CHAPTER 5

Introduction to Fractals

Larry S. Liebovitch

Center for Complex Systems and Brain Sciences, Center for Molecular Biology and Biotechnology, & Departments of Psychology and Biomedical Sciences
Florida Atlantic University
777 Glades Road
Boca Raton, FL 33431
U. S. A.
E-mail: liebovitch@clifford.ccs.fau.edu
http://walt.ccs.fau.edu/~liebovitch/larry.html

Lina A. Shehadeh

Center for Complex Systems and Brain Sciences
Florida Atlantic University
777 Glades Road
Boca Raton, FL 33431
U. S. A.

INTRODUCTION

This chapter is a *Word* version of the *PowerPoint* presentation given by Dr. Larry S. Liebovitch at the NSF Nonlinear Methods in Psychology Workshop, October 24-25, 2003 at George Mason University, Fairfax, VA. The *PowerPoint* presentation itself is also available as a part of this web book. Here the notes which can be seen on the *PowerPoint* presentation by using "Normal View" are presented as text around their respective *PowerPoint* slides. The concept here is to try to reproduce the look and feel of the presentation at the workshop. Therefore, this is not, and is not meant to be, your usual "print" article. The form of the language here is more typical of spoken, rather than written, English. The form of the graphics is sparser, larger pictures captioned with larger fonts, that is more typical of PowerPoint presentations than printed illustrations. We hope that this experimental format may provide a simpler introduction to fractals than that of a more formal presentation. We also hope that the availability of the PowerPoint file will be of use in teaching these materials and may also serve as a starting point for others to customize these slides for their own applications.

This chapter is about "fractals". Objects in space can have fractal properties. Time series of values can have fractal properties. Sets of numbers can have fractal properties. Much of the statistics that you are familiar with deals with the "linear" properties of data. Fractals can help us describe some "non-linear" properties of data.

Most data are characterized by the mean and standard deviation, like 45.3 ± 0.3. You'll learn here that if the data are fractal, those means and standard deviations are meaningless! A pretty basic change in the simplest way we handle data.

Fractals are important because they CHANGE the most basic ways we analyze and understand experimental data.

We'll start with objects. Let's first see the difference between the non-fractal and fractal objects.

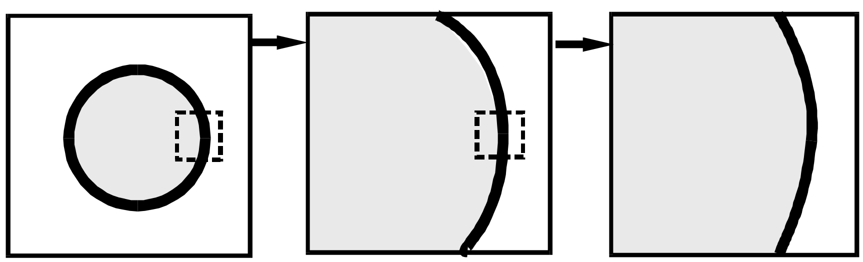
Properties of Objects in Space

Non-Fractal and Fractal Objects are different.

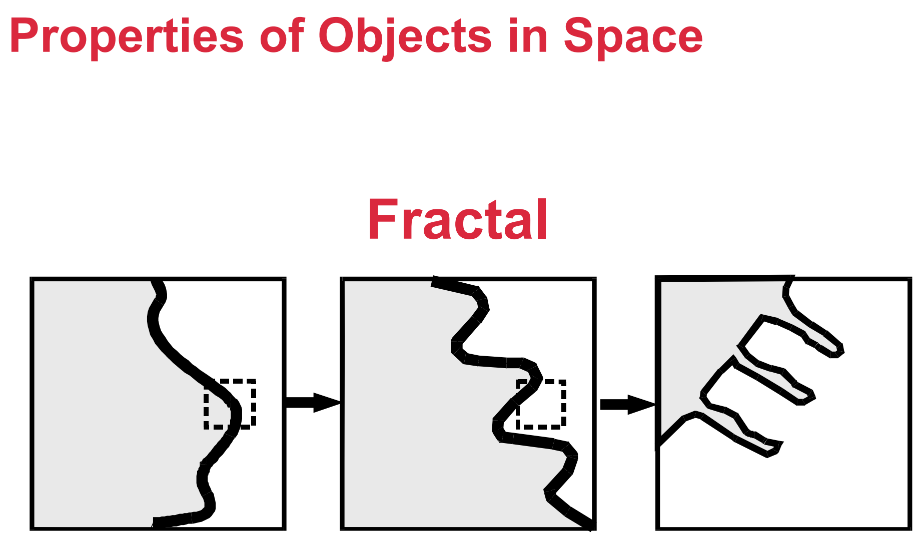
As we enlarge a non-fractal object, no new details appear.

Properties of Objects in Space

Non-Fractal



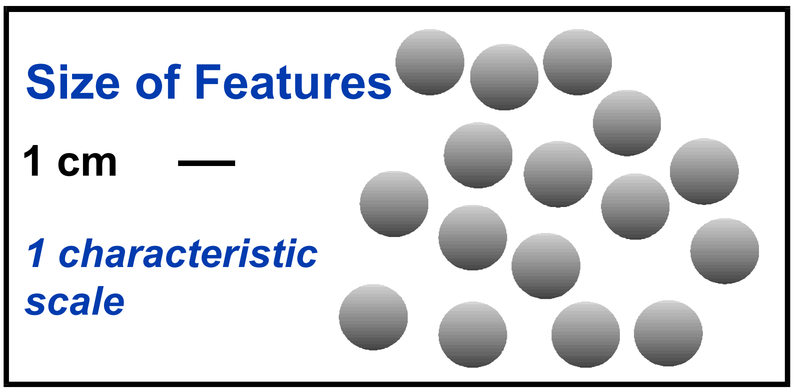
But, as we enlarge a fractal object we keep seeing ever smaller pieces. For example, this series of pictures could show first the inside of the intestine, then the crypts between the cells, then the microvilli on each cell. The smaller pieces are copies of the larger pieces. They are not exact smaller copies, but they are smaller replicas that are kind of like the larger pieces.



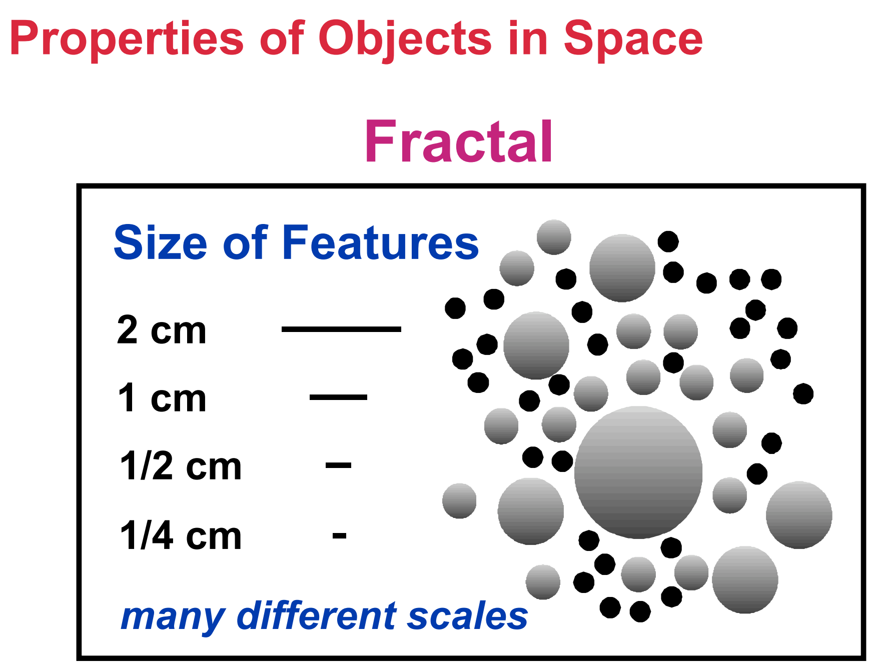
A non-fractal object has most pieces that are about the same size.

Properties of Objects in Space

Non - Fractal



A fractal object has pieces of all different sizes. The variation in the size of the pieces of fractal objects is much larger than the variation in the size of the pieces of non-fractal objects. Typically, there are a few big pieces, some medium-sized pieces, and very many tiny pieces.



Fractal objects have interesting properties. Here we describe those properties very briefly. Then later, we will describe them in more detail.

Properties of Fractal Objects

Self-Similarity.

The little pieces are smaller copies of the larger pieces.

Scaling.

The values measured depend on the resolution used to make the measurement.

Statistics.

The “average” size depends on the resolution used to make the measurement.

A tree is fractal. It has a few large branches, some medium-sized branches, and very many small branches. A tree is self-similar: The little branches are smaller copies of the larger branches. There is a scaling: The length and thickness of each branch depends on which branch we measure. There is no average size of a branch: The greater the number of smaller branches we include, the smaller is the "average" length and thickness.

This tree is from http://www.feebleminds-gifs.com/trees23.jpg.

Example of a Fractal

A tree is fractal

from http://www.feebleminds-gifs.com/trees23.jpg

The pattern of lightning in the sky is fractal. It has a few large branches, some medium-sized branches, and very many small branches. The lightning pattern is self-similar: The little branches are smaller copies of the larger branches. There is a scaling: The length of each branch depends on which branch we measure. There is no average size of a branch: The greater the number of smaller branches we include, the smaller is the "average" length and thickness.

Example of a Fractal

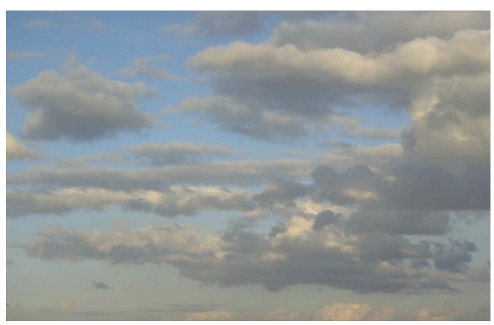
Lightning is fractal

from http://bobqat.com/Mazama/Sky/013.html

The pattern of clouds in the sky is fractal. They are made up of a few big clouds, some medium-sized clouds, and very many small clouds. The cloud pattern is self-similar: The little clouds are smaller copies of the larger clouds. There is a scaling: The size of each cloud depends on which cloud we measure. There is no average size of a cloud: The greater the number of smaller clouds we include, the smaller is the "average" size of a cloud.

Example of a Fractal

Clouds are fractal



From http://www.feebleminds-gifs.com/cloud-13.jpg

The pattern of paint colors in a Jackson Pollack painting is fractal. The pattern is made up of a few big swirls, some medium-sized swirls, and very many small swirls. The pattern is self-similar: The little swirls are smaller copies of the larger swirls. There is a scaling: The size of each swirl depends on which swirl we measure. There is no average size of a swirl: The greater the number of smaller swirls we include, the smaller is the "average" size of a swirl.

Example of a Fractal

A Pollock Painting is Fractal

Autumn Rhythm, 1950, oil on canvas, 266.7 cm by 525.8 cm

From R. P. Taylor. 2002. Order in Pollock's Chaos, Sci. Amer. Dec. 2002

Fractals

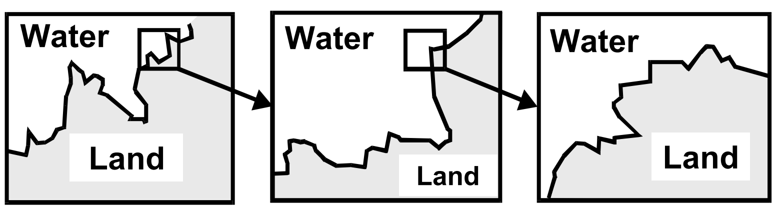
Self-Similarity

Self-similarity: Objects or processes whose small pieces resemble the whole.

The coastline, the fractal border between the land and the sea, has many bays and peninsulas. As you magnify the coastline you see ever smaller bays and peninsulas. The structure at a large scale is similar to the structure at a small scale. It is similar to itself at different scales. This is called self-similarity.

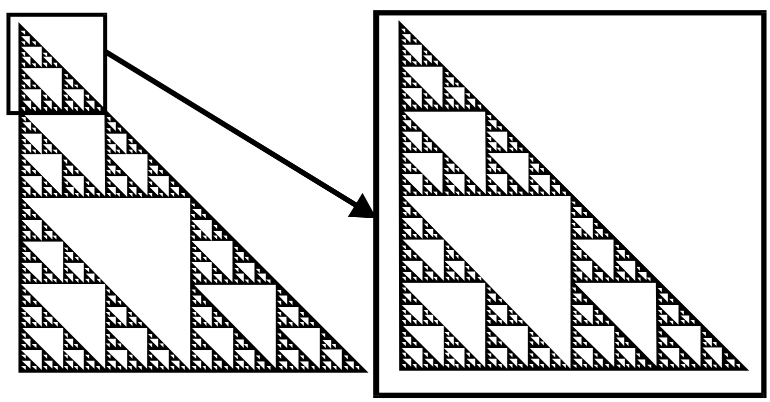
Self-Similarity

Pieces resemble the whole.



