Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates

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Abstract

This article presents a framework for studying the role of recovery on defaultable debt prices (for a wide class of processes describing recovery rates and default probability). These debt models have the ability to differentiate the impact of recovery rates and default probability, and can be utilized to invert the market expectation of recovery rates implicit in bond prices. Empirical implementation of these models suggests two central findings. First, the recovery concept that specifies recovery as a fraction of the discounted par value has broader empirical support. Second, parametric debt valuation models can provide a useful assessment of recovery rates embedded in bond prices. This article has attempted to model recovery and comprehend their impact on debt values.

This article analyzes the effect of recovery on the price of defaultable debt. We specifically pose two empirical questions using defaultable debt models:

- Which concept of recovery is supported in market price of corporate coupon bonds? and,
- When Does recovery (in default) have an identifiable impact on debt prices? If so, what can be learned about recovery rates embedded in market debt prices?

Our next central purpose is to theoretically investigate whether recovery risk is priced: Are risk-neutral expected recovery rates lower or higher than its physical counterpart? These issues are potentially crucial for understanding the role of recovery in default risk modeling.

Our theoretical analysis reveals that cross-sectional variations in risk-neutral expected recovery rates are possibly impacted by differences in counterparty risk aversion and firm-specific moments of the physical recovery density. We derive two principal results. First, our developments suggest that the risk-neutral default probabilities are not only higher than physical default probabilities but also inversely related to the level of recovery. Second, the derived risk-neutral recovery density is such that the risk-neutralized expected recovery rates are lower than their physical counterpart. Firms with more volatile or left-skewed (physical) recovery densities may incur further reductions in risk-neutral expected recoveries. These features are possibly reflected in the term structure of credit spreads.

For the empirical application involving parametric debt models and stochastic recovery, we appeal to the framework presented in Duffie and Singleton (1999), Jarrow and Turnbull (1995) and Lando (1998). We explicitly derive defaultable coupon bond prices under the assumption that (1) the bondholders recover a fraction of the face value of the bond (the *"Recovery of Face Value"* model), or (2) the bondholders recover a fraction of the present value of face (the *"Recovery of Treasury"* model), and (3) the bondholders recover a fraction proportional to the pre-default market value (the *"Recovery of Market Value"* model). In so doing, our set-up possibly provides two incremental contributions. First, the approach allows for a flexible correlation structure between the risk-free interest rate, the default probability and the recovery rate. Second, in our model, the recovery rate can be functionally linked to default probability or to other fundamental factors that proxy default risk. The debt model can therefore be utilized to separate credit spreads into their default probability and recovery related components. From a practical viewpoint, some of the recovery models can be applied to infer the market's expectation of recovery rate implicit in bond prices. Finally, standard model assessment tools can be employed to gauge which recovery concept has broader empirical support. Even if all recovery concepts are adequately supported by the data, they could fundamentally depart in the quality of recovery levels implied by market prices. As such, these differences could be relevant for the design and management of recovery sensitive instruments.[1]

To illustrate the empirical potential of various defaultable debt models, they are applied separately to a sample of BBB-rated corporate bonds. The closed-form defaultable bond models we test rely on three assumptions. One, guided by our theoretical results, we specify that recovery rates are negatively associated with default probability. This parametric link is also inspired by an empirical study by Fitch that shows that when aggregate defaults tend to rise, actual recovery rates often decline (WSJ; March 19, 2001). Two, we posit a family of hazard rates that are linear in the short interest rate (Duffee (1999) and Duffie and Singleton (1997, 1999)). Lastly, it is assumed (for tractability) that the interest rate process is governed by single-factor diffusion. In this modeling environment, our theoretical characterizations show that different recovery conventions can have distinct implications for defaultable bonds. The recovery and default risk underpinnings of defaultable debt prices is apparent from our analytical characterizations.

Our empirical exercises provide several key insights. Between the recovery of face value model, the recovery of treasury model, and a version of the Duffie-Singleton model, the bond data better supports the notion of recovery based on the recovery of treasury model. For example, the out-of-sample absolute yield basis point errors and the absolute dollar pricing errors are consistently lower for the recovery of treasury model (and statistically significant). Adopting the recovery of treasury model is also found to reduce the dispersion of pricing errors. Among other possible explanations, a superior out-of-sample fit of the recovery of treasury model has the implication that bondholders are not overly optimistic about immediate cash recovery of the face value of the bond. The underlying assumptions of the recovery of treasury model may more closely approximate the implicit valuation process of the market-place.

