_2 e^{-\frac{\pi}{a} \lambda_1 y} \right) \cos \frac{\pi}{a} x \\ \tau_{xy} &= p \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left(e^{-\frac{\pi}{a} \lambda_2 y} - e^{-\frac{\pi}{a} \lambda_1 y} \right) \sin \frac{\pi}{a} x \end{aligned} \quad (9)$$

For Case 2 the stresses are:

$$\begin{aligned} \sigma_x &= \frac{p}{k_2} e^{-\frac{\pi}{a} k_1 y} \left(k_2 \cos \frac{\pi}{a} k_2 y - k_1 \sin \frac{\pi}{a} k_2 y \right) \cos \frac{\pi}{a} x \\ \sigma_y &= \frac{p}{k_2} e^{-\frac{\pi}{a} k_1 y} \left(k_2 \cos \frac{\pi}{a} k_2 y + k_1 \sin \frac{\pi}{a} k_2 y \right) \cos \frac{\pi}{a} x \\ \tau_{xy} &= \frac{p}{k_2} e^{-\frac{\pi}{a} k_1 y} \sin \frac{\pi}{a} k_2 y \sin \frac{\pi}{a} x \end{aligned} \quad (10)$$

In the isotropic half-plane under identical loading, the stresses are:

$$\begin{aligned} \sigma_x &= p \left(1 - \frac{\pi}{a} y \right) e^{-\frac{\pi}{a} y} \cos \frac{\pi}{a} x \\ \sigma_y &= p \left(1 + \frac{\pi}{a} y \right) e^{-\frac{\pi}{a} y} \cos \frac{\pi}{a} x \\ \tau_{xy} &= p \frac{\pi}{a} y e^{-\frac{\pi}{a} y} \sin \frac{\pi}{a} x \end{aligned} \quad (11)$$

RESULTS AND DISCUSSION

From equations (9) and (10) it is clear that the rates of decay of the stresses with distance from the surface of the orthotropic half-plane depend not only on a , the half-period of the sinusoidal loading, but also on the parameters λ_1 and λ_2 in Case 1, and on k_1 and k_2 in Case 2. In both cases, of course, the parameters are functions of material properties. In contrast, as is evident in equations (11), the rates of decay of the stresses in the isotropic half-plane depend on only a .

To illustrate some of the influences of anisotropy on stress decay, the stresses $\sigma_y(0,y)$ and $\tau_{xy}(\frac{a}{2},y)$ have been calculated for the isotropic half-plane, two unidirectional graphite/epoxy filamentary composite half-planes, and two $(+45^\circ)_s$ graphite/epoxy filamentary composite half-planes. (Two additional sets of unidirectional half-plane results are obtained by rotating the original unidirectional materials by 90° with respect to the (x,y) system). The material properties are listed in Table 1. Materials A and A' (A rotated 90°) are representative of a typical graphite/epoxy laminate (see ref. 4); materials B and B' (B rotated 90°) are representative of a unidirectional graphite/epoxy laminate containing a higher-modulus graphite (see ref. 5). Materials \bar{A} and \bar{B} are $(+45^\circ)_s$ laminates composed of layers of materials A and B, respectively.

Figures 1 and 2 contain graphs of $\sigma_y(0,y)$ in the isotropic (I) and the four unidirectional half-planes. (Note that the stresses in the unidirectional half-planes fall into Case 1.) When the applied loading is normal to the fiber direction (Materials A and B), $\sigma_y(0,y)$ in the unidirectional materials differs only moderately from its counterpart in the isotropic material. However, when the loading is parallel to the fibers (materials A' and B'), the differences are pronounced. For example, in Figure 2(a), which is for the higher-modulus graphite, $\sigma_y(0,y)$ in the isotropic material decays to 20% of its maximum value at

$\frac{\pi}{a}y = 3$, but in material B' only to 38% of its maximum value at $\frac{\pi}{a}y = 8$, evincing large differences in decay rate with distance from the boundary.

Material-related differences are also evident in Figures 1(b) and 2(b), where $\tau_{xy}(\frac{a}{2}, y)$ is graphed for the five half-planes. Results for the isotropic and orthotropic half-planes differ in the magnitude and location of the maximum shear stress, and in rate of decay with distance from the boundary. Typically, maximum shear stress is greater in the orthotropic material when the loading is normal to the fibers (materials A and B), and its location is somewhat nearer the surface. The decay rates in the three half-planes are comparable.

On the other hand, when the loading is parallel to the fibers (materials A' and B'), the maximum shear stress is lower in the orthotropic material, and the peak occurs somewhat farther from the surface. In addition, the rates of decay with distance from the surface are markedly different. In Figure 2(b), for example, the shear stress in the orthotropic material is still at nearly 50% of its maximum value at $\frac{\pi}{a}y = 8$. In contrast, the shear stress in the isotropic material falls to 50% of its maximum value at $\frac{\pi}{a}y \approx 2.6$, after having attained a much greater maximum value.