This is the Sierpinski Triangle. In this mathematical object each little piece is an exact smaller copy of the whole object.

Sierpinski Triangle

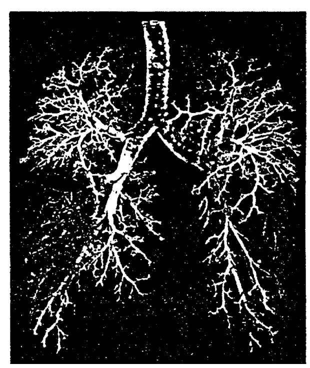


The blood vessels in the retina are self-similar. The branching of the larger vessels is like the branching of the smaller vessels. The airways in the lung are self-similar. The branching of the larger airways is like the branching of the smaller airways. In real biological objects like these, each little piece is not an exact copy of the whole object. It is kind of like the whole object which is known as *statistical self-similarity*.

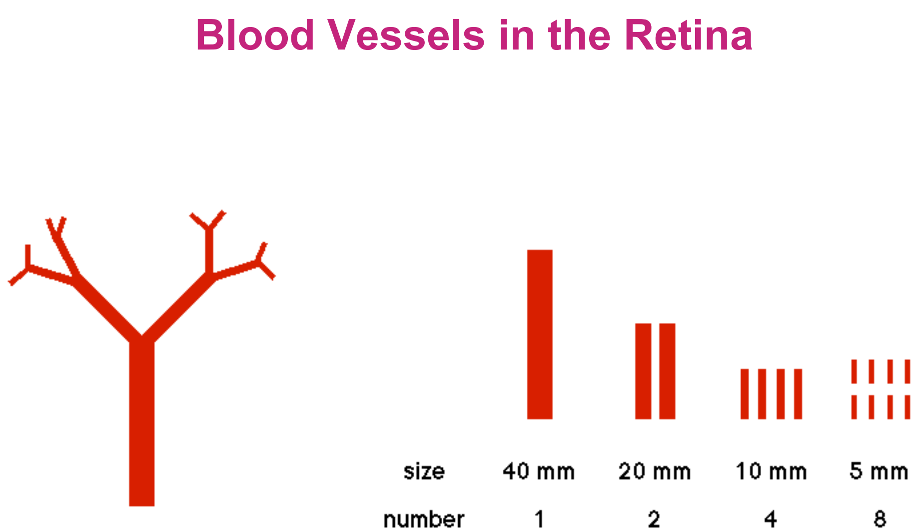
Branching Patterns

blood vessels
in the retina
Family, Masters, and Platt 1989
Physica D38:98-103
Mainster 1990 Eye 4:235-241

air ways
in the lungs
West and Goldberger 1987
Am. Sci. 75:354-365



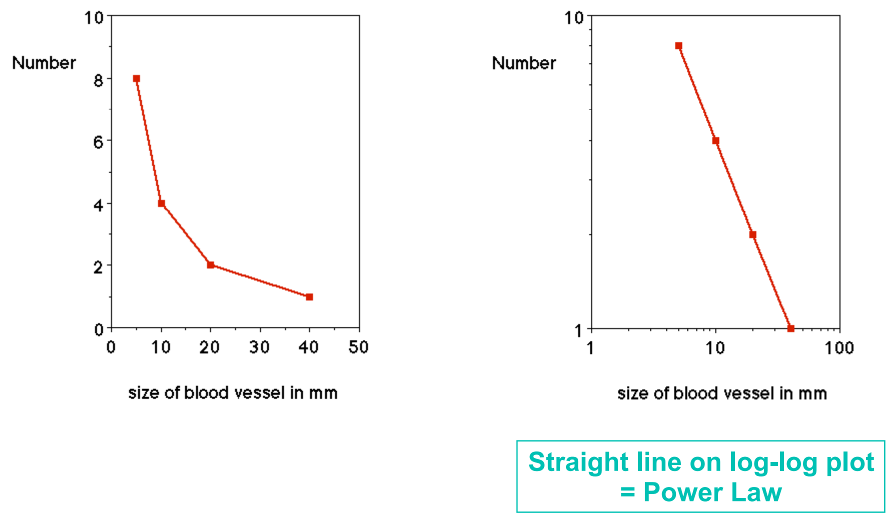
Let's try to understand statistical self-similarity. Here is an unrealistically simplified picture of the blood vessels in the retina. If we ask how many vessels are there of each different size we see that there is one that is 40mm long, two that are 20mm long, four that are 10mm long, and eight that are 5 mm long.



We can plot how many vessels there are of each size. This is called the Probability Density Function (PDF). A power law distribution is evidenced in a straight line on a plot of log (number) vs. log (size).

PDF - Probability Density Function

HOW OFTEN there is THIS SIZE



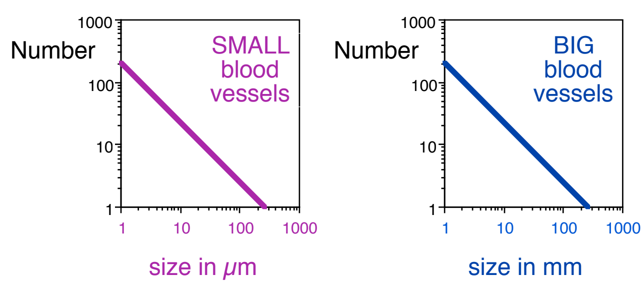
The PDF of the large vessels is a straight line on a plot of log (number) vs. log(size). There are a few big-big vessels, many medium-big vessels, and a huge number of small-big vessels.

The PDF of the small vessels is also a straight line on a plot of log(Number) vs. Log(size). There are a few big-small vessels, many medium-small vessels, and a huge number of small-small vessels.

The PDF of the big vessels has the same shape (i.e., is similar to) the PDF of the small vessels. The PDF is a measure of the statistics of the vessels. So, the PDF (the statistics) of the large vessels is similar to the PDF (the statistics) of the small vessels. This is statistical self-similarity. The small pieces are not exact copies of the large pieces, but the statistics of the small pieces are similar to the statistics of the large pieces.

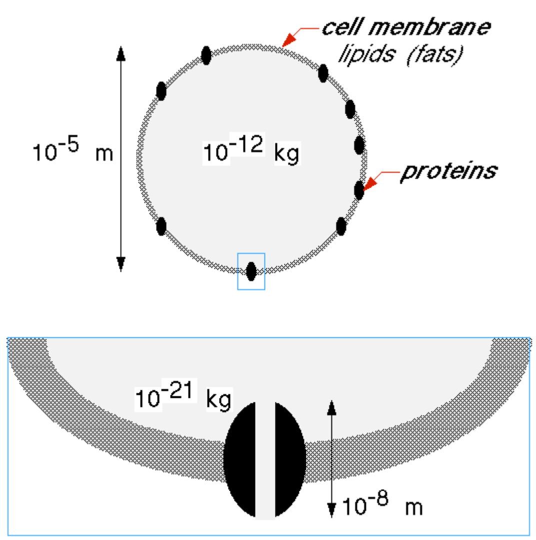
Statistical Self-Similarity

The statistics of the big pieces is the same as the statistics of the small pieces.

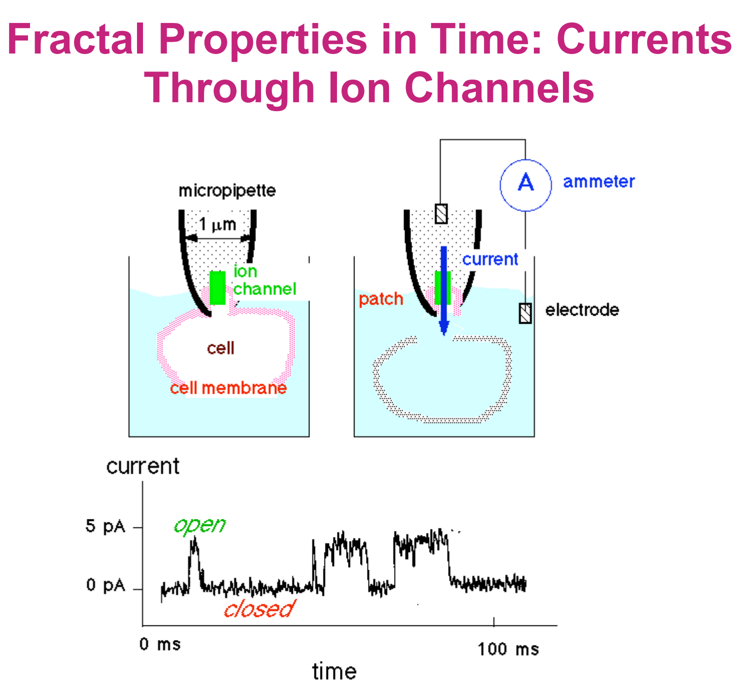


Fractals are not only objects in space, but can also be processes in time. There are proteins, called "ion channels," in the fatty membranes of living cells that let ions, like sodium and potassium, enter or exit the cell.

Fractal Properties in Time: Currents Through Ion Channels



A small pipette can suck up a small piece of cell membrane with only one ion channel in it, and even it tear it off and away from the cell. The movement of sodium or potassium through the ion channel produces an electrical current that can be measured. It's a pretty small current, a picoAmp, which is about one billionth (1/1,000,000,000) of the current from a "D" battery. This is called the "Patch Clamp." What's really interesting is that these ion channel proteins act like little electrical switches. Either they are either fully open or fully closed to the movement of sodium or potassium. They switch, all the time, between these fully open and fully closed states. It's impressive to watch this technology measure the changes in a single molecule at a time.

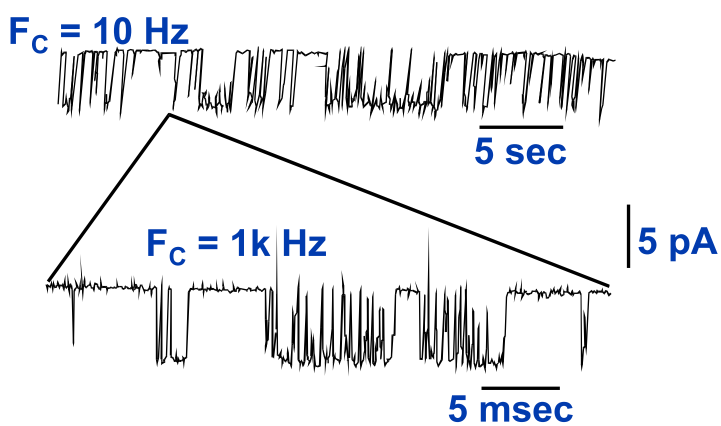


These open and closed times are fractal! If you record them and play them back slowly you see a sequences of open and closed times. But if you take one segment of time, and play it back at higher resolution, you see that it actually consists of many briefer open and closed times. It is self-similar in time.