The second insight that emerges is that the recovery of face value model and the recovery of treasury model are both informative about (risk-neutral) expected recovery rates. These estimates provide a useful benchmark for comparing expected recovery rates over time and across credit ratings. Finally, we observe that the expected recovery rates are much less volatile for the recovery

[1]The recovery of market-value model of Duffie and Singleton (1999) has led to considerable insights about credit risks. In their debt valuation model, they adopt the assumption that recovery in default is proportional to the pre-default market value of the defaultable security. For this case, they prove that the defaultable debt value is the discounted expectation of the promised payments, where the discount rate combines time value, default arrival and recovery rates. Although this framework has facilitated tractable closed-form characterizations of defaultable securities and it has generated realistic credit spreads, the simplicity and elegance of this approach comes at a cost of confounding the effect of recovery-in-default from the probability of default. For this reason, determining recovery concepts that allow debt prices to reveal information about expected recovery levels can prove useful in the design of recovery sensitive credit derivatives. The language of the Basle Accord, for instance, requires distinct assessments of the credit risk events of default and loss given default.

of face value model. The recovery of face value convention may well be a more suitable contracting choice for writing contingent claims on recoveries.

This paper is organized as follows. Section 1 explain how risk-neutral recovery rates are related to the density of the physical recovery, and counterparty risk aversion. In section 2, we present a framework for pricing defaultable coupon bonds under broad recovery specifications. The next section develops closed-form models of defaultable bonds that allow recovery to vary with default probability. Our empirical applications and estimation strategy is outlined in Sections 4 through 7. In particular, we present the out-of-sample pricing error results and the estimates of the market implied recovery rates. Section 8 summarizes our main results and concludes.

1 Recovery, Default, and the Market Price of Recovery Risk

Most default risk models directly specify an adapted process for the expected level of recovery under the risk-neutral measure (starting with Duffie and Singleton (1999) and Jarrow and Turnbull (1995)). By implication there is the recognition that the actual/physical level of recovery is a random variable, given default, with an expected value that can be modeled for valuation purposes. To clarify how firm-specific risk-neutral recovery rates are impacted by risk aversion and moments of the physical recovery density, let w denote the proportion of value recovered in default. Suppose that the physical recovery rate given default has a density with support in the unit interval, say $\overline{f}[w]$, $0 \leq w \leq 1$ satisfying $\int_0^1 \overline{f}[w]\, dw = 1$.

Let $\overline{h}$ be the probability of default under the physical probability measure. The risk-neutral probability h is also abstractly linked to its physical counterpart, and possibly recovery. Our objective is to address two questions: (i) When is physical expected recovery rate at least as large as the risk-neutral expected recovery rate? (ii) When is the risk-neutral probability of default at least as large as the physical probability of default? We argue that these issues are crucial for understanding the price of default risk and/or recovery risk, and for interpreting model-implied recovery rates.

To derive an explicit relationship between expected recovery rates under the risk-neutral and physical probability measures, consider a problem where a firm could default on its obligation. In such a scenario, many counterparties would be affected and as such would have an interest in buying insurance against this event. Naturally, the lower the expected recovery rate, the greater its impact on counterparties and the more they are willing to pay to cover against such an eventuality. For a counterparty wealth level of W_0, their post-default wealth may be written as: $W_0 - (1-w)\, F$, where, for simplicity, F is the notional principal at stake. Therefore, the expected utility of wealth, for utility function $U[.]$, conditional on default, assuming no hedging, is given by:

$$E\{U[w]\} = \int_0^1 U[W_0 - (1-w)\,F\,]\,\overline{f}[w]\,dw \tag{1}$$

where $E\{.\}$ is expectation under the physical probability measure.

