CHAPTER 5

Analyzing Students' Thinking

In this chapter, we examine the type of professional development experience in which teachers analyze student thinking as revealed in students' written assignments, think-aloud problem-solving tasks, class discussions and clinical interviews. Within this kind of professional development sessions, teachers learn to observe various types of student mathematical activity and to interpret what they observe, with the ultimate goal of enhancing their students' learning opportunities.

Theoretical rationale and empirical support

In Chapter 1, we discussed the research evidence that supports teachers learning about students' mathematical thinking. We argued that doing so can help teachers develop not only a knowledge base about students' conceptions and problem-solving strategies that they can use in planning instruction but also skills for listening to students and interpreting their thinking.

Professional development that helps teachers analyze students' mathematical work is a logical vehicle to achieve these goals. First, it is consistent with the professional development principle that teachers should engage actively in concrete activities close to their own practice, not just abstract discussions. Second, according to Simon's (1994) Learning Cycles model, analyzing student artifacts creates the context necessary to start a learning cycle focusing on students' thinking. As groups of teachers examine artifacts together, they can engage in active learning, experience cognitive dissonance as different interpretations are proposed and construct new meanings. Third, examining students' work and thinking is precisely what we want teachers to do as part of their everyday teaching practice. Therefore, engaging in these tasks with the guidance of an expert is a valuable way to learn to do the same tasks independently (Collins, Brown, & Newman, 1989).

Research shows that analyzing student thinking can promote instructional practices that result in higher student achievement. Evidence supporting this claim comes from several research studies on outcomes of professional development programs for elementary teachers based on a Cognitive Guided Instruction (CGI) model (e.g., Carpenter & Fennema, 1992; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996), as well as research conducted by the *Integrating Mathematical Assessment* (*IMA*) project involving middle school students (Gearhart, Saxe & Stipek, 1995).

Moreover, comparison studies between teachers who had participated in CGI training and those who had not showed that CGI teachers were using some highly effective practices in their classroom teaching:

> [T]eachers who had been in CGI workshops spent more time having children solve problems, expected multiple solution strategies from their children, and listened to their children more than did control teachers. (Fennema, Carpenter & Franke, 1997, p.194)

Case studies of teachers who participated in CGI programs (Fennema Carpenter, Franke, & Carey, 1992; Fennema, Franke, Carpenter, & Carey, 1993) also show that these teachers gained a better understanding of student thinking and expressed views about the learning and teaching of mathematics consistent with the goals of school mathematics reform. We attribute these results mainly to the teachers' analysis of student thinking, as this was the key professional development activity used in the CGI programs.

Illustration 3: Building a classification of addition/subtraction problems from the analysis of a videotaped problem-solving session.

This illustration depicts a typical 2-hour-long session in a CGI program. We adapted this vignette from the description provided in Fennema, Carpenter, Levi, Franke & Empson (1999). In this session, teachers viewed a videotape of a first-grade child solving four word problems. The goal was for them to identify different types of problems involving addition and subtraction. From this activity, the teachers were able to reconstruct the "Classification of Word Problem Chart" (shown later in Figure 9) that is part of the research model informing the CGI program.

The session opened with teachers discussing three mathematical word problems:

1. *Lucy has 8 fish. She wants to buy 5 more fish. How many fish would Lucy have then?*
2. *TJ has 13 chocolate chip cookies. At lunch she ate 5 of these cookies. How many cookies did TJ have left?*
3. *Janelle has 7 trolls in her collection. How many more does she have to buy to have 11 trolls?*

These problems represent different types of addition and subtraction problems. At first glance, problem 1 seems to involve addition, and problems 2 and 3 seem to require subtraction. However, problems 1 and 3 can also be characterized as having to do with "joining" two sets, while problem 2 is about "separating" an original set into two subsets. Characterizing problems in this way suggests that subtraction may not be the only approach to solving problem 3, for example.