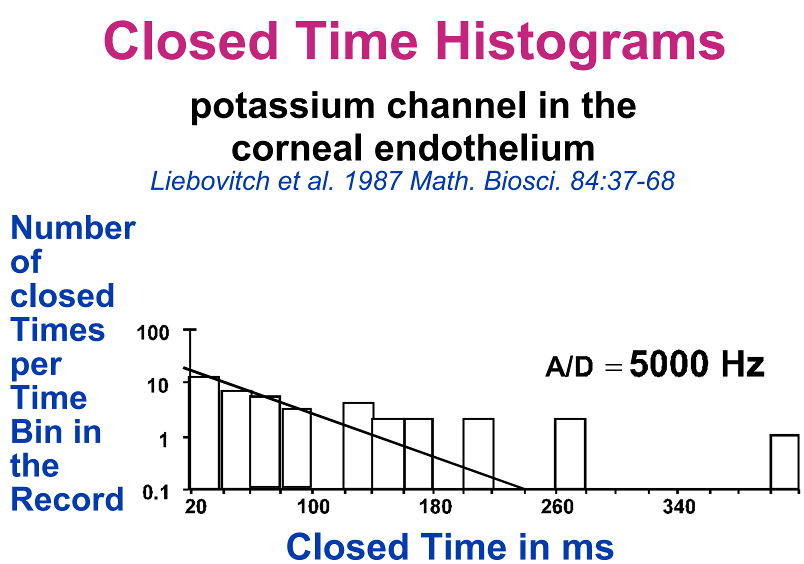
Currents Through Ion Channels

ATP sensitive potassium channel in β cell from the pancreas

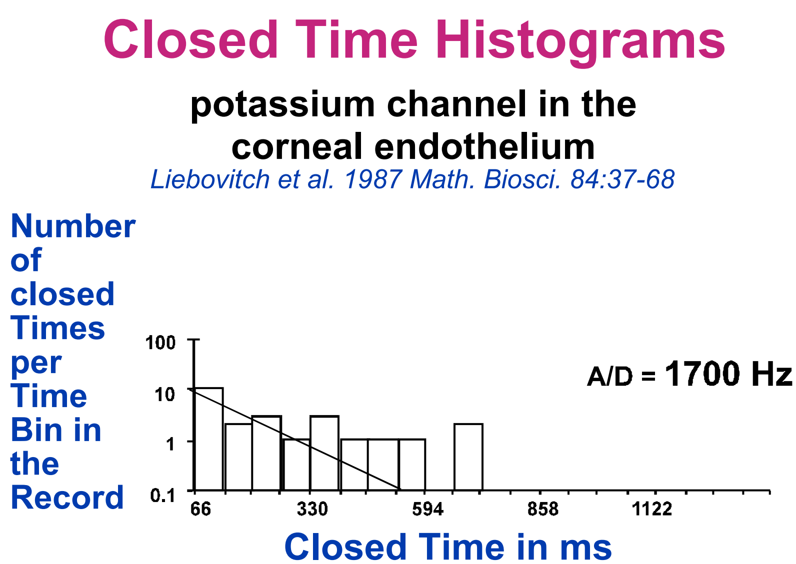
Gilles, Falke, and Misler (Liebovitch 1990 Ann. N.Y. Acad. Sci. 591:375-391)



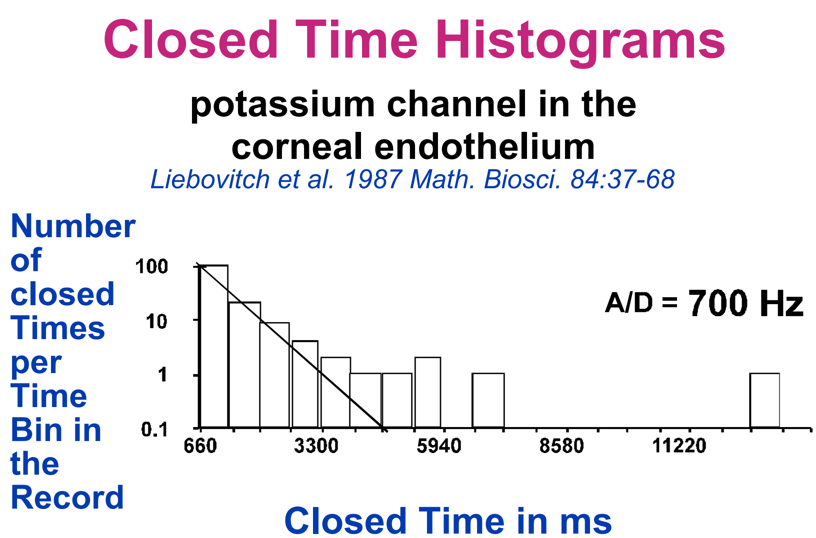
Here is a histogram of the times (in ms) that one channel was closed. The recording was made at the fastest time resolution, allowing the briefest closed times to be recorded. The PDF is mostly a straight line on this log (number) versus time (t) plot, but with an occasional longer closed time. Data with fractal properties often have unusual events that occur more often than expected from the usual "Bell Curve." Those occasional longer closed times are a hint that these data might be fractal.



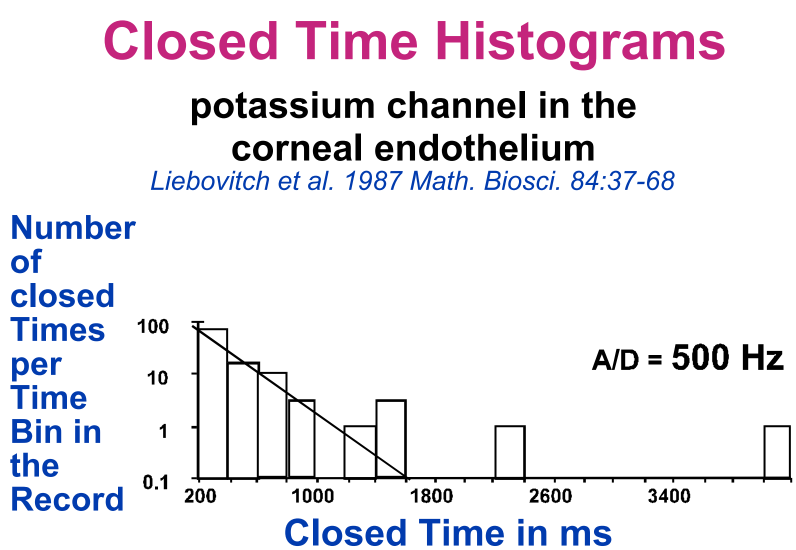
Here is another histogram of the closed times (in ms) of that same ion channel. This recording was made at a little slower time resolution and so longer closed times were recorded. The PDF is mostly a straight line on this log (number) versus time (t) plot, but with an occasional longer closed time.



Here is another histogram of the closed times (in ms) of that same ion channel. This recording was made at an even slower time resolution and so even longer closed times were recorded. The PDF is mostly a straight line on this log (number) versus time (t) plot, but with an occasional longer closed time.



Here is another histogram of the closed times (in ms) of that same ion channel. This recording was made at a much lower time resolution and so only the longest closed times were recorded. The PDF is mostly a straight line on this log (number) versus time (t) plot, but with an occasional longer closed time. The PDF looks similar at different time resolutions. The PDF is a measure of the statistics. So, the statistics is similar to itself at different time resolutions. This is statistical self-similarity in time.

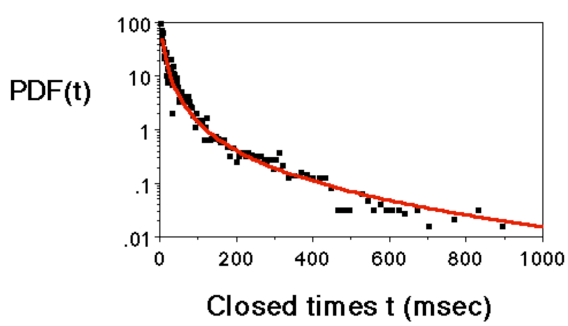


Each of those histograms of the closed times is measured at its own time resolution, the time width of each bin. Wouldn't it be nice to see all those different time scales at once? We can't do that with a histogram, but we can covert each histogram into its PDF and then combine those PDFs. Here is the PDF of all those histograms combined. Now we can see that there is a simple relationship (red line) between all the different closed times. Thus, there is a relationship between the closed times as short as a millisecond and those as long as a second. This relationship is called a scaling relationship.

Closed Time PDF

potassium channel in the corneal endothelium

Liebovitch et al. 1987 Math. Biosci. 84:37-68



Fractals

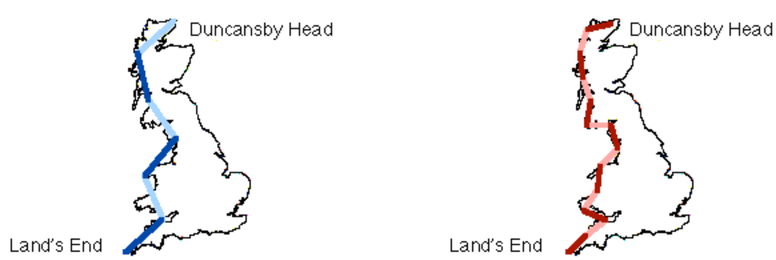
Scaling

Scaling: The value measured depends upon the resolution used to make the measurement.

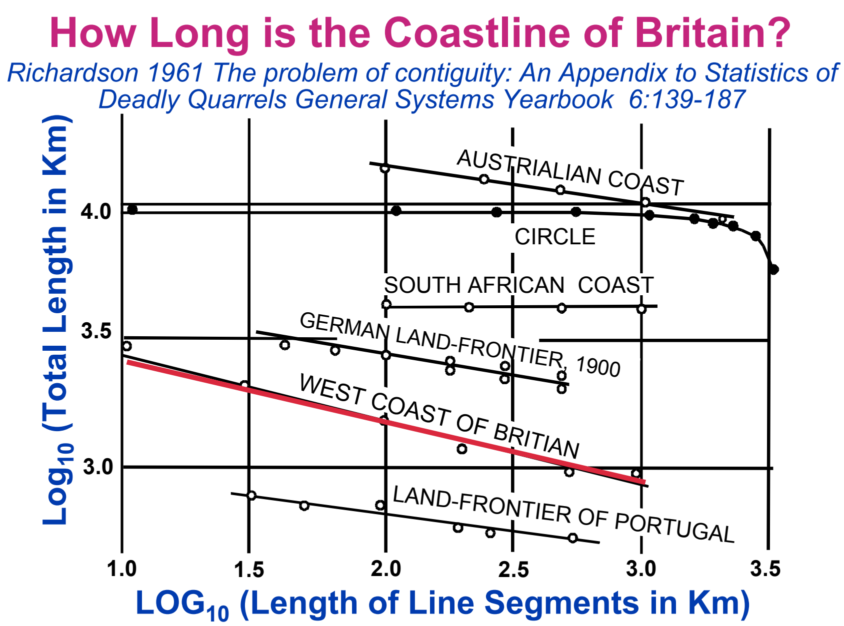
If we measure the length of the west coast of Britain with a large ruler, we get a certain value for the length of the coastline. If we now measure it again with a smaller ruler, we catch more of the smaller bays and peninsulas that we missed before, and so the coastline measurement is longer. The value we measure for the coastline depends on the size of the ruler that we use to measure it.

Scaling

The value measured depends on the resolution used to do the measurement.



Here is a plot of how the length of the west coast of Britain depends upon the resolution that we use to measure it. There is no one value that best describes the length of the west coast of Britain. It depends upon the scale (resolution) at which we measure it. As we measure it at a finer scale, we include the segments of the smaller bays and peninsulas, and the coastline is longer. This is one of the surprising way in which fractals change the most basic way that we analyze and understand our data. There is no one number that best describes the length of the west coast of Britain. Instead, what is important is how the length depends upon the resolution that we use to measure it. The more smaller bays and peninsulas, the more the length of the coast increases when it is measured at a finer resolution, and the steeper the slope on this plot. This plot therefore shows that the coast of Britain is rougher than that of Australia, which is rougher than that of South Africa, which is rougher than that of a plain circle.



Iannaccone and his colleagues study how organisms develop in order to understand and cure cancer in kids. They mix cells from another animal into an embryo so that the fate of these marker cells can be traced out as the animal develops. The cells that are added have a different enzyme, which attaches to a radioactive marker that blackens a photographic film to make a picture. On the following page are some of those pictures of the liver. Look, the added cells are not in one clump. They are in islands of all different sizes.

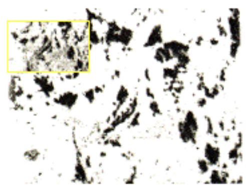
There is no one area that best describes the size of these islands. The area measured depends on the resolution used. This scaling relationship is a straight line on a plot of log (area) versus log (resolution).

There is no one perimeter that best describes the size of these islands. The perimeter measured depends on the resolution used. This scaling relationship is also a straight line on a plot of log (perimeter) versus log (resolution).

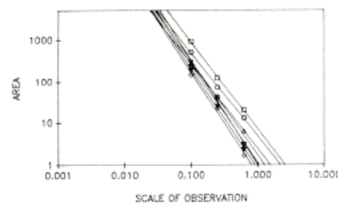
This is one of the surprising way in which fractals change the most basic way that we analyze and understand our data. There is no one number that best describes the area or perimeter of these islands. Instead, what is important is how the area or perimeter depends upon the resolution that we use to measure it.