An additional material-related phenomenon is evident in the stresses computed for the two $(+45^\circ)_s$ laminates (which require a solution of the form given by Case 2) and graphed, along with results for the isotropic case, in Figures 3 and 4. Besides exhibiting different decay rates, the stresses in the $(+45^\circ)_s$ laminates are seen to oscillate, behavior which is, of course, evident in equations (10).

It should also be noted that large decay-rate differences can exist not only between isotropic and orthotropic materials, but also between orthotropic materials within the same case. This phenomenon appears to be most strongly influenced by the material property ratio $\frac{E_y}{G}$.

With regard to the applicability of St. Venant's principle, another set of surface tractions which is statically equivalent to the present system is the set of zero tractions, which leads directly to the trivial problem of an unstressed half-plane. Though in the isotropic case it might still be argued that, at a distance greater than $2a$ from the surface, all stresses are (more or less) negligible and, hence, comparable to the trivial solution, this argument clearly would not hold for some of the orthotropic materials considered here, especially the unidirectional materials under loading parallel with the fibers.

CONCLUDING REMARKS

The examples presented herein illustrate only one of several pitfalls associated with a facile invoking of St. Venant's principle, and only one of the reasons that the principle is difficult to state generally enough to be widely applicable yet specifically enough to be maximally helpful. In the simply connected body, at least, the problem is more one of quantity than quality, since most statements of the principle contain such forgiving phrases as "sufficiently far removed" and "essentially the same;" however, the analyst is sometimes asked to justify its use numerically. When the required numerical values are easily obtained, the principle is largely superfluous. Only when the calculations are difficult does the analyst have a vital interest in the principle's applicability which, in such cases, is difficult to establish. The present solution suggests that, at least in problems involving some orthotropic materials, the utility of traditional forms of St. Venant's principle can be quite limited.