Now consider a contingent claim written on the recovery rate w and paying the claim-holder the dollar amount $C[w]$ were default to occur with a recovery rate of w. Let π denote the price of this claim. The counterparty wealth of buying η units of the contingent claim is given by $W_0 - \eta\,\pi$ with probability: $(1-\overline{h})$, and with probability $\overline{h}$, it is: $W_0 - (1-w)\,F \;+\; \eta\,C[w] - \eta\,\pi$. The expected utility of the counterparty buying insurance is:

$$E\{U[w]\} \equiv (1-\overline{h})\,U\,[W_0 - \eta\,\pi] + \overline{h}\int_0^1 U\,[W_0 - (1-w)\,F \;+\; \eta\,C[w] - \eta\,\pi]\,\overline{f}[w]\,dw. \tag{2}$$

By a variational argument, the partial derivative of $E\{U[w]\}$ with respect to η at the optimum, evaluated at $\eta = 0$, should be zero. This condition yields the price of insurance as (U' is the partial derivative of the counterpart utility function with respect to w):

$$\pi = \frac{\overline{h}\int_0^1 U'\,[W_0 - (1-w)\,F] \times C[w] \times \overline{f}[w]\,dw}{(1-\overline{h})\,U'\,[W_0] + \overline{h}\int_0^1 U'\,[W_0 - (1-w)\,F] \times \overline{f}[w]\,dw} \tag{3}$$

$$= h\int_0^1 \left\{ \frac{U'\,[W_0 - (1-w)\,F]}{\int_0^1 U'\,[W_0 - (1-w)\,F]\;\overline{f}[w]\,dw} \right\} \times C[w] \times \overline{f}[w]\,dw \tag{4}$$

where the **risk-neutral default probability**, h, in (4) can be defined as:

$$h \equiv \frac{\overline{h}\int_0^1 U'\,[W_0 - (1-w)\,F]\;\overline{f}[w]\,dw}{(1-\overline{h})\,U'\,[W_0] + \overline{h}\int_0^1 U'\,[W_0 - (1-w)\,F]\;\overline{f}[w]\,dw} \tag{5}$$

which is a valid probability function since $h \in [0,1]$. Equation (5) provides the economic intuition that the risk-neutral default probability is related to (i) the physical probability of default $\overline{h}$, (ii) the counterparty risk aversion as reflected in the marginal utility of wealth, and (iii) the distribution of physical recovery.

Focus first on the full recovery situation with $w = 1$. It follows that $\int_0^1 U'\,[W_0 - (1-w)\,F]\;\overline{f}[w]\,dw$ $= U'\,[W_0]$ so that the physical and the risk-neutral probability of default are the same regardless of the counterparty risk aversion (i.e, $h = \overline{h}$).

Consider now the more plausible scenario of partial recovery. For a sufficiently wide class of utility functions, it is reasonable to expect that

$$\frac{U'\left[W_0\right]}{\int_0^1 U'\left[W_0-(1-w)\,F\right]\,\overline{f}[w]\,dw}<1. \tag{6}$$

Substituting this restriction into (5), we obtain:

$$h>\overline{h}. \tag{7}$$

With non-zero risk aversion and partial recovery, the risk-neutral default likelihood is therefore an amplified version of physical default probabilities. Equations (5)-(7) suggest that the amplification of the risk-neutral default probability is positively related to the tails of the loss distribution. This result is also relevant for modeling recovery rates.

From (4), it can be determined that the **risk-neutral density of the conditional recovery rate** is given by:

$$f[w]\equiv\frac{U'\left[W_0-(1-w)\,F\right]\times\overline{f}[w]}{\int_0^1 U'\left[W_0-(1-w)\,F\right]\,\overline{f}[w]\,dw} \tag{8}$$

which connects the risk-neutral recovery density to the physical counterpart and the degree of risk aversion. Absent risk aversion, equation (8) shows that $f[w]=\overline{f}[w]$, so that the risk-neutral and physical densities coincide.

The next proposition proposes a relationship between the expected recovery rates under the risk-neutral and the physical probability measures.

Proposition 1 *Consider the risk-neutral expected recovery rate:*

$$E^Q\{w\} \equiv \int_0^1 w\,f[w]\,dw, \tag{9}$$

$$= \int_0^1 w\left\{\frac{U'\left[W_0-(1-w)\,F\right]\,\overline{f}[w]}{\int_0^1 U'\left[W_0-(1-w)\,F\right]\,\overline{f}[w]\,dw}\right\}dw. \tag{10}$$

The risk-neutral expected recovery rate is less than the physical expected recovery rate, $E\{w\}$*,*

$$E^Q\{w\}\leq E\{w\} \tag{11}$$

since $\int_0^1 w\,U'\left[W_0-(1-w)\,F\right]\,\overline{f}[w]\,dw-\left(\int_0^1 w\,\overline{f}[w]\,dw\right)\left(\int_0^1 U'\left[W_0-(1-w)\,F\right]\,\overline{f}[w]\,dw\right)=$ $Cov\left(w,U'\left[W_0-(1-w)\,F\right]\right)\leq 0$*, for all concave counterparty utility functions.*