After the participants had a chance to solve the three problems on their own, the facilitator initiated the discussion by asking, "Which of these two problems are most alike and why?" Besides noticing that problems 2 and 3 involved subtraction, a teacher also commented that problem 3 would be harder for his/her students. After a brief discussion of this point, the facilitator introduced the videotape, in which a first-grade child, Rachel, solves the same three problems. (The videotape is available in the CGI professional development support materials available from the authors.)

The facilitator invited the participants to watch how the child solved these problems and to think about how *the child* perceived these problems in terms of similarity, difference and level of difficulty. Rachel's approach surprised the teachers, as Rachel solved the third problem by "joining," while most teachers had solved the same problem by subtraction. In the ensuing discussion, problems involving a joining action were distinguished from ones involving a separating action. To clarify the difference, the facilitator asked teachers to write a problem of each kind and then to share and discuss these problems with the group. In the course of the discussion, participants also agreed that problem 3 must have been more difficult for Rachel because "the child just can't go step by step through the problem and do what it says."

The facilitator then introduced the next segment of videotape, in which Rachel solves yet another addition/subtraction problem:

4. *Max had some money. He spent $9.00 on a video game. Now he has $7.00 left. How much money did Max have to start with?*

The follow-up discussion on the child's solution of this problem led the group to realize that this problem, too, could not be easily solved "step by step." In addition, this problem could be even harder to approach because the child would not know where to start.

Building on these observations, the facilitator pointed out that addition and subtraction problems may vary not only according to the type of action involved in solving them (i.e., "joining" or "separating") but also according to where the unknown appears in the story. After some discussion, the leader suggested that the following variables could be used to organize the four problems:

A. Involve joining
B. Involve separating

and,

i) The unknown is introduced at the end of the word problem
ii) The unknown is introduced in the middle of the problem
iii) The unknown is introduced at the beginning of the problem

The group then used these variables to create the 2x3 matrix reproduced in Figure 9. When the matrix was completed, the leader also introduced the "official names" used in the CGI project to refer to each of these six types of addition and subtraction problems (highlighted in boldface in Figure 9).

Figure 9
CGI classification of word problems chart

1. Lucy has 8 fish. She wants to buy 5 more fish. How many fish would Lucy have then? **(join-result unknown)**	*3. Janelle has 7 trolls in her collection. How many more does she have to buy to have 11 trolls?* **(join-change unknown)**	**join-start unknown**
2. TJ has 13 chocolate chip cookies. At lunch she ate 5 of these cookies. How many cookies did TJ have left? **(separate-result unknown)**	**(separate-change unknown)**	*5. Max had some money. He spent $9.00 on a video game. Now he has $7.00 left. How much money did Max have to start with?* **(separate-start unknown)**

The session concluded with further discussion about each type of problem.

Illustration 4: Supporting teachers in analyzing the results of a test on area

The episode we report in this section occurred in the Making Mathematics Reform a Reality (MMRR) project described in Chapter 2. It was part of the field experiences that took place in the first year of the professional development program. In the MMRR project a mathematics teacher educator was assigned to each school as school facilitator to support participating teachers as they implemented innovative instructional experiences in their classes. The professional development activity described below took place while one of the school facilitators worked with two 7th grade teachers implementing their first inquiry unit, an adaptation of the inquiry on area described in Chapter 1.

The two teachers had designed a comprehensive paper-and-pencil test to assess what their students had learned about area at the end of the unit. This test included items to assess whether students could compute the area of different figures, describe the strategies they used to solve these problems and show understanding of some basic concepts about area. The teachers had already graded these tests, but when the school facilitator asked them to say what they thought their students actually learned about area and what aspects of area might still be a problem, neither teacher felt able to respond.

The facilitator then suggested that each teacher select three or four student papers that presented interesting differences in students responses and re-examine these tests to determine what each student knew or did not know about area. In the after-school meeting scheduled to discuss their findings, both teachers expressed surprise at the challenge this analysis presented, especially since grading the test had been rather straightforward. In several cases, they came to the meeting with just a guess about why a student might have answered a question in a certain way. The discussion that developed as everyone tried to make sense of such puzzling responses was very informative. It often clarified some mathematical points about area, uncovered the student's thinking process and helped teachers further articulate their instructional goals for the unit. Since some student work revealed particular misconceptions, the facilitator also asked both teachers to brainstorm ideas about how to help each student gain a better understanding, either in individual after-school sessions or in future classroom instruction.