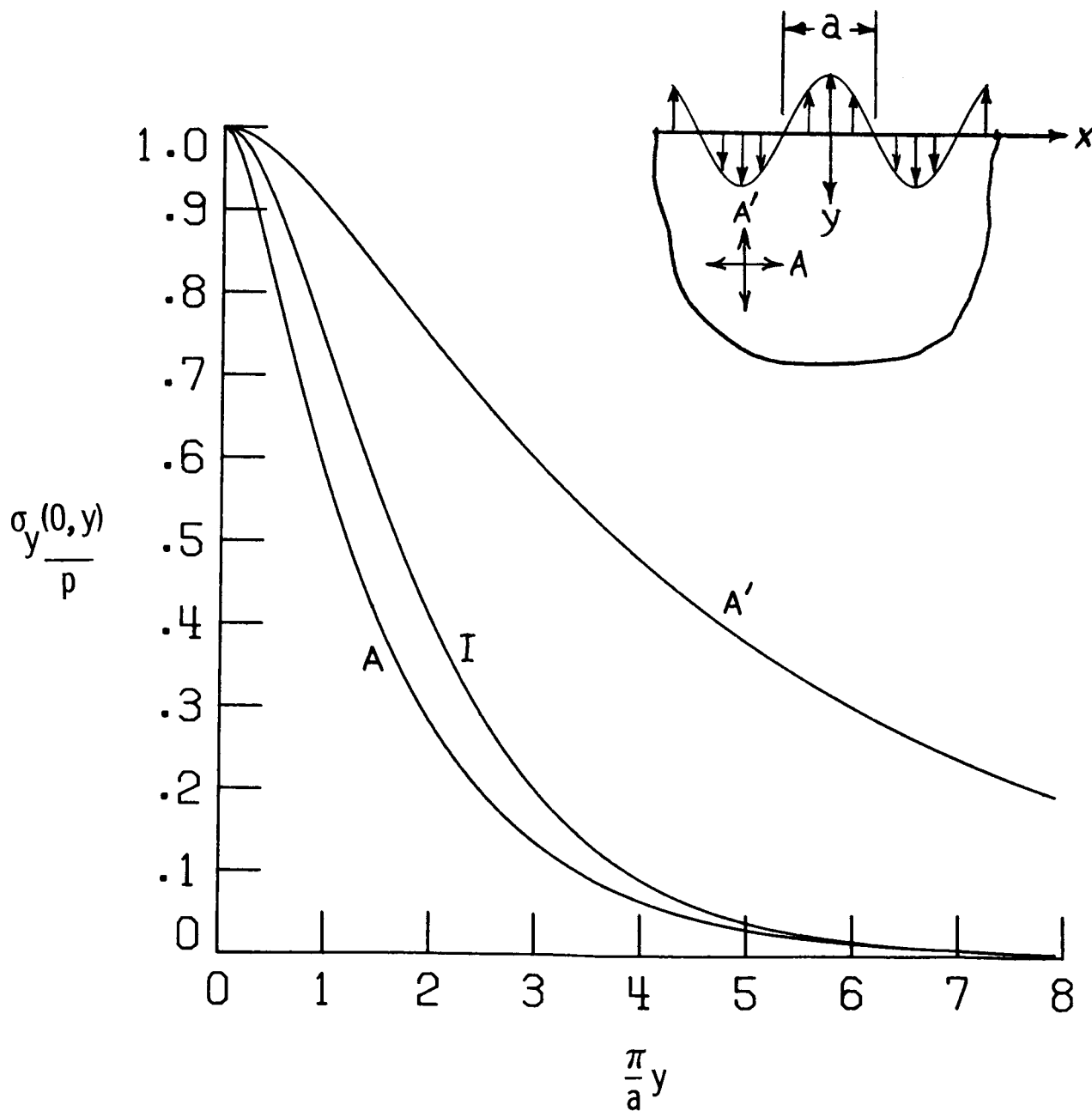
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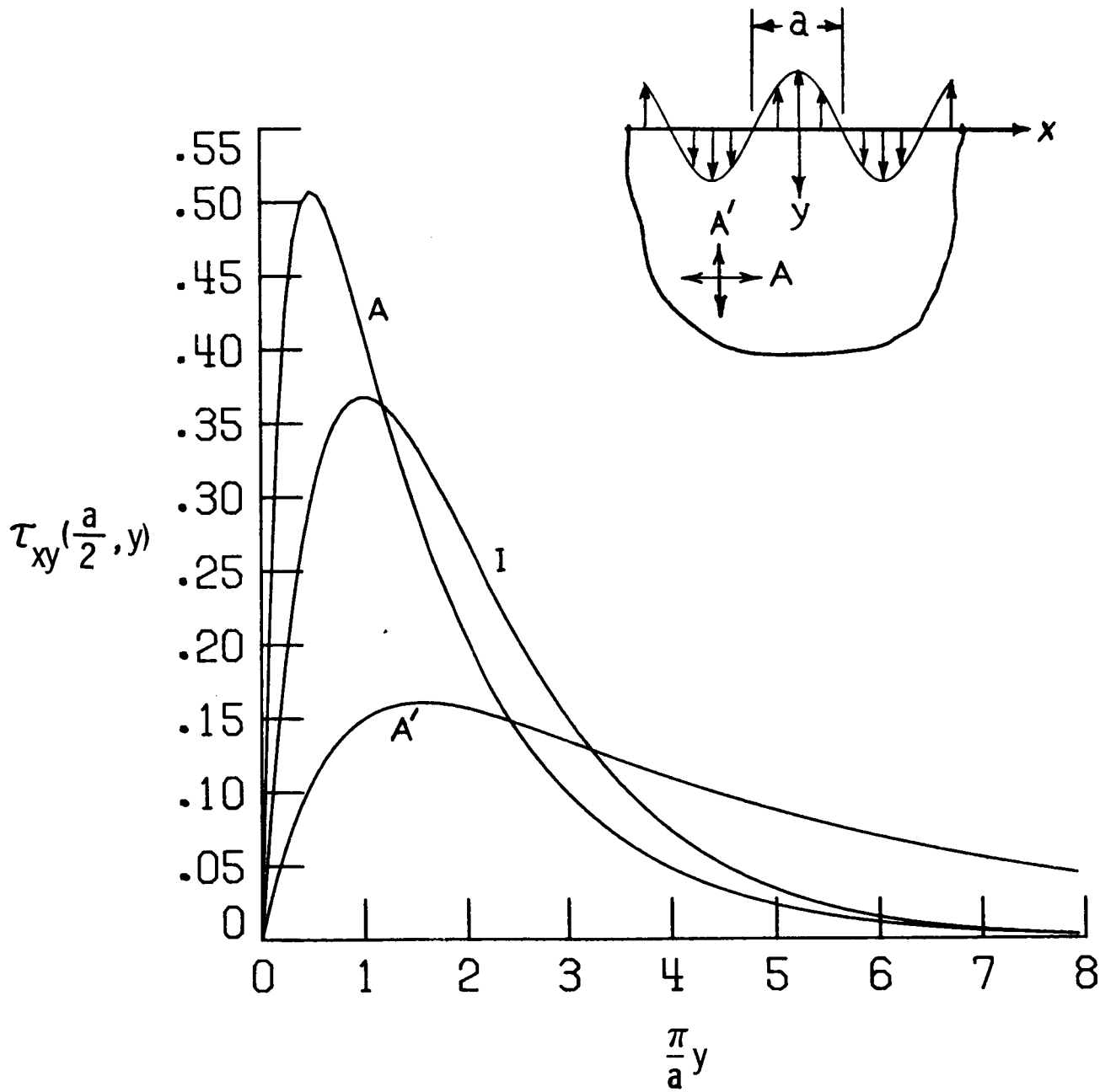
TABLE 1 - HALF-PLANE MATERIAL PROPERTIES