From the perspective of default risk modeling, the central result from Proposition 1 is that the risk-neutral expected recovery rate is generally lower than the physical recovery rate. Suppose the counterparty has a utility function belonging to the constant absolute risk aversion class with risk aversion γ. In the case of positive risk aversion we assert that $\int_0^1 w\, f[w]\, dw < \int_0^1 w\, \overline{f}[w]\, dw$. This is because

$$\frac{\partial E^Q\{w\}}{\partial \gamma}|_{\gamma=0} = -\int_0^1 w\ (W_0 - F + F\, w)\, \overline{f}[w]\, dw + E\{w\} \int_0^1 (W_0 - F + F\, w)\, \overline{f}[w]\, dw \tag{12}$$

$$= -F \left\{ \int_0^1 w^2\, \overline{f}[w]\, dw - \left(\int_0^1 w\, \overline{f}[w]\, dw \right)^2 \right\} < 0, \tag{13}$$

and the two expected recovery rates are equal when $\gamma = 0$. When combined with (5), equations (11) and (13) establish that the impact of risk aversion is to both increase risk-neutral default probabilities and simultaneously lower risk-neutral expected recovery rates. In particular, it is unjustified to directly employ market implied recovery rates as forecasts of expected recoveries without adjusting for risk aversion (unless it is argued that counterparty exposure is too minimal for risk aversion to be an issue).

It can be shown that firms with more uncertain or left-skewed physical recovery densities are likely to experience greater divergence between the risk-neutral and physical recovery rates. For further details on these higher moment connections and ways to generalize to a broader family of marginal utilities, the reader is referred to Bakshi, Kapadia, and Madan (2000). Proposition 1 has implications for extracting physical recovery information from market-based studies.

2 Recovery in Default, and Defaultable Debt Modeling

This section presents the necessary framework for pricing defaultable coupon bonds for a broad class of recovery definitions. These pricing results hold for a wide specification of processes describing the evolution of the recovery rate, the default probability and the interest rate. As such, the defaultable bond models are intended to achieve the desired balance between default risk and recovery. At the outset we must note that our primary content is the empirical assessment of recovery concepts with the theoretical framework supportive of the empirical work.

2.1 Defaultable Debt Pricing under Stochastic Recovery Rates

To address theoretical and empirical issues, we appeal to the default risk models presented in Duffie and Singleton (1999), Jarrow and Turnbull (1995) and Lando (1998). Specifically for the case of defaultable coupon bonds and *random recovery*, it can be shown that its price, $P(t,T)$, is:

$$
\begin{aligned}
P(t,T) \quad = \quad & E_t^Q \left\{ \int_t^T \exp\left(- \int_t^u [r(s) + h(s)]\, ds \right) \times c(u)\, du \right\} \\
& + E_t^Q \left\{ \exp\left(- \int_t^T [r(s) + h(s)]\, ds \right) \right\} \times F \\
& + E_t^Q \left\{ \int_t^T \exp\left(- \int_t^u [r(s) + h(s)]\, ds \right) \times y(u) \times h(u)\, du \right\}
\end{aligned}
\tag{14}
$$

where,

> *F is the promised face value of the defaultable bond maturing at date T, and $\{c(u) : u > t\}$ is the promised continuous-coupon payment;*
>
> *$E_t^Q \{.\}$ represents conditional expectation under the risk-neutral probability measure Q and $r(t)$ is the economy-wide spot interest rate;*
>
> *$y(u)$ is the recovery payout in the event of default and $h(u)$ is the hazard rate. Typically, the recovery payout $y(u)$ is significantly below remaining promised payments.*

Equation (14) suggests that the intrinsic value of a defaultable security is the expected discounted value of the underlying payoff, when the discounting factor is increased to the sum of the interest rate and the hazard rate. When recovery payout is positive, the defaultable bond price has three distinct components. For instance, the first conditional expectation accounts for the receipt of coupons prior to default. The second term is due to the promised face value in the absence of default. Finally, the last integral determines the value of the single recovery payout if the firm defaults. This conditional expectation involves some novelties arising from the product of recovery and hazard rate. Intuitively, the underlying payoff embodies recovery with probability $h(u)\, du$ in the entire no-prior default time-domain. The next subsection presents a technique that analytically and simultaneously determines all the three components required to value defaultable coupon bonds.