Although not planned as part of the professional development program, this experience was an eye-opener for the both the teachers and the school facilitator. Among other things, it engendered a greater appreciation for the importance of analyzing students' work, and it also called into question the grading process that the teachers had so far taken for granted as a viable way to measure student learning.

Main elements and variations

As stated at the beginning of the chapter, analyzing students' thinking involves primarily the in-depth examination and discussion of selected artifacts of students' mathematical activity. Effective implementations of this type of professional development also require the following:

- ***Worthwhile student artifacts for analysis.*** Discussions around the selected artifact will be rich only when the mathematical task(s) assigned to the students admit more than one solution and/or methods of solution, and result in partial or incorrect solutions by some students.
- ***Alternative interpretations to be examined.*** As teachers first analyze the artifacts, they should be requested to generate a variety of hypotheses about possible interpretations. The group can then examine each *hypothesis* for its likelihood of being correct.

Although analyzing students' thinking may at first appear straightforward, our illustrations show that there is not just one way to implement this kind of professional development. Considerable variations can occur depending on the kind of student artifacts available, who provides them, and how teachers analyze them.

For example, teachers can analyze productively the following kinds of student artifacts:

- ***Written work*** students produce in response to homework assignments or assessments.
- ***Videotaped "clinical interviews***," where the interviewer presents a student with a mathematical task and asks probing questions about what the child is doing and why.
- Videotaped excerpts and/or written transcripts of ***actual lessons*** in which students actively discuss a mathematical topic, solve problems in a group or report on the results of individual and/or small-group work.
- "***Cases***" or narratives of classroom experiences created to highlight the mathematical thinking and activities of selected students.

The suitability of each type of artifact depends on the goals of the professional development experience. For example, among the artifacts listed above, written work may reveal the least because it is only a product of student thinking, and even the student's written explanation of his/her solution may not always be enlightening. On the other hand, this kind of artifact presents some unique advantages, as teachers can quickly skim through the work of several different students, noting similarities and differences that can generate interesting questions and speculations. Clinical interviews are more likely to reveal the thinking processes of an individual student working to solve a problem alone. Video excerpts from a mathematics lesson may instead allow teachers to analyze the interaction among several learners working on a mathematical task. Finally, while videos and/or transcripts of a problem-solving session can capture the actual dialogue of students working on mathematical tasks, they do not provide background information on the individual learners or the instructional context to support interpretations of the learning event. Cases, or classroom narratives, on the other hand, usually do offer such information, but they are necessarily based on the writer's interpretation of the event, which may unduly influence the teachers' analysis of the students' thinking and reasoning.

Who provided the artifacts to be examined can also affect the implementation of this type of professional development. The main options in this case are as follows:

- The ***facilitator*** provides the artifacts, or
- The ***teachers*** themselves collect the artifacts from their own students.

Once again, each option has its strengths and weaknesses. Only when the facilitator provides the artifacts can these be carefully selected beforehand to illustrate specific kinds of student strategies or misconceptions. Also, some teachers may feel somewhat uncomfortable and defensive when using their own students' work. On the other hand, teachers may be more interested and motivated in analyzing their own students' work. Moreover, collecting and making sense of their own students' work apprentices teachers immediately to the daily process of analyzing student thinking. Several programs, cognizant of the benefits and limitations of each option, do both. That is, teachers experience a guided analysis of pre-selected artifacts first, and then they collect and analyze student work from their own classroom.

How the artifacts are analyzed also varies, depending on the main goals of the professional development experience. The most interesting variations occur along the following dimensions:

- The extent to which the facilitator structures and focuses the analysis.
- The role the facilitator plays in the analysis and/or discussion of the artifacts.
- The role that research-based knowledge of student thinking about the mathematical topic plays in the analysis. It is worth noting that, while using research is always highly desirable, to date there are only a few mathematical topics for which substantial research on student thinking is available.