Material	E_x , psi	E_y , psi	ν_{xy}	G, psi
A (Unidirectional) graphite/epoxy	19×10^6	1.89×10^6	0.38	0.93×10^6
A'	1.89×10^6	19×10^6	0.038	0.93×10^6
B (Unidirectional) graphite/epoxy	94×10^6	3.6×10^6	0.25	1.6×10^6
B'	3.6×10^6	94×10^6	0.01	1.6×10^6
A ($(+45^\circ)_S$ Laminate) of material A	3.20×10^6	3.20×10^6	0.718	4.93×10^6
B ($(+45^\circ)_S$ Laminate) of material B	6.01×10^6	6.01×10^6	0.879	24.0×10^6



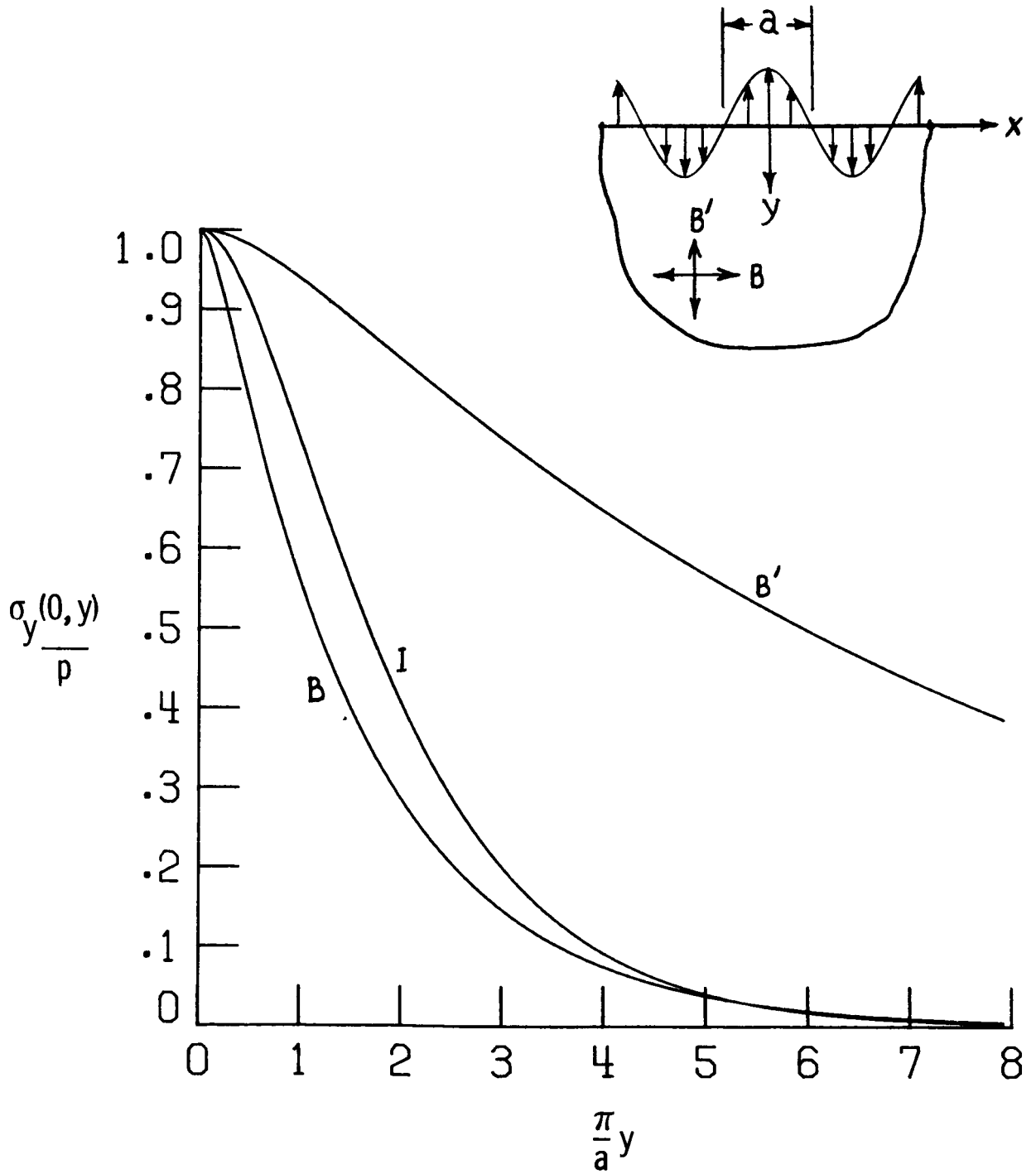
(a) Normal stress

Figure 1 - Stress decay in unidirectional Material A(A').