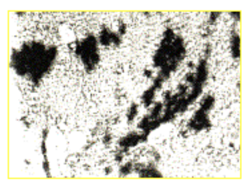
Genetic Mosaics in the Liver

P. M. Iannaccone. 1990. FASEB J. 4:1508-1512.
Y.-K. Ng and P. M. Iannaccone. 1992. Devel. Biol. 151:419-430.

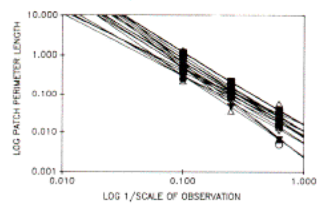


area vs. resolution





perimeter vs. resolution



So far, we've seen fractal scaling in space. There are also fractal scaling in time. The usual way to measure the switching of an ion channel is the "kinetic rate constant." That tells us the probability that the ion channel switches between open and closed states. But the ion channel must be closed (or open) long enough for us to see it as closed (or open). A more appropriate measure is the probability that the ion channel switches between open and closed states, given that it has already remained in a state for a certain amount of time. That certain amount of time defines the time resolution at which we measure the switching probability. We called that probability the "effective kinetic rate constant" (k_{eff}),

$$k_{eff} = \Pr\ (T{=}t,\ t{+}\Delta t \mid T > t_{eff})\ /\ \Delta t, \qquad [5.1]$$

which is the probability (Pr) for the ion channel to open (or close) during the time interval T = (t, t+Δt), given that it has already remained closed (or open) for a time $T \geq t_{eff}$. In the branch of statistics called renewal theory, k_{eff} is called the "age specific failure rate," for example, the probability that a light bulb fails in the next second given it has already burned for t_{eff} hours. In the branch of statistics used in epidemiology and insurance, k_{eff} is called the "survival rate," for example, the probability that a patient dies of cancer this year, if they have already had cancer for t_{eff} years.

Fractal Kinetics

Liebovitch et al. 1987 Math. Biosci. 84:37-68.

Kinetic Rate Constant:

k = Prob. to change states in the next dt.

Effective Kinetic Rate Constant:

k_{eff} = Prob. to change states in the next dt, given that we have already remained in the state for a time k_{eff}.

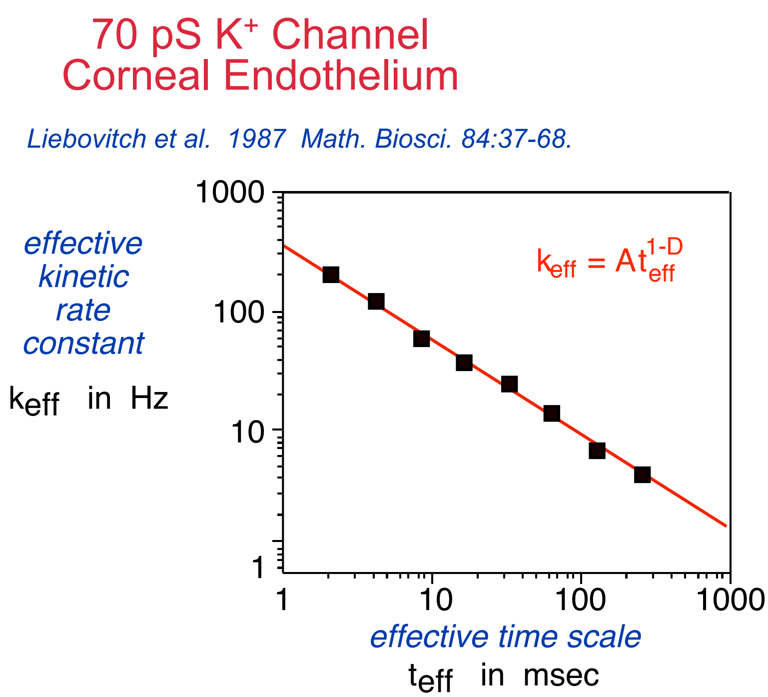
age-specific failure rate

$$\mathbf{k}_{eff} = \Pr(T=t, t+dt \mid T > t_{eff}) / dt$$

$$= -\frac{d \ln P(t)}{dt}$$

P(t) = cumulative dwell time distribution

We measured the open and closed times for an ion channel in the cells in the cornea, the clear part in the front of the eye that you look through to see these words. The effective kinetic rate constant is a straight line on a plot of log (effective kinetic rate constant) versus log (effective time used to measure it). This is a fractal scaling relationship in time. The faster we could look, the briefer open and closed times we would see.



Fractals have given us a new way to analyze data from the patch clamp measurements of the open and closed times of ion channels. Instead of measuring a property (the kinetic rate constant) at one time scale, we measure how a property (the effective kinetic rate constant) changes when we measure it at different time scales. We have been using the information in this fractal scaling relationship to give us clues about the structure and motions in ion channel protein. Specifically, we have been using the scaling relationship to calculate the energy difference between the open and closed states of the ion channel protein and how that energy difference varies in time. The picture of ion channels before fractals analysis was that they are firm, sharp, uptight things that go click, click, click, between a few, very different static states. The picture of ion channels after fractal analysis is that they are complex dynamic things, with many pieces of different size that move over different time scales, whose new shapes and movements determine what it's going to do next.

Fractal Approach

Liebovitch 1989 Math. Biosci. 93:97-115.
Liebovitch and Tóth 1991 Bull. Math. Biol. 53:443-455.
Liebovitch et al. 2001 Methods 24:359-375.

New viewpoint:

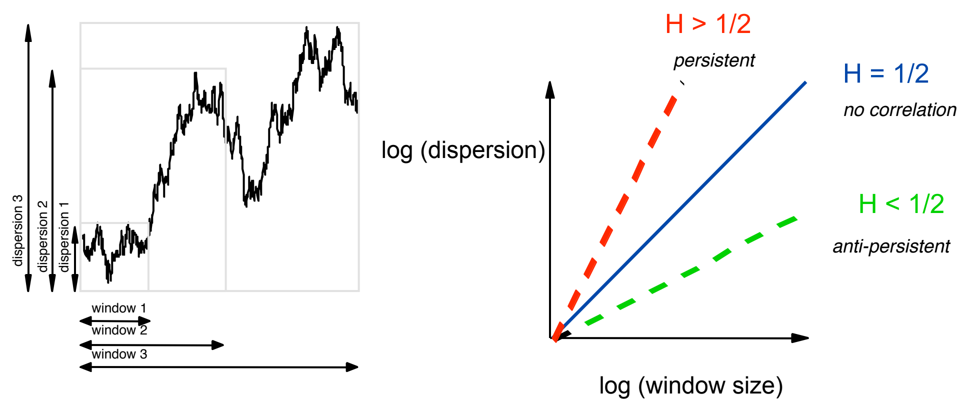
Analyze how a property, the effective kinetic rate constant, k_{eff}, depends on the effective time scale, t_{eff}, at which it is measured.

This Scaling Relationship:

We are using this to learn about the structure and motions in the ion channel protein.

We have seen examples of scaling relationships for measurements in space and time. There can also be scaling relationships for the correlations between measurements. Like the scaling relationships for measurements, the scaling relationship for the correlations between the measurements is often a power law, that is, a straight line on a logarithmic-logarithmic plot. For example, at the left in the figure on the following page is a measurement in time. It is self-similar—there are ever larger fluctuations over ever longer times. We can measure the dispersion, the variation in the value, over different windows of time. The dispersion is ever larger over ever longer time windows. The slope of this scaling relationship on a plot of log (dispersion) versus log(window size) is called the Hurst Exponent, H. When $H = 0.5$, the measurements are not correlated. When $H > 0.5$, the measurements are positively correlated. This is called persistence. An increase now is more likely followed by an increase at all time scales later. When $H < 0.5$, the measurements are negatively correlated. This is called anti-persistence. An increase now is more likely followed by a decrease at all time scales later. There are many different ways to find the correlational scaling relationship. One method is the Hurst Rescaled Range Analysis. Another method is Detrended Fluctuation Analysis.

Correlations



Measures of disperson:

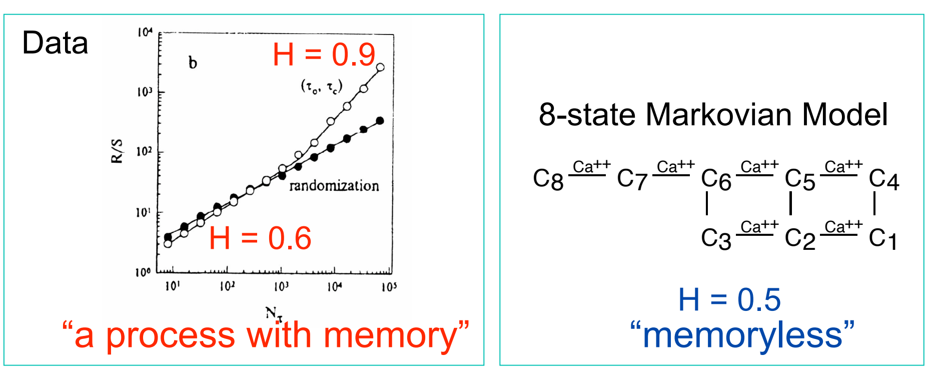
Hurst Rescaled Range Analysis: R/S

Detrended Fluctuation Analysis: DFA

On the left, the Hurst rescaled range analysis was used to measure the correlations in the open and closed times of an ion channel protein (open circles). At short times, H = 0.6, and at along times H = 0.9. These are very persistent correlations. The correlations disappear (black circles) when the order of the open and closed times was randomly shuffled. This means that there is a long term "memory," which gets stronger with time, in how the shape of the ion channel protein changes in time. Previous models of ion channels, as shown on the right, assumed that the channel switched between a few, discrete shapes, without any memory. This fractal analysis tells us that ion channels do not behave that way. Instead, the fractal analysis has enabled us to see that there are important, continuous dynamical processes, with memory, going on inside the ion channel protein.

Fractal Kinetics

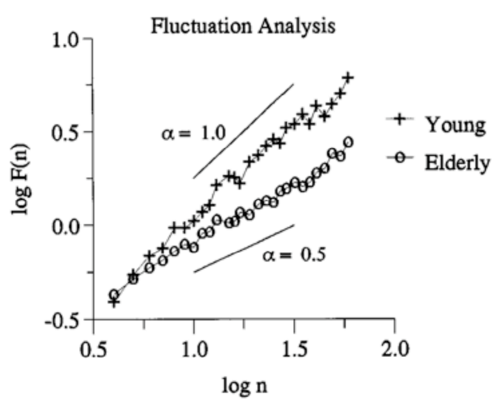
Kochetkov, et al. 1999. J. Biol. Phys. 25:211-222.



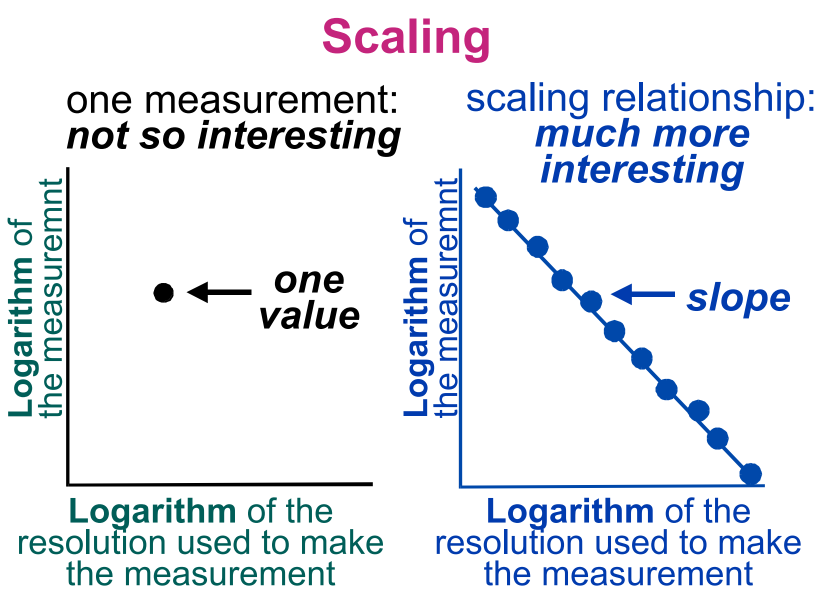
Here, the detrended fluctuation analysis was used to measure the correlations in the time between footsteps. This scaling relationship is also a power law, a straight line on a logarithmic-logarithmic plot. The scaling exponent of that power law is different for the young and the elderly person. These studies have given us insight into how the brain controls coordination and walking, and how that control depends on age and is changed by disease.

Fractal Walking

Hausdorff et al. 1997. J. Appl. Physiol. 82:262-269.



This is the take-home lesson: We are used to thinking that there is one measurement that best describes a property of an object. For a fractal object that extends over many scales, in space or time, a property depends on the scale at which it is measured. There is no one measurement that best describes the object. The object is best described by how the property measured depends upon the resolution at which it is measured. This relationship is characterized by a parameter called the fractal dimension. The fractal dimension can be calculated from the slope of this logarithmic-logarithmic graph.