Even though not emphasized in the preceding discussion, the primitive modeling variables $h(u)$, $y(u)$ and $r(u)$ can all be functions of the state of the economy. For example, the hazard rate $h(u)$ can exhibit complex dependencies on systematic as well as firm-specific factors (as empirically modeled in Bakshi, Madan, and Zhang (2000) and Duffee (1999)). Leave the precise (risk-neutral) evolution of these variables unspecified for now. As in Duffie and Singleton (1999) and Lando (1998), equation (14) provides an internally consistent framework to jointly model default risk and recovery payout.

While modeling the defaultable debt price in (14), three assumptions are typically made about recovery payout in default:

1. **Recovery payout is a fraction of the face value**:

$$y(u) = w(u) \times F. \tag{15}$$

   This recovery assumption is conceptually straightforward. When $w(u) \equiv w_0$, $0 \leq w_0 \leq 1$, the recovery payout is a constant proportion of the promised face value (Duffie (1999), Duffie and Singleton (1999) and Lando (1998)). For this reason, we refer to this model as the recovery of face value model (hereby, the **RFV model** as in Duffie-Singleton (1999)).

2. **Recovery payout is the fraction of the present value of the face**:

$$y(u) = w(u) \times B(u,T) \times F, \tag{16}$$

   where $B(u,T) = E_u^Q \left\{ \exp\left(-\int_u^T r(s)\, ds\right)\right\}$ is the time-u price of a default-free discount bond with maturity date T. This class of recovery payout is equivalent to the assumption that recovery is a fraction of the price of a treasury bond with the face value F and maturity T (see Longstaff and Schwartz (1995), Jarrow and Turnbull (1995) and Collin-Dufresne and Goldstein (2000)). We will refer to this model as the recovery of treasury model (hereby, the **RT model**).

3. **Recovery payout is a fraction of the pre-default debt value**:

$$y(u) = w(u) \times P(u_-, T). \tag{17}$$

   which is the **RMV model** of Duffie and Singleton (1999). In this model, the impact of recovery is subsumed within the defaultable discount rate.[2]

Several observations are in order about recovery conventions. One, the first two models assume no recovery of the unpaid coupons. At first glance, equations (15) and (16) can be seen as different ways of defining recovery rate $w(u)$ given the process for recovery $y(u)$. As mentioned also in Jarrow and Turnbull (1995), Longstaff and Schwartz (1995) and Duffie and Singleton (1999), the

[2]See, among others, the empirical applications in Bakshi, Madan, and Zhang (2000), Duffee (1999), Duffie and Singleton (1997), He (2000), Liu, Longstaff, and Mandell (2001), and Janosi, Jarrow, and Yildirim (2000). The Duffie-Singleton framework (with recovery as a fraction of pre-default debt value) has become an integral part of credit risk related research.

recovery definition (16) is relatively more complex and requires the knowledge of the term structure of default-free interest rates. If the process for recovery $y(u)$ is given, the definitions (15) and (16) have no pricing implications. However, the constructs (15) and (16) assume economic content when they are employed as building blocks for modeling the recovery payout $y(u)$.

While recovery rates may be of second-order consideration for high-rated corporate bonds, they are crucial determinants of bond values and yields in the default sensitive high-yield debt market. Existing evidence appears to suggest that default probabilities and recoveries are inversely related: the recovery payout tends to be low when actual defaults rise and the reverse (Altman (2001)). However, in the limiting case of no default risk (i.e., treasuries), the recovery rates become irrelevant. Many such properties of the recovery rate can be incorporated into equations (15)-(16) with little loss of generality. In light of the empirical evidence, one can postulate $w(u)$ to be a function of the hazard rate or other state variables. For example, one would not expect high recovery rates to coincide with high default probabilities. In fact, one such parameterization will be pursued in Section 3 and empirically implemented in subsequent sections. While easily relaxed, our recovery conventions assert that bonds by the same issuer have identical recovery rates regardless of their maturity.