- The extent to which instructional implications of the analysis are explicitly addressed.
- The nature and extent of follow-up experiences that could extend what teachers learn from analyzing the artifacts.

Analyzing students' thinking can occur in any of the formats we identified in Chapter 3: summer institutes, university courses, workshops, study groups, one-on-one interactions with a teacher educator, and independent work.

Facilitators for this type of professional development experience are most effective if they understand clearly the mathematics principles underlying the tasks being analyzed and know well the research on students' thinking in the particular mathematical topic.

Teacher learning needs addressed

At first, the activity of analyzing student thinking might seem to relate only to the teacher learning need we have called "understanding student thinking." While this is indeed a main goal of this kind of professional development experience, our two illustrations show that analyzing student mathematical activity can achieve much more than that. In this section, we discuss how this type of professional development experience can contribute to most of the teacher learning needs we identified in Chapter 1:

- ***Developing a vision and commitment to school mathematics reform.*** Although teachers focus on what students do and think in this type of experience, the act of examining students' mathematical activity in innovative learning situations can also contribute to teachers developing a vision and commitment to school mathematics reform. In this case, teachers can develop images of school mathematics reform in action from the instructional context that generated the student samples. The samples themselves can also show evidence of what students can accomplish when offered the kind of learning opportunities promoted by reform. This may then lead teachers to challenge traditional learning goals and practices and to experience a felt need for instructional change. The potential for this type of experience to engender a vision of reform, however, depends on the artifacts chosen and the structure and facilitation of the experience. If participants are to draw larger implications for the teaching and learning of mathematics, facilitators must help them move beyond the specifics of the learning situation they are analyzing and encourage the discussion to develop in that direction.

- ***Strengthening one's knowledge of mathematics.*** As our examples illustrate, analyzing student thinking can lead teachers to a better understanding of mathematical ideas. This is especially true when the facilitator carefully selects and sequences artifacts around a "big mathematical idea" and then focuses part of the conversation on uncovering and examining that idea. Teachers' learning of new mathematics can further be enhanced through presentations or follow-up reading assignments on the mathematical idea examined.

- ***Understanding the pedagogical theories that underlie school mathematics reform.*** Analyzing student thinking can also introduce teachers to the constructivist theories of learning that inform the current recommendations for school mathematics reform. However, in order to truly meet this teacher learning need, the analysis of students' artifacts should be supplemented by readings and/or presentations about the theoretical foundations and empirical research supporting a constructivist perspective. This component is missing in both our illustrations.

- ***Understanding students' mathematical thinking.*** Understanding students' mathematical thinking is obviously at the core of this kind of professional development experience. As both examples illustrate, examining specific examples of students' mathematical activity in depth gives teachers valuable insights about the many different ways in which students at different grade levels approach problems or develop specific concepts or skills. Even more importantly, it can help teachers learn to conduct a similar analysis of their own students' work, to both understand where students might be in their development of key mathematical ideas and to devise learning experiences to best help them progress. This second goal, however, calls for teachers to collect and analyze artifacts from their own classes.

- ***Learning to use effective teaching and assessment strategies.*** While learning new teaching practices is not an explicit goal of this kind of professional development experience, there are two notable exceptions. First, teachers can learn strategies for encouraging students to share their thinking and approaches to solutions. Second, teachers can learn to interpret students' work. We argue that both these strategies are at the core of school mathematics reform.

Supporters of this kind of professional development experience would also argue that these practices are likely to result in better instruction. Knowing how their students' think can empower teachers to make informed instructional decisions and to devise effective assessments. As the vignette on examining the results of a test on area (Illustration 4) shows, even well-designed assessment tools can prove ineffective unless teachers learn to interpret the results and use them to inform instruction.

Finally, we should not forget that teachers, whenever they examine student thinking that takes place in reform mathematics classrooms, are exposed to other teachers' worthwhile teaching practices.

- ***Becoming familiar with exemplary instructional materials and resources.*** Becoming familiar with exemplary instructional materials and resources is not typically a goal of analyzing student thinking. One exception occurs when teachers examine student work in lessons adapted from exemplary instructional materials. In this case, the analysis of the students' work can become an effective vehicle to examine the potential outcomes and goals of the materials.