Fractals

Statistics

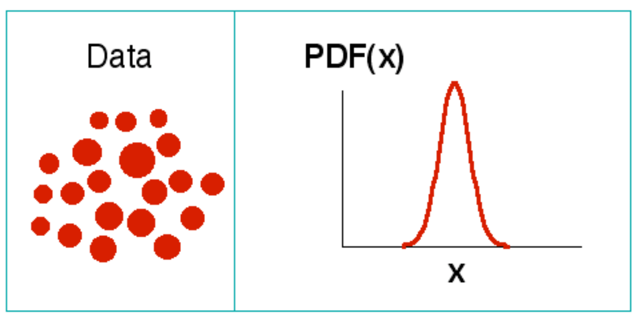
Fractals have some unique statistical properties. The “average” size depends on the resolution used to make the measurement. What is important is not the average, but how the average depends on the resolution used to make the measurement.

Here is a set of numbers; maybe they are the values measured from an experiment. I have drawn a circle to represent each number. The diameter of the circle is proportional to the magnitude of the number. Here is a non-fractal set of numbers. Most of them are about the size of an average number. A few are a bit smaller than the average. A few are bit larger than the average.

Not Fractal

Here is the PDF of theoe non-fractal numbers. The PDF is how many numbers there are of each size. The PDF here is called a "Bell Curve," a "Gaussian Distribution," or a "Normal Distribution." It's strange that someone chose to call this a "normal" distribution. We are about to see that much of the world is definitely not like this kind of "normal."

Not Fractal



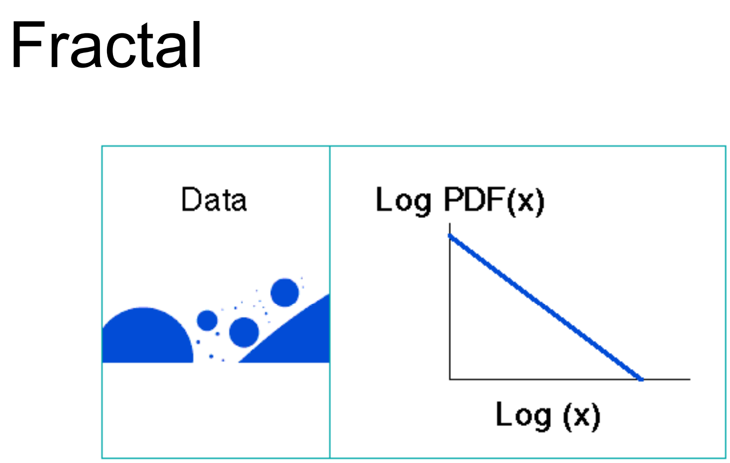
Here is a picture of Gauss on the old 10 Deustche Mark German bill. He has now been replaced by the 5 Euro. You can see his curve and even the equation for it on this bill! There are no equations on American money. (There is a scientist on American money. Do you know who it is?)

Gaussian
Bell Curve
“Normal Distribution”

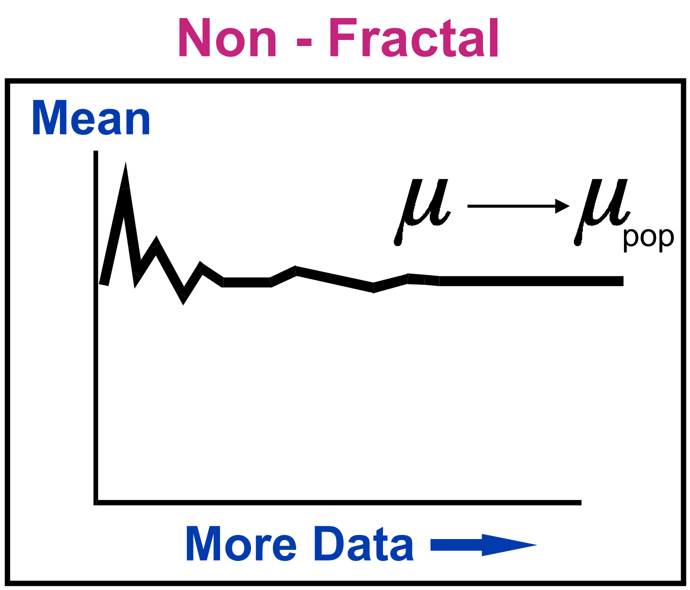
Here is a set of numbers from a fractal distribution. The diameter of each circle is proportional to the size of the number. These numbers could be from the room around you. Look around your room. There are few big things (people and chairs), many medium-sized things (pens and coins), and a huge number of tiny things (dust and bacteria). It is not at all like that "Normal" distribution. Sets of data from many things in the real world are just like this. We call this a fractal distribution of numbers because it has the same statistical properties as the sizes of the pieces in fractal objects.

Fractal

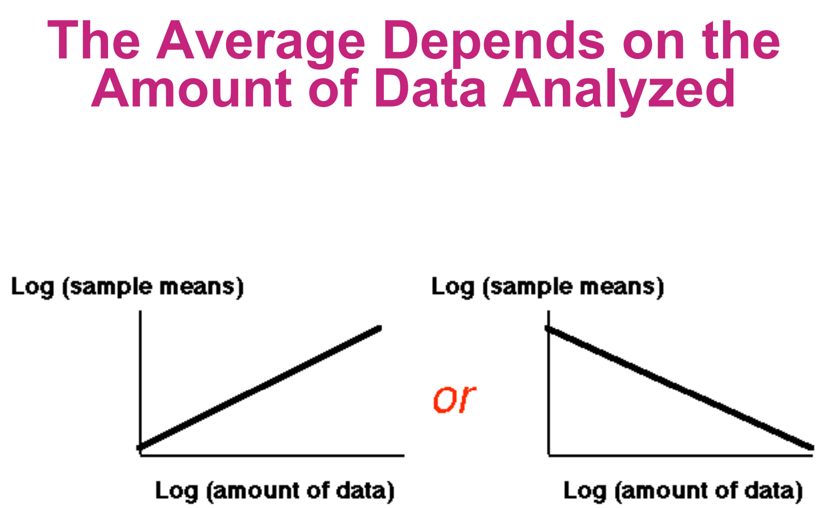
Here is the PDF of these fractal numbers. The PDF is how many numbers there are of each size. There are a few big numbers, many medium sized numbers, and a huge amount of tiny numbers. The PDF is a straight line on a plot of log(How Many Numbers; the PDF) versus log(value of the numbers).



The statistics of a fractal set of numbers is very different from the statistics of "normal" numbers that they taught you about in Statistics 101. The statistics you learned in Statistics 101 is only about non-fractal numbers. Take the average of a sample of non-fractal numbers. This is called the Sample Mean. As you include ever more data, the sample means, shown here as μ, get ever closer to one value. We call that value the Population Mean, shown here as μ_{pop}. We think that the population mean is the "real" value of the mean.



The statistics of fractal numbers is very different. Take the average of a sample of fractal numbers. This is called the Sample Mean. As you include ever more data, the sample means do NOT get ever closer to one value. Either the sample means keep increasing OR the sample means keep decreasing as you include more data. THERE IS NO Population Mean. There is NO one value that best describes the data. The data extend over a range of many different values.



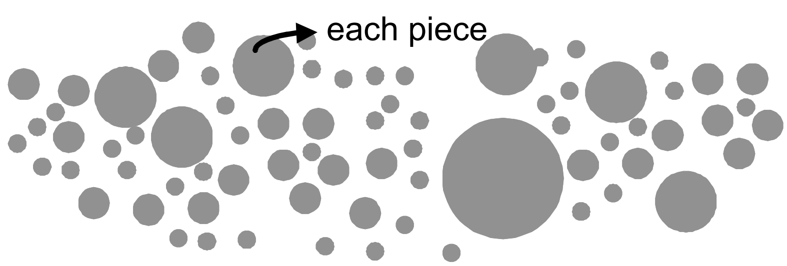
Here is why that happens. Again, here is a set of fractal numbers. The diameter of the circles are proportional to the size of the numbers. As you include ever more numbers one of two things will happen:

1. If there is an excess of many small values, the sample means get smaller and smaller.

2. If there is an excess of a few big values, the sample means get larger and larger

Whether 1 or 2 happens depends on the ratio of the amount of small numbers to the amount of big numbers. That ratio is characterized by a parameter called the Fractal Dimension.

The Average Depends on the Amount of Data Analyzed



Let's play a non-fractal game of chance. Toss a coin, if it comes up tails we win nothing, if it comes up heads we win \$1. The average winnings are the probability of each outcome times how much we win on that outcome. The average winnings are (1/2) x (\$0) + (1/2) x (\$1) = 50¢. Let's go to a fair casino to play this game. Fair casinos exist only in math textbooks; "fair" means the bank is willing only to break even and not make a profit. We and the casino think it's fair for us to be charged 50¢ to play one game. That seems reasonable; half the time we win nothing, half the time we win \$1, so if it costs 50¢ to play each time, on average, we and the casino will break even.

Ordinary Coin Toss

Toss a coin. If it is tails win \$0, If it is heads win \$1.

The average winnings are:

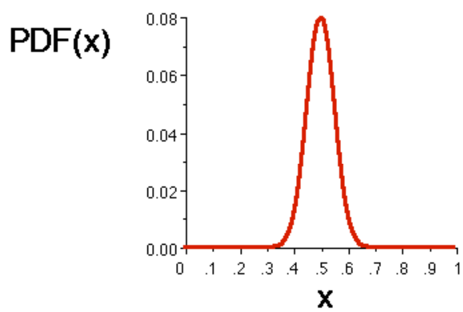
$$2^{-1}.1 = 0.5$$

$$\boxed{\mu \longrightarrow 1/2}$$

Non-Fractal

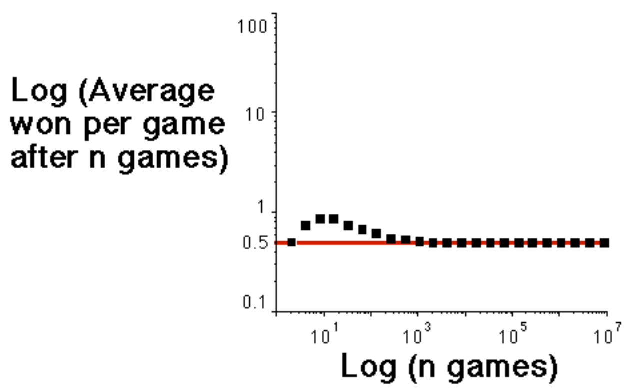
Here is the PDF of that non-fractal game of chance. It shows how often (the PDF on the vertical axis) you will win how much money (the x value on the horizontal axis) if you play 100 times. It's a Bell Curve—a Gaussian, Normal distribution—just the kind of distribution they taught you about in Statistics 101.

Ordinary Coin Toss



Here's what happens when I played that non-fractal game, over and over again. A computer (actually a Macintosh Plus running Microsoft BASIC!) picked a random number to simulate flipping the coin. Here, the average winnings per game is shown after n games. For a while I (the Mac) was lucky. I was winning more than an average 50¢ in each game. But, as you might suspect (this is called the Law of Large Numbers), after a while my luck ran out. In the long run, I was winning exactly an average of 50¢ in each game.

Ordinary Coin Toss



Now, let's play a fractal game of chance. This game was invented by Niklaus Bernoulli who lived in St. Petersburg, Russia, and was published by his uncle Daniel Bernoulli who lived in Germany, about 350 years ago. Here, we toss a coin UNTIL it comes up heads. If it comes up heads on the first toss, we win \$2. If it comes up tails first, and then heads on the second toss, we win \$4. If it comes up tails twice, and then heads on the third toss, we win \$8. And so on.

The average winnings are the probability of each outcome times how much we win on that outcome. The average winnings are $(1/2) \times (\$2) + (1/4) \times (\$4) + (1/8) \times (\$8) + (1/16) \times (\$16) + \ldots. = 1 + 1 + 1 + 1 \ldots = \infty$. We say to the casino, "Half the time we'll win \$2; the median winnings of this game is \$2" because half the time the coin comes up heads on the first toss and we win \$2. "So, we think it is very fair to put up twice the median winnings, \$4, to play each game". To our surprise, the now angry casino owner says, "No!" He adds, "The average winnings of this game are infinite, you must put up more than all the money in the universe to play this game, even once!"

This game became known as the St. Petersburg Paradox, because we and the casino cannot agree on the fee to play this game. It was called a "paradox" because it was so surprising, and difficult for many people to believe, that the player and the casino owner could disagree on what is fair for such a simple game. This game is well known amongst mathematicians, which means it's well known amongst the people who know it well. But, because of its unusually mathematical character, it was not one of the threads of probability theory that was woven into the fabric of statistics that became popular in the natural sciences and that is taught in Statistics 101. Now, with the popularity of fractals, it is being rediscovered and its importance in analyzing and understanding real data increasingly appreciated.