2.2 Characterizing Defaultable Bond Prices

In order to learn about recovery rates from the market bond prices, we need to validate the specific bond pricing model that arises from equation (14) when we adopt the recovery assumptions (15)-(16). In this subsection we outline a sufficiently wide framework under which equation (14), when combined with assumptions (15)-(16), is analytically tractable. For these general characterizations, we impose assumptions about the structure of recovery rates, the hazard rates, the interest rates, and the default-free discount bond prices (B(t,T)). For the purpose at hand, define, for convenience, the vector, $X(t) \equiv (r(t), h(t), w(t), \log(B(t,T)))'$. As is traditionally done, assume that the $X(t)$ dynamics is governed by a Markov-Ito process:

$$dX(t) = \mu[X(t), t]\, dt + \sigma[X(t), t]\, d\omega(t) \tag{18}$$

where the drift coefficient, $\mu[X(t), t]$, is a vector of expected instantaneous changes in $X(t)$ and the diffusion coefficient, $\sigma[X, t]$ is a full-rank local covariance matrix between changes in the (vector) standard Brownian motion $d\omega(t)$.

To solve the relevant conditional expectations in (14) for the RFV model and the RT model (i.e., the recovery payout assumptions in (15)-(16)), define the characteristic function of the future uncertainty $\nu \equiv (\int_t^u [r(s) + h(s)]\, ds, h(u), w(u), \log(B(u,T)))$, with density $q(\nu)$, as shown below (see Bakshi and Madan (2000) and Duffie, Pan, and Singleton (2000)):

$$\begin{aligned} J\,(t,u;\phi,\varphi,\upsilon) &\equiv E_t^Q\left\{\exp\left(-\int_t^u [r(s)+h(s)]ds + i\phi h(u) + i\varphi w(u) + i\upsilon \log(B(u,T))\right)\right\} \quad (19) \\ &= \int \exp\left(-\int_t^u [r(s)+h(s)]ds + i\phi h(u) + i\varphi w(u) + i\upsilon \log(B(u,T)\right) q(\nu)\, d\nu \end{aligned}$$

where ϕ, φ and υ are some transform parameters, $i = \sqrt{-1}$ and the transforms parameter for the first uncertainty $\int_t^u [r(s)+h(s)]\, ds$ has been set at i. With this substitution for the first transform parameter, the characteristic function, $J(t,u;\phi,\varphi,\upsilon)$, is the time-$t$ price of a hypothetical defaultable claim that pays $\exp\left(i\phi h(u) + i\varphi w(u) + i\upsilon \log(B(u,T)\right)$ at time u. This observation is immediate by noting that $\exp\left(-\int_t^u [r(s)+h(s)]\, ds\,\right) \times q(\nu)$ is the defaultable state-price density. As emphasized in Bakshi and Madan (2000), the spanning properties of the characteristic functions facilitates the spanning and pricing of most claim payoff that rely on ν. In what follows, assume that the characteristic function $J(t,u;\phi,\varphi,\upsilon)$ is sufficiently differentiable in its arguments. By a standard argument, $J(t,u;\phi,\varphi,\upsilon)$ satisfies the partial differential equation:

$$\frac{1}{2}\,\mathrm{trace}[\sigma\sigma' J_{XX}] \;+\; \mu\, J_X + J_t - (r+h)\;J = 0 \qquad (20)$$

subject to the boundary condition: $J(u,u) = \exp\left(i\,\phi\, h(u) + i\,\varphi\, w(u) + i\,\upsilon\;\log(B(u,T)\right)$. We now prove a correspondence between the characteristic function $J(t,u)$ and defaultable bond prices.

Proposition 2 *Suppose the characteristic function, $J(t,u)$, is analytically known by solving the conditional expectation (19) or the partial differential equation (20). With recovery assumptions (15)-(16), we have:*

1. *For the class of recovery rates displayed in (15), the defaultable coupon bond price can be obtained from the characteristic function (19) as follows:*

$$P(t,T) = \int_t^T J(t,u;0,0,0)\, c(u)\, du + J(t,T;0,0,0) \times F - F \int_t^T J_{\phi\,\varphi}(t,u;0,0,0)\, du, \quad (21)$$

where $J_{\phi\,\varphi}(t,u;0,0,0)$ denotes the second-order partial derivative of the characteristic function with respect to ϕ and φ, and evaluated at $\phi = 0$, $\varphi = 0$ and $\upsilon = 0$.

2. *For the recovery rate considered in (16), the defaultable coupon bond price is given by the expression:*

$$P(t,T) = \int_t^T J(t,u;0,0,0)\, c(u)\, du + J(t,T;0,0,0) \times F - F \int_t^T J_{\phi\,\varphi}(t,u;0,0,-i)\, du, \quad (22)$$

where $J_{\phi\varphi}(t,u;0,0,-i)$ is now evaluated at $\phi=0$, $\varphi=0$ and $\upsilon=-i$.