- ***Understanding equity issues and their implications for the classroom.*** Analyzing student thinking can be powerful for exploring issues of equity in learning mathematics in schools. Teachers have reported being surprised by the reasoning skills that students from disadvantaged backgrounds and students with disabilities reveal when given the opportunity to explain their solutions. These experiences can challenge teachers' biases against students with different learning styles or cultural backgrounds. At the same time, knowing how differently students may approach a task alerts teachers to the influence that race, class, gender and disability may have on students' mathematical performance. We need to keep in mind, however, that to capitalize on this potential, the selected artifacts must represent a wide-range of abilities and socio-cultural backgrounds.

- ***Coping with the emotional aspects of engaging in reform.*** While coping with the emotional aspects of engaging in reform is not an explicit goal of experiences that analyze students' thinking, some teachers may need help dealing with the discomfort and frustration this kind of professional development activity may generate. It is not uncommon for teachers to feel overwhelmed as they realize how powerful, yet time consuming, it is to examine the thinking process of each of their students in-depth. Therefore, facilitators should watch for and be ready to address these feelings. Although there is no easy way to resolve the time constraints teachers must live with, facilitators can discuss realistic expectations for analyzing students' thinking as part of everyday practice and suggest some concrete strategies to make it a possibility.

One of the most desirable outcomes of examining student thinking is that teachers develop the habit of paying careful attention to students' work.

- ***Developing an attitude of inquiry towards one's practice.*** As we mentioned earlier, one of the most desirable outcomes of examining student thinking is that teachers develop the habit of paying careful attention to students' work. Teachers can then determine what students already know and do not know and make better instructional decisions. In other words, developing an attitude of inquiry toward students' work is a central goal of this type of professional development experience, although it may not necessarily invite teachers' inquiry on other aspects of their practice.

Summary

Although analyzing students' thinking might seem at first to be a rather narrowly focused strategy, our analysis reveals that this type of professional development experience is complex and powerful. The analysis of students' thinking can take a number of different forms, depending on what kind of artifacts are examined and who provides them. The implementation of this activity also depends on how the facilitator focuses the process of analysis, the specific tasks that enable the analysis, and the role the facilitator plays in both the design and the implementation of the professional development experience. The choices that the facilitator makes on each of these dimensions determines which different teacher learning needs can be met.

Suggested follow-up resources

If you are interested in learning more about exemplary professional development materials that can help teacher educators plan and facilitate the analysis of student thinking, we recommend the following resources:

Fennema, E., Carpenter, T., Levi, L., Franke, M.L., and Empson, S.B. (1999). *Children's mathematics: Cognitively guided instruction. Professional development materials*. Portsmouth, NH: Heinemann. (videotapes available from the University of Wisconsin at Madison).

The creators of CGI offer a detailed and varied set of materials to support teacher educators in implementing a professional development program based on this approach. These materials provide first of all a description of the research model for studying students' thinking about numbers and operations that informs the program. They also include suggestions for planning a comprehensive professional development program designed to introduce this research model, invite teachers to examine their own students' thinking, and help them make instructional decisions accordingly. Facilitators of such program can also find examples of lesson plans for specific sessions, problems sets and students' work to use with participants, and tips about various implementation issues. Videotapes of students' problem solving are not included in the published materials, but they are available directly from the authors.

Schifter, D., Bastable, V., and Russell, S. J. (1999). *Developing mathematical ideas (DMI)* (casebooks + facilitator's guides + videos) Parsippany, NJ: Dale Seymour.

This set of materials for teacher educators supports the implementation of an entire professional development program for elementary teachers who want to focus on numbers and operations. The sixteen 3-hour sessions that comprise this program have the analysis of students' thinking at their very core – whether the analysis is conducted through a written "case," video images of students engaged in mathematical activities, or student work the participants collect from their own classes. In each session, the Facilitator's Guide provide concrete suggestions about how to analyze the student artifacts and develop productive discussions about them.