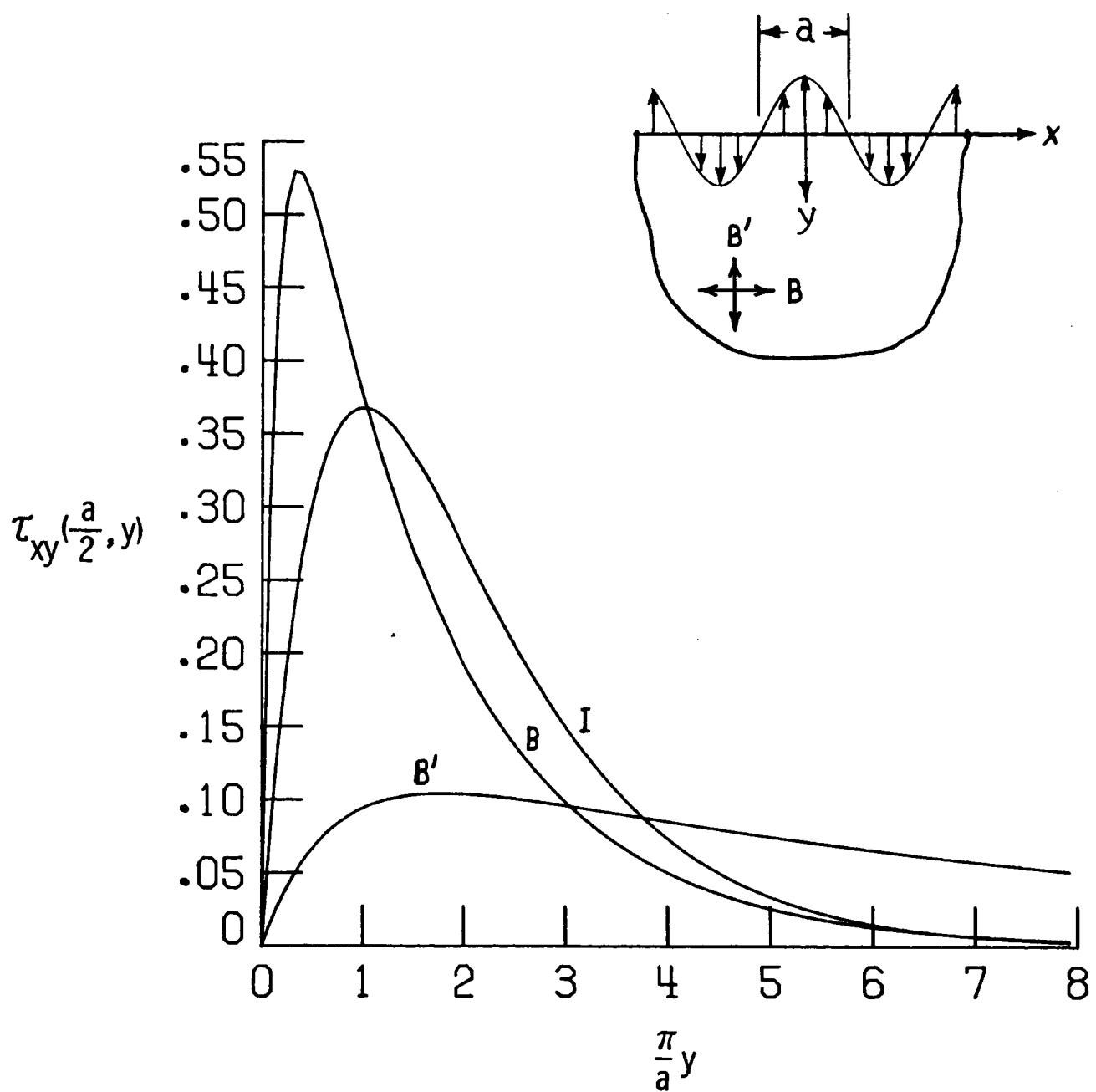
(b) Shear stress

Figure 1 - Concluded.