In (21) and (22), each defaultable coupon bond price component is obtained from the characteristic function by combining the operations of differentiation and/or evaluation.

The characteristic function effectively synthesizes the bond valuation problem in (14). Of particular note is the use of second-order partial derivative of the characteristic function with respect to ϕ and φ that renders the analytical tractability of the third term in (14). Additionally, we observe that on account of the second-order cross-partial derivative in equations (21) and (22) that closed-form expressions for this final conditional expectation would typically be hard to conjecture even under simple parameterizations of $r(t)$, $h(t)$ and $w(t)$.

Comparing the results in Proposition 2 with the RMV model of Duffie-Singleton (1999), several fundamental differences emerge from the perspective of modeling debt prices and inferring recovery rates. Recall that in the Duffie-Singleton model

$$P(t,T)=E_t^Q\left\{\int_t^T \exp\left(-\int_t^u R(s)\,ds\right)\,c(u)\,du+F\times\exp\left(-\int_t^T R(s)\,ds\right)\right\}. \tag{23}$$

Although the RMV model is flexible in modeling credit spreads via modeling the defaultable discount rate $R(s)\equiv r(s)+(1-w(s))\,h(s)$, it is virtually impossible to empirically differentiate the effect of recovery and hazard rates. In contrast, the recovery conventions (15)-(16) lead to identifiable impacts on credit spreads. Hence the market expectation of recovery rates can be inverted from the term structure of credit spreads.

Notice that even though recovery and hazard rate appear multiplicatively in equation (14), it is the presence of the discounted face value that differentiates the impacts of hazard rate and recovery. One may view the second term (i.e., discounted face) as identifying the hazard rate leaving the recovery rate to be identified by the term that discounts payment on default. The ability to differentiate the impacts of recovery rates and hazard rates is especially useful in pricing and hedging credit derivatives that are contingent on the default event and/or recovery levels given default. Therefore, the general recovery formulations presented here are critical to wider applications in the field of credit risk.

While there is overlap with respect to RFV model with Duffie and Singleton (1999) and Duffie, Pan, and Singleton (2000), the proposition also presents in complete the RT model that the Duffie-Singleton (1999) paper puts aside as computationally burdensome especially for stochastic recovery (page 701, second paragraph).

To illustrate the general tractability of the approach outlined in Proposition 2, we consider three specifications for the recovery process. First, suppose that recovery rates are independent of the interest rate and the hazard rate. For example, one may write the recovery rate as:

$$w(u) = 1 - e^{-z}, \qquad\qquad z \in [0, \infty], \tag{24}$$

where z is independently and identically gamma-distributed with parameters α and β (with mean α/β and variance α/β^2). Here the recovery rate is strictly between zero and one. Under assumption (16) and the independence of recovery, it follows that equation (22) of Proposition 2 takes the form:

$$P(t,T) = \int_t^T J(t,u;0,0,0)\,c(u)\,du + J(t,T;0,0,0) \times F - F \times \left(\frac{\beta}{1+\beta}\right)^{\alpha} \int_t^T i\,J_\phi(t,u;0,0,-i)\,du \tag{25}$$

where $J_\phi(t,u;0,0,-i)$ is the first-order partial derivative of the characteristic function with respect to ϕ evaluated at $\phi = 0$ and $v = -i$. In this parametric special case, the parameters α and β are not separately identified and it is just the expected recovery that could be identified.

The previous formulation assumes that the recovery rates are unrelated to the underlying hazard rate. However, a little reflection is suggestive of dependencies in that hazard rates near zero should be associated with full recovery. Equally relevant is the situation of large hazard rates when default is likely. But when default is likely, one would expect the recovery rates to be low. Therefore, one might model $w(u)$ in any of the definitions of recovery as functionally related to the hazard rate $h(u)$, whereby we may specify:

$$w(u) = \Psi[h(u)]. \tag{26}$$

Barring extreme counterexamples, some desirable properties of the function $\Psi[.]$ are that $\Psi[0] = 1$ while $\Psi[\infty] = 0$. We also expect that $\Psi[h]$ be a decreasing function of the hazard rate. A wide class of such functions is the collection of completely monotone functions defined as Laplace transform of a positive function, $f[z]$, on the positive half line, or that