St. Petersburg Game (Niklaus Bernoulli)

Toss a coin. If it is heads win \$2, if not, keep tossing it until it falls heads.

If this occurs on the N-th toss we win \$$2^N$.

With probability 2^{-N} we win \$$2^N$.

H	\$2
TH	\$4
TTH	\$8
TTTH	\$16

The average winnings are:

$$2^{-1}2^{1} + 2^{-2}2^{2} + 2^{-3}2^{3} + \ldots =$$

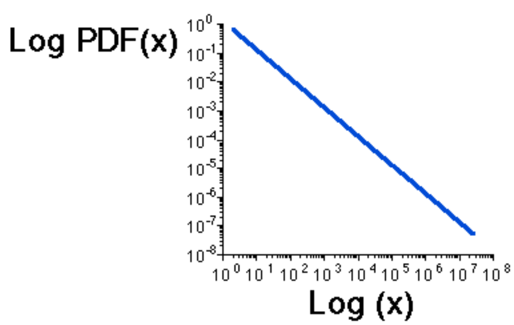
$$1 + 1 + 1 + \ldots = \infty$$

$\mu \downarrow \infty$

Fractal

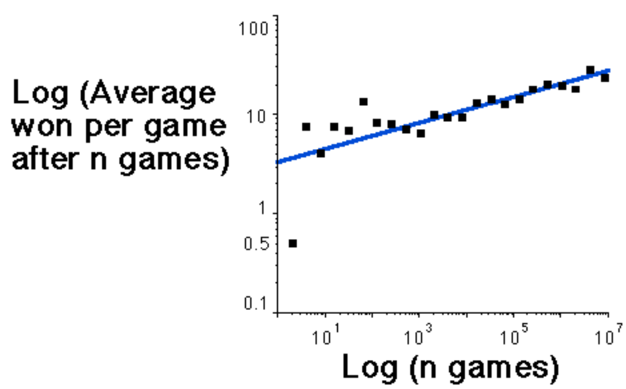
Here is the PDF of that fractal game of chance. It shows how often (the PDF on the vertical axis) you will win how much money (the x value on the horizontal axis). It's NOT a Bell Curve, Gaussian, or Normal distribution. Most often you win only a small amount, more often you win a bigger amount, very rarely you win a huge amount. It is just like the fractal PDF of the blood vessels in the retina, or any fractal object! It is a straight line on a plot of log (How Often) versus log(How Much).

St. Petersburg Game (Niklaus Bernoulli)



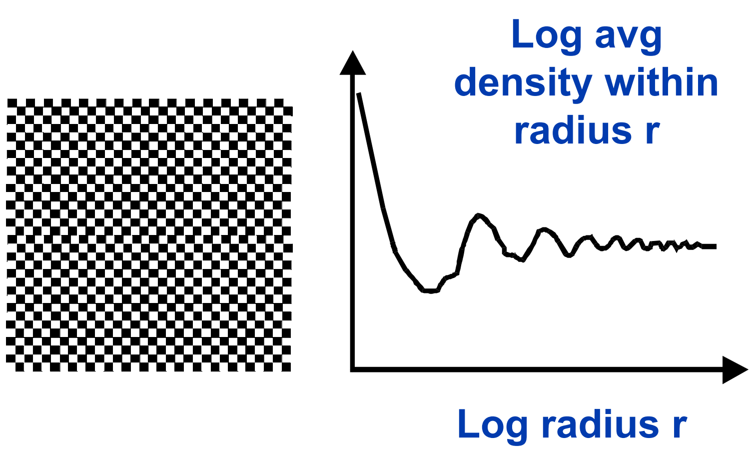
Here's what happens when I played that fractal game over and over again. Here, the average winnings per game is shown after n games. The more I played, the more often there was sometimes a lot of tails before that first head. When there are a lot of those tails, I won a huge jackpot. As more and more of those jackpots happened, the average winnings per game kept increasing. There is no average (population mean) for this game. The more I played, the more the average kept changing. They told you in Statistics 101 that the more data you have, the closer the sample means are to the population mean. Not here! There is no population mean. The more data we have (the more games I played) the more the sample means keep changing. The few exponentially large wins keep pushing the sample mean up, which is very different than what you learned in Statistics 101. Welcome to fractals.

St. Petersburg Game (Niklaus Bernoulli)



Here is a non-fractal object. It is a checkerboard. Actually, I'm only showing you a piece of it; it should really extend forever in each direction. Place a circle on it. Count all the black pixels in that circle, and divide by the total number of pixels. That is the average density within that circle. The graph shows how that density changes as the circles get bigger and bigger. The average density fluctuates a bit; after all, we are putting a round circle over a square grid. But, as the circles get bigger and bigger, the average density gets closer and closer to 1/2. This seems reasonable because the checkerboard is 1/2 black and 1/2 white.

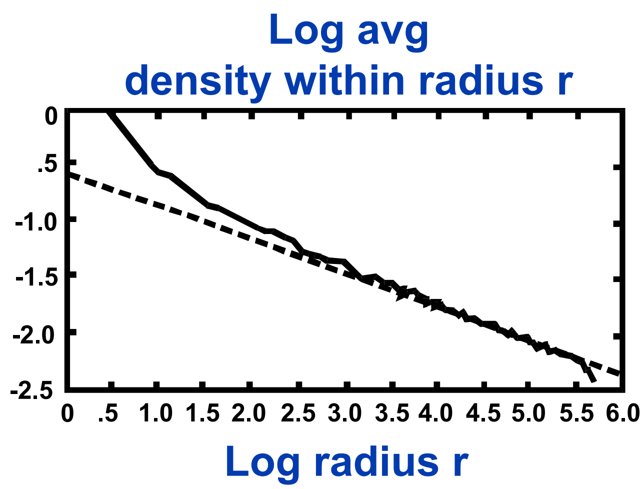
Non-Fractal



The figure on the next page is a fractal object. It is called a Diffusion Limited Aggregation (DLA). It is statistically self-similar. It has little spaces between its little arms, medium spaces between its medium-sized arms, and large spaces between its large arms. We're only showing you a piece of it, but it should also really extend forever in each direction. Place a circle on it. Count all the black pixels in that circle, and divide by the total number of pixels. That gives the average density within that circle. The graph shows how that density changes as the circles get bigger and bigger. As the circles get bigger we catch more of the ever larger spaces between the arms, and so the density gets smaller. As the circles get ever bigger, the density gets ever smaller. There is no one density that describes this object. What's more, the local density on a big arm is very high. The local density between big arms is very low. Yet, the same mechanism makes the arms and the spaces between them. Based upon our Statistics 101 training, we are used to thinking that when the local average changes, when there is a difference in the mean value between an experiment and a control, or between now and then, that the system must have changed. Here, fractals, with infinite variance, have moments, such as the mean, that can be very different in space and time or between experiments and controls, even though the basic process has not changed at all!

Fractal

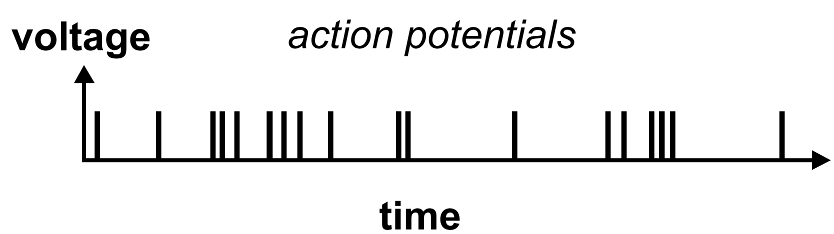
Meakin 1986 In On Growthand Form: Fractal and Non-Fractal Patterns in Physics Ed. Stanley & Ostrowsky, Martinus Nijoff Pub., pp. 111-135



Here is yet another example of fractal data. Data from many experiments have fractal properties. Here are the action potentials, the little electrical sparks, that encode information sent down the nerves in your body. Teich et al. measured them in the auditory nerve, which brings information about sounds from your ear to your brain.

Electrical Activity of Auditory Nerve Cells

Teich, Jonson, Kumar, and Turcott 1990 Hearing Res. 46:41-52



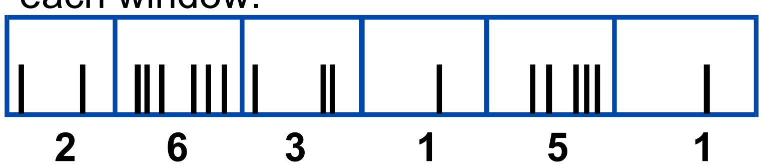
They divided the time record into windows and counted the number of action potentials in each window. Here, two in the first window, six in the second, and so on. The "firing rate" is the number of action potentials in each window, divided by the time duration of that window.

Electrical Activity of Auditory Nerve Cells

Teich, Jonson, Kumar, and Turcott 1990 Hearing Res. 46:41-52

Divide the record into time windows:

Count the number of action potentials in each window:



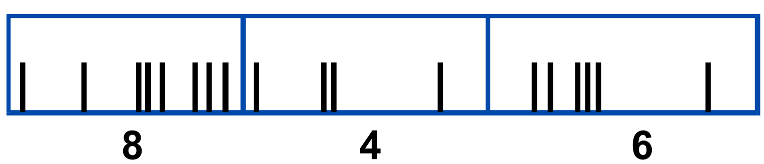
Firing Rate = 2, 6, 3, 1, 5,1

Here they made the windows twice as long in time, and counted the number of action potentials in each window. Again, the "firing rate" is the number of action potentials in each window, divided by the time duration of that window.

Electrical Activity of Auditory Nerve Cells

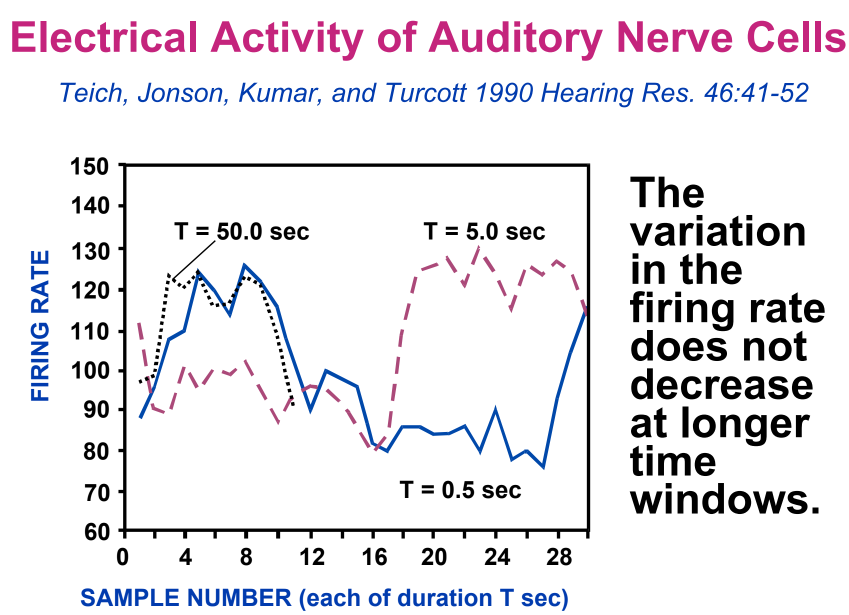
Teich, Johnson, Kumar, and Turcott 1990 Hearing Res. 46:41-52

Repeat for different lengths of time windows:



Firing Rate = 8, 4, 6

In Statistics 101 they taught you that as you collect more data, the fluctuations average out. You were taught to expect that the fluctuations in the firing rate should be less as the time windows get longer. But look here—the variations don't change much as the time windows go from 0.5 s to 5.0 s to 50.0 s! [Actually, the real deal here is that the variance of the fluctuations falls much slower than 1/sqrt(n)]. You include more data, but you don't get any closer to the real firing rate. There is no one single value, like a population mean, that best describes the firing rate. The increase in variation at longer time windows is real. It represents correlations in the action potentials which may tell how information is encoded in the timing of the action potentials.