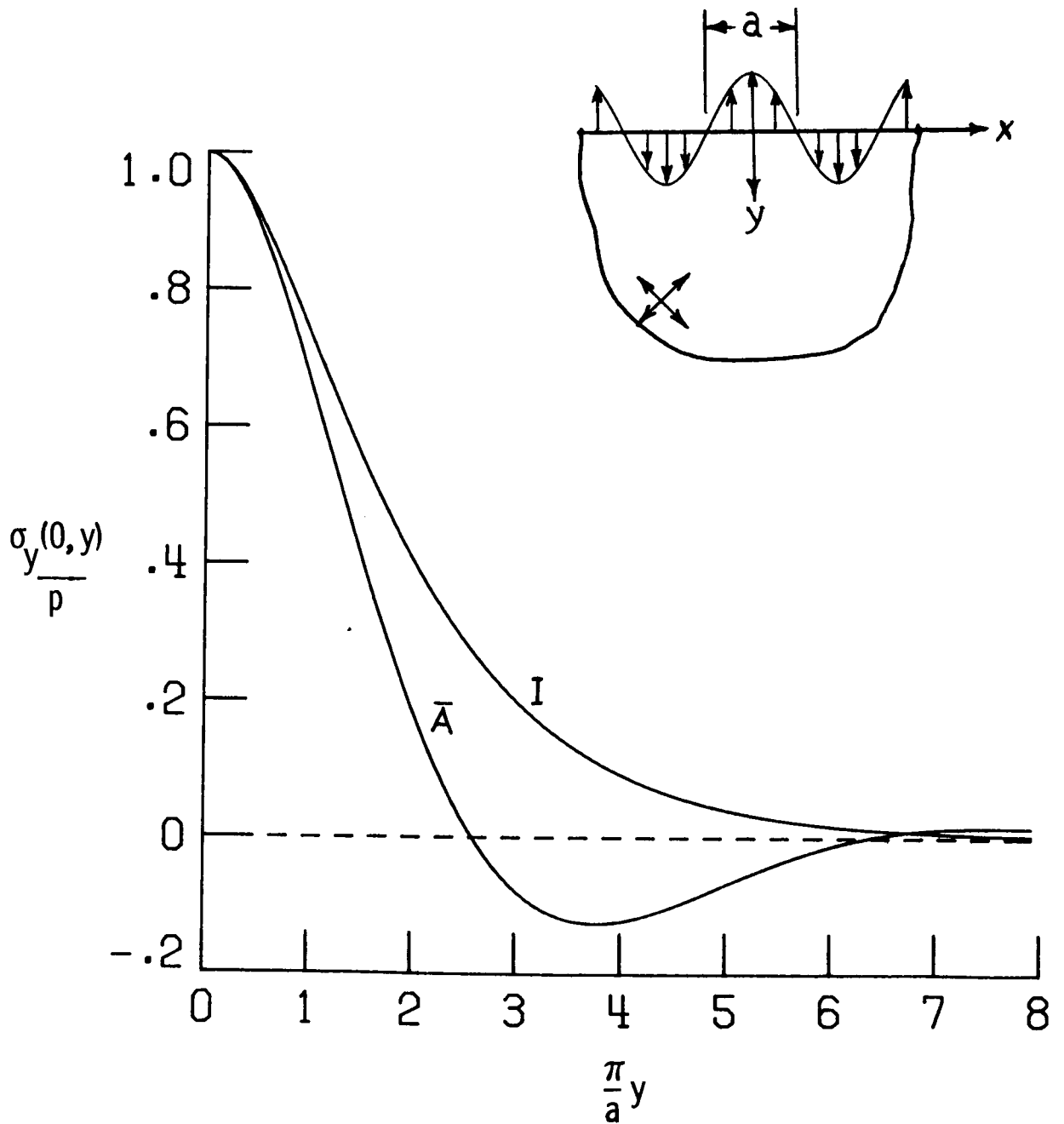
(a) Normal stress

Figure 2 - Stress decay in unidirectional Material B(B').