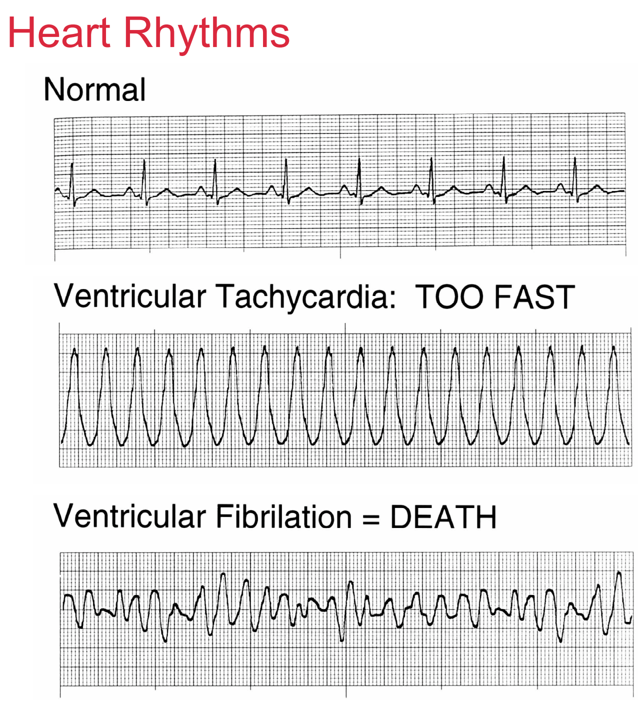


Fractals

Power Law PDFs

PDFs: Fractal data have a characteristic PDF form called a Power Law.

These are electrocardiograms (EKGs) that record the voltage from the heart. The first chart strip shows a normal heart. The second chart strip shows a heart that is beating dangerously fast (ventricular tachycardia). This is dangerous because it can lead to ventricular fibrillation, where the heart no longer contracts in a regular way and can result in death in 3 minutes.



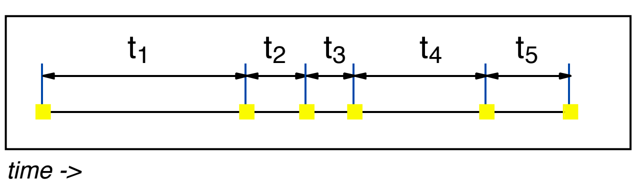
To prevent ventricular tachycardia from leading to fibrillation and death, a small device called a "cardioverter defibrillator" can be placed just under the skin in the chest. Small wires are snaked around through the blood vessels into the heart. It listens, electronically, carefully to the heartbeat. When the heart beats 3 times its normal rate for 5 beats, it sends an electrical shock strong enough to kick the heart back into a normal rhythm. But, it's also a computer, with a small memory. It can remember when it is triggered. Back in the hospital, a small coil is placed on the patient's chest. Over this radio link, the cardioverter defibrillator can play back when it was triggered. We have been analyzing the times between when the device was triggered. Just to be clear, we are not analyzing the time between heartbeats. We are analyzing the durations between consecutive triggerings of the cardioverter defibrillator.

Inter-event Times

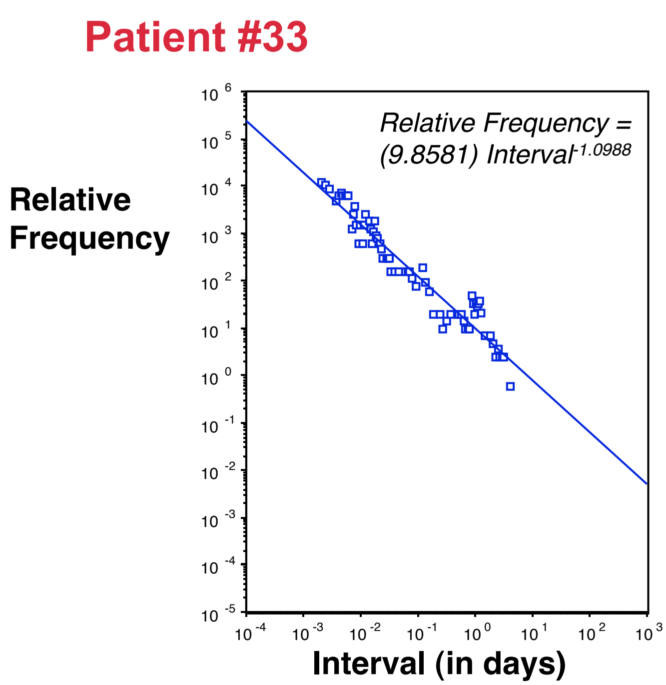
Liebovitch et al. 1999 Phys. Rev. E59:3312-3319.

Cardioverter Defibrillator

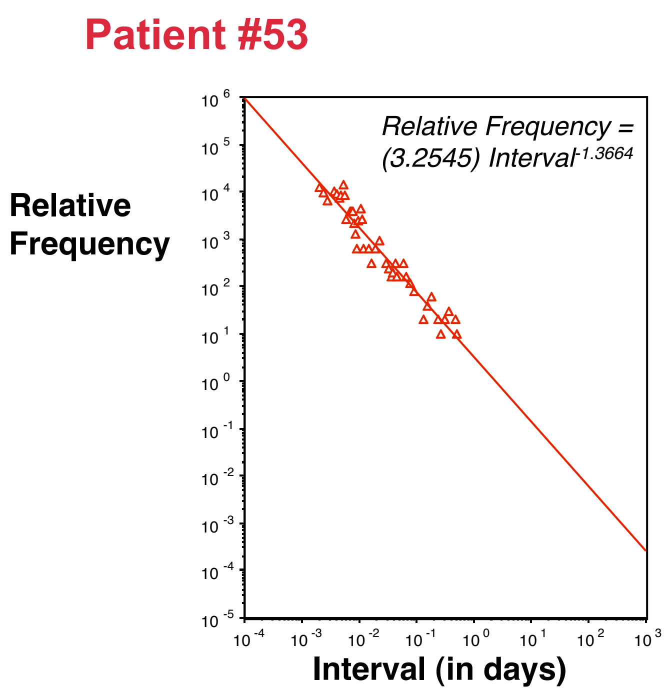
- Episodes of Ventricular Tachycardia (v-tach)



Here, from one patient, is the relative frequency, the number of occurrences of a given duration between the defibrillator events. The PDF is proportional to the relative frequency. The plot is a straight line on a log (how often) versus log (time between events) scale. This is fractal scaling. The events of rapid heart rate happen with a fractal timing.



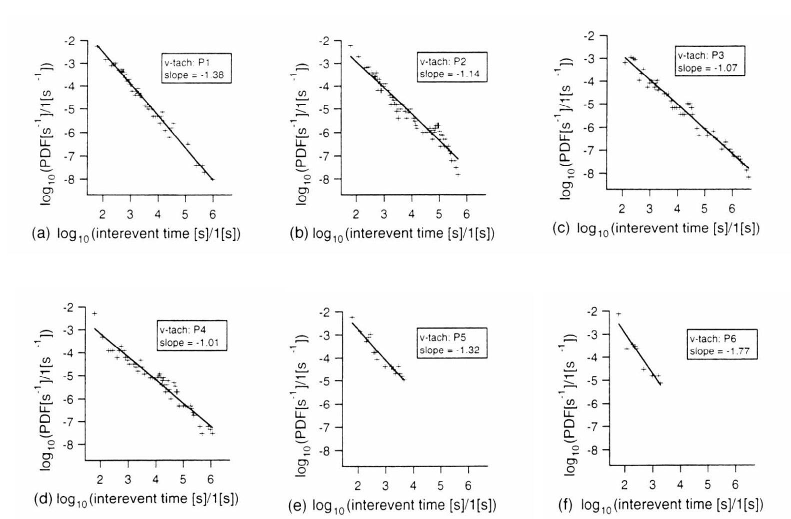
Here again, from another patient, is the relative frequency. The PDF is proportional to the relative frequency. The plot is also a straight line on a log (how often) versus log(time between events) scale. This is also fractal scaling. The events of rapid heart rate for this patient also happen with a fractal timing.



In fact, for many patients events of ventricular tachycardia happen with a fractal pattern. Most of the times between events are short, sometimes they are longer, and very rarely they are very long, which is typical of fractals. There is no average time between events. If you measured the "average rate" of events you would get a different answer if you measured them over a day or a week or a year. There is no one number that best describes the time between these events. The time between events happens over many different time scales. We are working on other ways, fractal ways, of characterizing these times to assess the status of patients and the effectiveness of medical therapies. For example, we are seeing whether the slope or intercept of these PDFs is a good indicator of diagnosis or treatment outcome.

6 Patients

Liebovitch et al. 1999 Phys. Rev. E59:3312-3319.



We are also analyzing the times at which different e-mail viruses arrive at the gateway into an internet service provider. On the picture on the following page are the events—the arrival times of e-mail viruses. We are looking at the duration of times between the arrival of each virus. We have studied 4 viruses:

1. *AnnaKournikova* doesn't have a picture of her, it's a file that you wouldn't want to open.

2. *Magistr* can erase sectors on your hard disk or your cmos/bios. If you don't know what the cmos/bios is, you don't want us to tell you what happens if it gets erased.

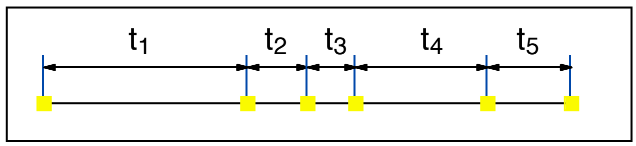
3. *Klez* puts together messages by joining fragments of phrases that it contains.

4. *Sircam* tempts you to open and execute its attached file.

Much is known about the structure of the Internet. Less is known about the dynamics of the Internet. The arrival times of these viruses depend on both the structure and dynamics of the Internet. We are hoping that our study of these arrival times will tell us how the structure interacts with the dynamics if the Internet.

Inter-arrival Times of E-mail Viruses

Liebovitch and Schwartz 2003 Phys. Rev. E68:017101.



time ->

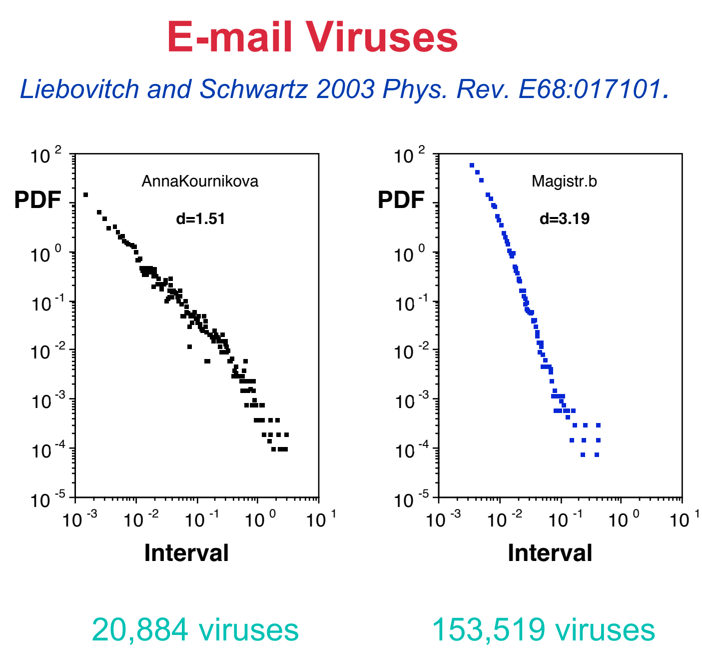
AnnaKournikova
"Hi: Check This!" AnnaKournikova.jpg vbs.

Magistr
Subject, body, attachment from other files: erase disk, cmos/bios.

Klez
E-mail from its own phrases: infect by just viewing in Outlook Express.

Sircam
“I send you this file in order to have your advice.”

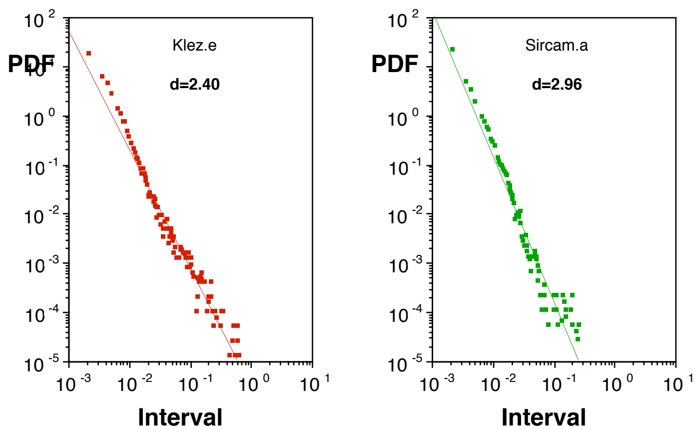
We have data, unfortunately, from many, many virus events. Here are the PDFs from *AnnaKournikova* and *Magistr*, how often the different times between the arrivals occurred. These plots are straight lines on a log (how often) versus log(time between arrivals) scale. These are fractal scalings. The arrival of these viruses happens with a fractal timing.



Here are the PDFs from *Klez* and *Sircam*, how often the different times between the arrivals occurred. These plots are also straight lines on a log (How Often) versus log(time between arrivals) scale. These are also fractal scalings. The arrival of these viruses also happens with a fractal timing.