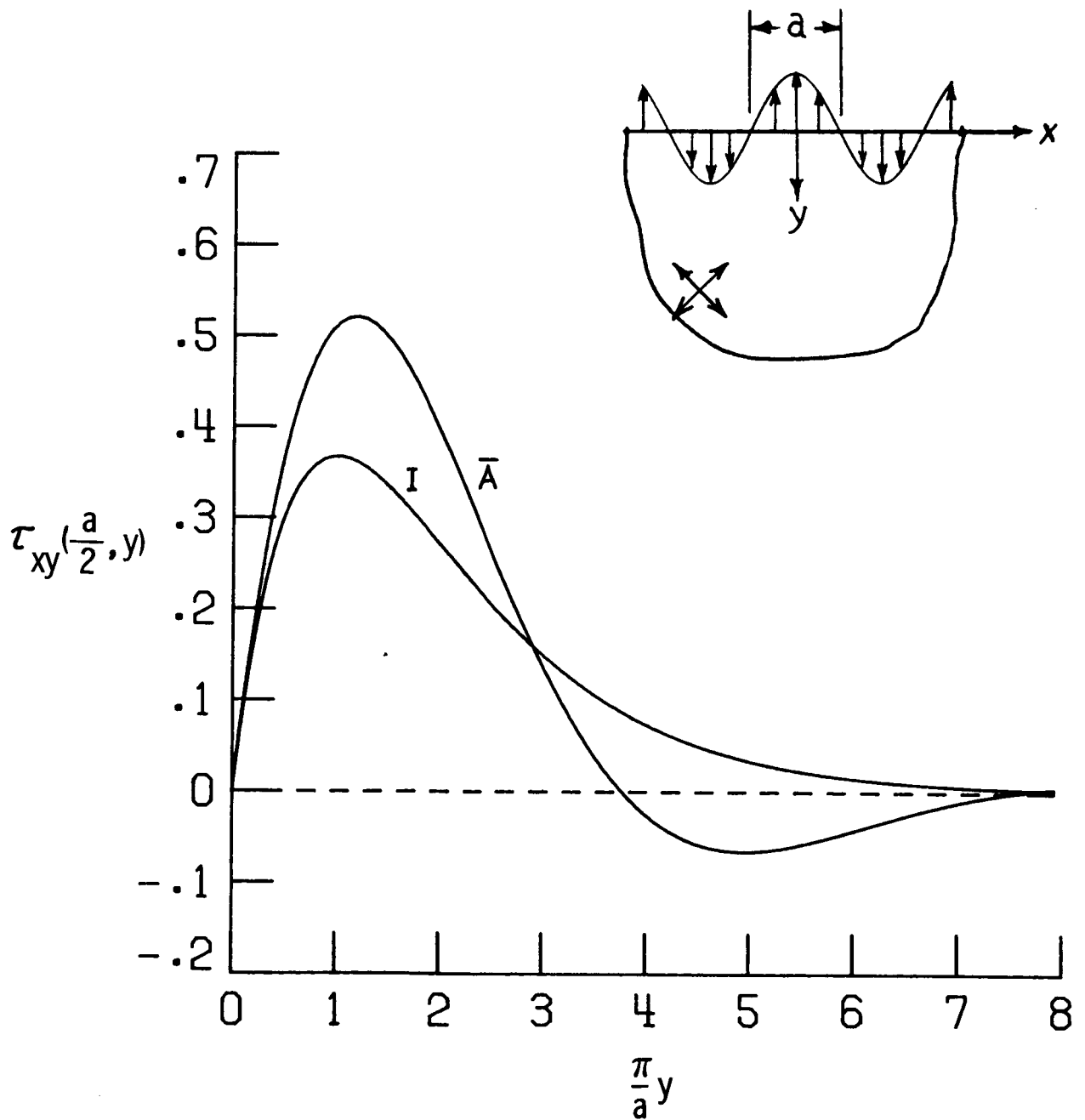
(b) Shear stress

Figure 2 - Concluded.