E-mail Viruses

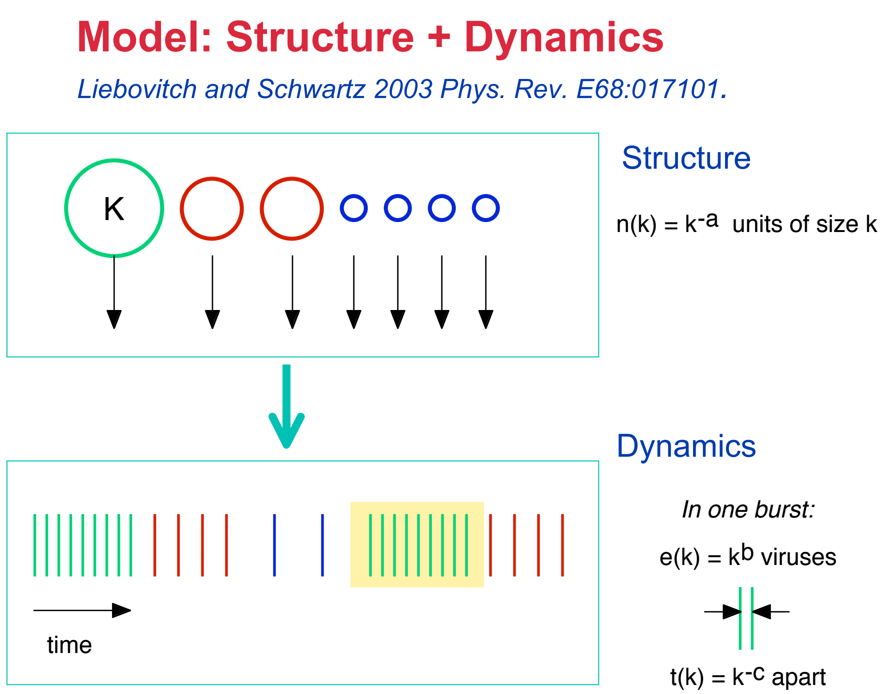
Liebovitch and Schwartz 2003 Phys. Rev. E68:017101.



413,183 viruses   781,626 viruses

We made a simple model to study how the structure and the dynamics of the Internet are interrelated.

Structure: There are networks of computers of different sizes.
Dynamics: When each network sends out viruses, the number and time between them depend on the size of the network.



There were some interesting results from these models. First, the model predicts a power law distribution in the PDF of the times between the arrival of the viruses. Second, the exponent of that power law tells us whether relatively more viruses are sent from the small number of larger networks or the larger number of small networks.

Model: Results

$$PDF(t) = t^{-d}$$

$$\text{where } d = 1 - a/c + b/c$$

The relative number of viruses from all units of size k $\sim k^{b-a}$

$$d = 1 - a/c + b/c \longrightarrow (b-a) = c\,(d-1)$$

When $d > 1$: $(b-a) > 0$,
relatively more viruses come from the larger units,
as seen in the data.

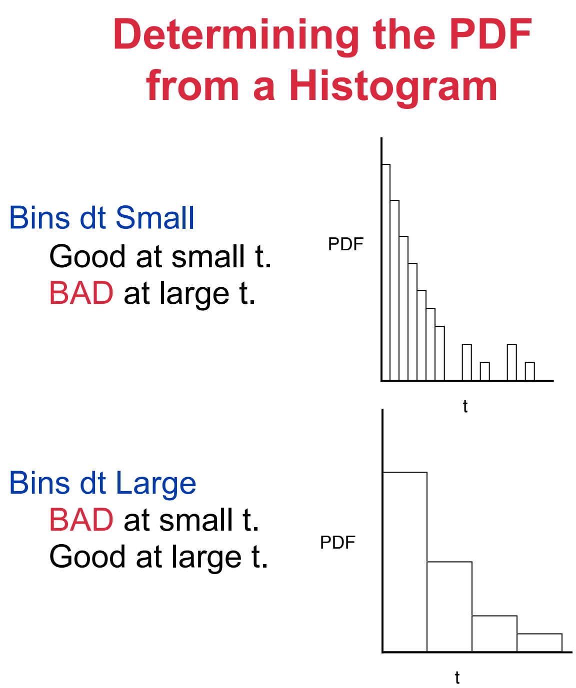
When $d < 1$: $(b-a) < 0$,
relatively more viruses come from the smaller units.

Fractals

Methods for Determining the PDFs

The PDF is an important tool in determining if experimental data have fractal properties. A power law PDF is characteristic of fractal behavior. The standard method for evaluating the PDF is to make a histogram of the data. That method is very good at determining the PDF when the data are not fractal. It is less good at determining the PDF when the data are fractal. Next, we'll see other ways of determining the PDF, and how they compare to the histogram method.

It's not so easy to make a good PDF from the histogram of How Many events there are in each time bin. If we choose the bins narrow, we get good resolution for the small events. But, at long times, because the bins are so narrow, we may see only 1 or 2 events, or even 0 events, in some of those bins. If we try to fix this by making the bins wider, we do get more events in the bins at longer times. But, at short times, we now have poor resolution.



We figured out a nice algorithm to get a better PDF. Narrow bins are good at short times. Wide bins are good at long times. So, we make histograms of different bin sizes. But, we cannot combine histograms of different bin sizes. However, we can compute the PDF from each histogram and then combine the PDFs. For each histogram, the PDF(t) is N(t), the number of values in the bin that covers (t, t+dt), divided by Nt, the total number of values in that histogram, divided by dt, the width of the bins in that histogram. The histograms with narrow bins give us good resolution in the PDF at short times. The histograms with wide bins give us good values in the PDF at long times. We've found that this method yields accurate and reliable PDFs for tails of many different kinds of distributions. See Liebovitch et al. 1999 for details.

Determining the PDF

Liebovitch et al. 1999 Phys. Rev. E59:3312-3319.

Solution:
Make ONE PDF
From SEVERAL Histograms of DIFFERENT Bin Size

$$PDF = \frac{N(t)}{N_{tot} dt}$$

$N(t)$ = number in [t, t+dt]

N_{tot} = total number in each histogram

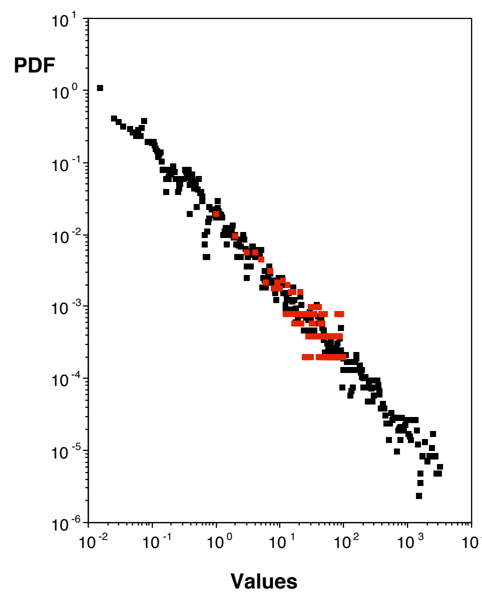
dt = bin size

Choose dt = 1, 2, 4, 8, 16 … seconds

Here, PDFs were measured from a set of fractal data. The red boxes indicate the PDF made in the usual way from one histogram. You can see where there are only 1 or 2 events in the largest bins. The black boxes indicate the PDF generated from the same data using the new multi-histogram method to make the PDF. Pretty impressive difference.

Determining the PDF

- New multi-histogram
- Standard fixed dt



Fractals

Summary

Self-Similarity

Definition: Pieces of an object in space, or parts of a process in time, are smaller versions of the whole object or process.

Examples: The Sierpinski Triangle in space and the times between the arrival of e-mail viruses.

Methods: A power law distribution of the PDF of the pieces of an object in space or the parts of a process in time is indicative of fractal behavior.

Importance for data analysis: There is no single scale, in space or time, that characterizes such data that extends over many scales.

Summary of Fractal Properties

Self-Similarity

Pieces resemble the whole.

SCALING

Definition: The value measured for a property depends on the scale, in space or time, over which it is measured.

Examples: The length of the west coast of Britain and the closed times of ion channel proteins in the cell membrane.

Methods: A power law scaling of the measured values or the correlation between the measured values is indicative of fractal scaling behavior.

Importance for data analysis: Since no one value properly characterizes the data, what is important is how the value measured depends on the resolution used to make the measurement.

Summary of Fractal Properties

Scaling

The value measured depends on the resolution.

STATISTICS

Definition: The PDF is a power law. The population mean and population standard deviation don't exist.

Examples: The winnings in the St. Petersburg game and the variation in the times between action potentials recorded from auditory nerve cells in the ear.

Methods: A power law distribution of the PDF or a power law scaling relationship for the moments is indicative of fractal behavior.

Importance for data analysis: When the mean depends on the spatial scale, the temporal scale, or how much data we analyze, then the mean is meaningless. What is meaningful is how the sample means, or another scaling property, depend on the spatial scale, the temporal scale, or how much data we analyze, which is described by the fractal dimension.

Summary of Fractal Properties

Statistical Properties

Moments may be zero or infinite.

Probability theory started from solving gambling problems about 400 years ago. About 200 years ago, those results were used to develop basic statistics. Most of the statistical tests we use were developed less than 100 years ago. We show you this to emphasize that statistics is NOT a dead science, although it's often presented like that in Statistics 101. It has changed a lot. It is still changing. It will change even more in the future. The statistical properties of fractals are examples of new ideas that are now being incorporated into and are changing statistics.

Statistics is NOT a dead science.

400 years ago:
Gambling Problems → Probability Theory

200 years ago:
Statistics → How we do experiments.

100 years ago:
Student's t-test, F-test, ANOVA

Now:
Still changing

The take-home lesson here is not that fractals some arcane super-sophisticated mathematical tool that only needs to be used in some strange circumstance. Fractals change the most basic way we look at experimental data. They allow us to analyze and make sense out of the huge amount of real data that "just ain't a bell curve." The most common use of mathematics and statistics in all science is *means* ± *s.e.m*. Fractals tell us that if the data are fractal, those means are meaningless! That's a pretty basic change in the simplest way we handle data. That's what revolutions in science are about—not about changing the complex stuff, but about changing the simplest stuff. The stuff that we were taught so firmly that we never thought it would change.

Fractals CHANGE the most basic ways we analyze and understand experimental data.

Fractals

No Bell Curves
No Moments
No mean ± s.e.m.

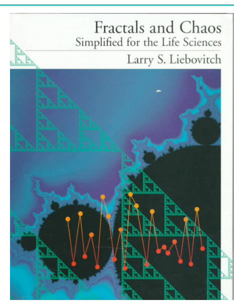
Measurements over many scales.

What is real is not one number, but how the measured values change with the scale at which they are measured (fractal dimension).

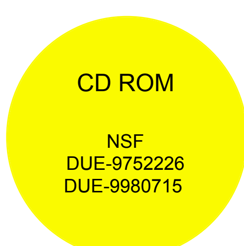
TO LEARN MORE ABOUT FRACTALS

1. A book called *Fractals and Chaos Simplified for the Life Sciences* (Liebovitch, 1998). This book consists of facing pages, where the left page is text and the right page is a picture. It leads you, one concept at a time, through the material.

2. A CD-ROM of curricula materials for a mathematics course for college students who never liked and never did well in math (funded, in part, by the National Science Foundation, Division of Undergraduate Education). The materials emphasize what mathematics is, how mathematicians do mathematics, and how mathematics is used in science. We're almost finished with it and would be happy to send you a free demo (contact information is on the first page of this chapter).



Fractals and Chaos and Simplified for the Life Sciences
Larry S. Liebovitch
Oxford Univ. Press, 1998



The Mathematics and Science of Fractals
Larry S. Liebovitch and Lina Shehadeh
www.ccs.fau.edu/~liebovitch/larry.html

We have concentrated here (and in the references noted on the previous page) on providing an introduction to fractal concepts, their importance, and what can be learned from them. Here are some books that describe the mathematical details of these methods and give examples of how scientists have used them.

Technical Details

J. Feder. 1988. *Fractals.* Plenum Press.

J. B. Bassingthwaighte, L. S. Liebovitch and B. J. West. 1994. *Fractal Physiology.* Oxford University Press.

P. M. Iannaccone and M. Khokha. 1996. *Fractal Geometry in Biological Systems.* CRC Press.

A. Bunde and S. Havlin, eds. 1994. *Fractals in Science.* Springer-Verlag.

Acknowledgements

The development of some of these materials was supported, in part, by NSF grants DUE-9752226 and DUE-9980715.