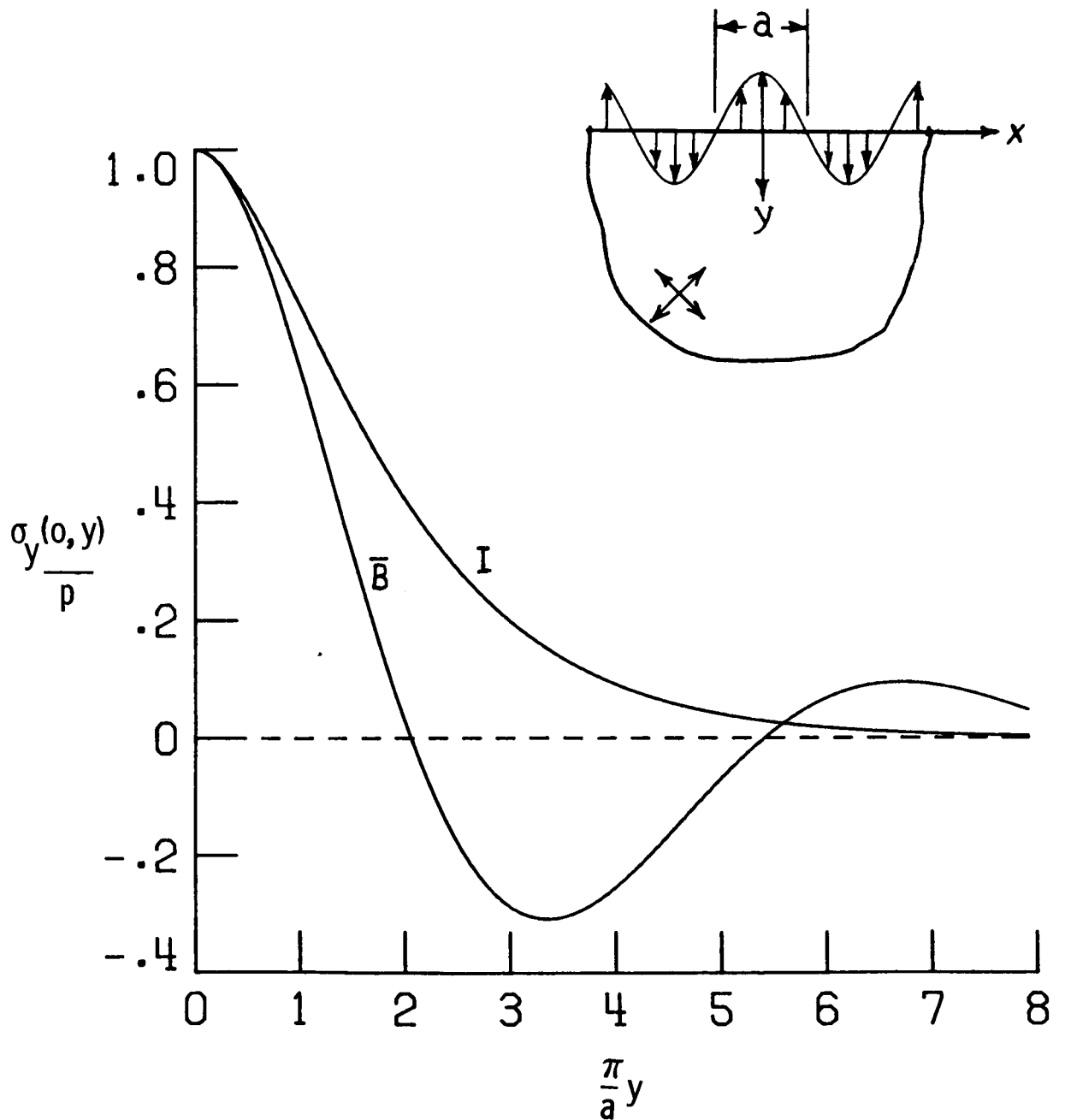


(a) Normal stress

Figure 3 - Stress decay in $(+45^\circ)_s$ Material \bar{A} .

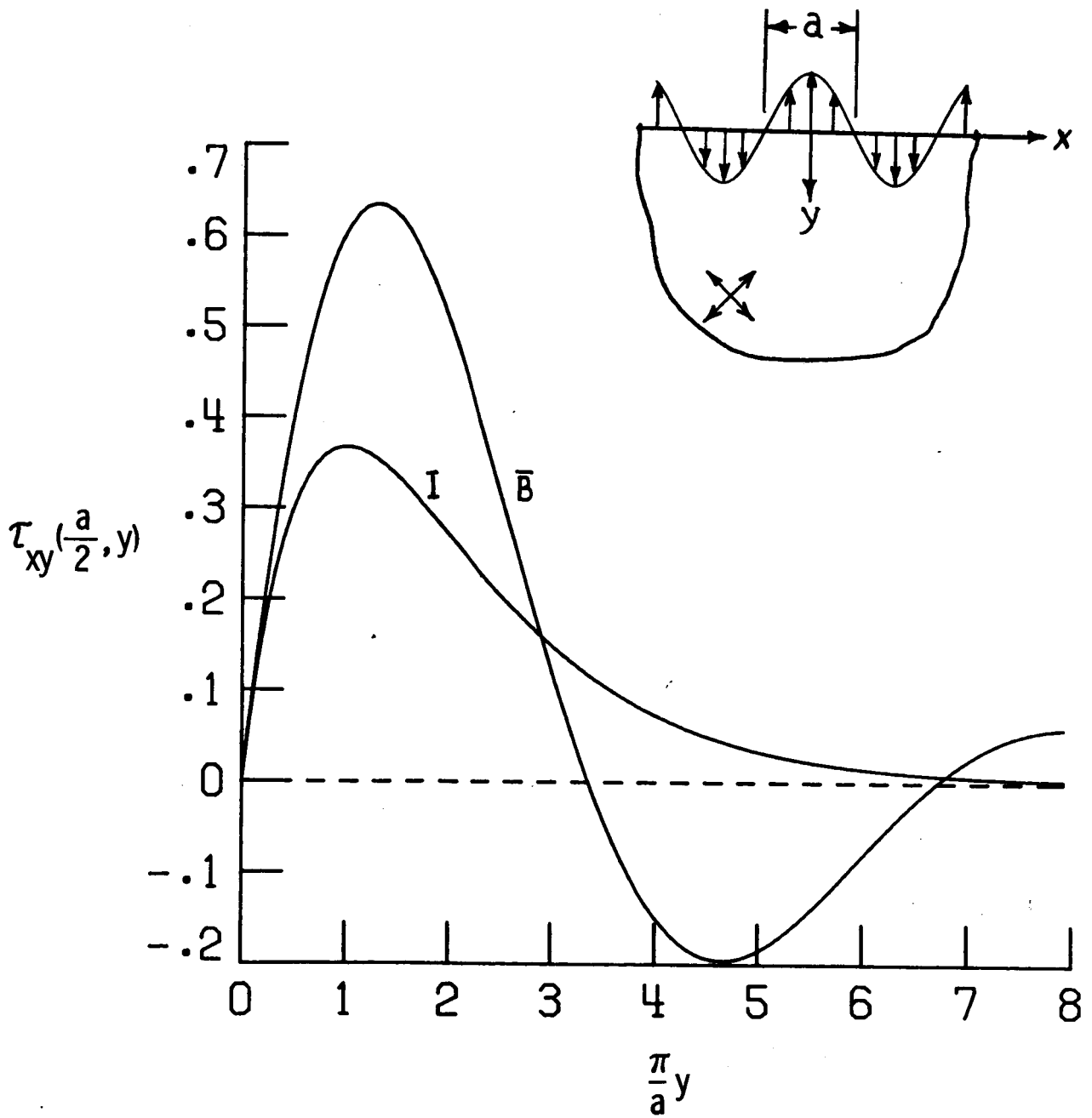


(b) Shear stress



(a) Normal stress

Figure 4 - Stress decay in $(+45^\circ)_s$ Material \bar{B} .



(b) Shear stress

Figure 4 - Concluded.

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