$$w(u) = \Psi[h(u)] = \int_0^\infty e^{-z\,h(u)}\,f[z]\,dz. \tag{27}$$

The condition $\Psi[0] = 1$ imposes the further restriction that $f[z]$ be a (risk-neutral) density on the positive half line. A robust two parameter family of such densities is again the gamma density so that

$$\Psi[h(u)] \;=\; \int_0^\infty e^{-z\,h(u)}\,f[z]\,dz \tag{28}$$

$$= \left(\frac{\beta}{\beta + h(u)}\right)^{\alpha} \tag{29}$$

since $f[z] = \frac{\beta^{\alpha}\, z^{\alpha-1}\, e^{-\beta\, z}}{\Gamma(\alpha)}$, and $\Gamma(.)$ is the mathematical gamma function. Thus, in this family of recovery rates, the counterpart to (22) is of the form:

$$P(t,T) = \int_t^T J(t,u;0,0,0)\, c(u)\, du + J(t,T;0,0,0) \times F - F \int_0^{\infty} f[z]\, dz \int_t^T J_{\phi}(t,u;iz,0,-i)\, du. \tag{30}$$

Therefore, even when a closed-form solution for the characteristic function is available, one may have to use Gauss-Laguerre quadrature to replace the integral with respect to z by a finite sum, an extension we intent to pursue in the future. Unlike equation (25), the impact of recovery parameters α and β on debt prices of different maturities is differentiated on account of the stochastic evolution of the hazard rate. This potentially permits the identification and estimation of the recovery parameters from the bond prices.

Up until now, we have considered defaultable bond models where recovery is immediately after default. However, recovery often occurs at a random time after default which is a *double-stopping* time problem. To briefly describe how this post-default recovery case can be accommodated within the framework of Proposition 2, define another stopping time $\chi^*(t) > \chi(t)$ where $\chi(t)$ is the default time process defined in Section 1.1, and $\chi^*(t)$ is the recovery time process. The hazard rate corresponding to the recovery stopping time takes the form: $\chi(u_-) \times (1 - \chi^*(u_-)) \times h^*(u)$, for default conditional recovery hazard rate $h^*(u)$. Notice that the combined term is zero until the occurrence of default; after default, the recovery hazard rate is positive until the recovery time. Guided by Bakshi, Madan, and Zhang (2000, Equation 2), the generic third term in (14) should then be modified as (ignoring discounting for now): $E_t^Q \int_t^T (1 - \chi(u_-))\, d\chi(u) \int_u^T (1 - \chi^*(u_-))\, y(s)\, d\chi^*(s)$, to account for recovery at a random time after default. Under suitable assumptions, this double-stopping time problem reduces to evaluating the conditional expectation:

$$E_t^Q \left\{ \int_t^T \exp\left(-\int_t^u [r(\mathrm{v}) + h(\mathrm{v})]\, d\mathrm{v}\right) h(u) \int_u^T \exp\left(-\int_u^s [r(a) + h^*(a)]\, da\right) y(s) \times h^*(s)\, ds\, du \right\} \tag{31}$$

with the first two terms remaining the same as in (14). This model is considerably more complex and involves default conditional recovery hazard rate modeling. With appropriate modifications to the characteristic function (19), one can develop defaultable bond prices for both the RFV model and the RT model. We do not empirically pursue the double-stopping time problem.

3 Closed-Form Models of Defaultable Debt

Building on results established in Proposition 2, this section presents analytical expressions for the price of defaultable coupon bonds. Our goal is to address the economic implications of recovery functions (15)-(16). The resulting closed-form models are the basis for the later empirical application. Our specific assumptions about the dynamics of the interest rate, the hazard rate and the recovery rate are as follows. First, we assume that the recovery rate is related to the underlying hazard rate as shown below:

$$w(u) \quad = \quad w_0 + w_1 \, e^{-h(u)}. \tag{32}$$

We may note that as $h \to 0$, $w \to w_0 + w_1$, and $h \to \infty$, $w \to w_0$. Thus, we require the restrictions that $w_0 \geq 0$, $w_1 \geq 0$ and $0 \leq w_0 + w_1 \leq 1$.

Assumption (32) is attractive from both theoretical and empirical viewpoints. First, consistent with extant empirical evidence, recovery is negatively related to default probability. That is, $\frac{\partial w(u)}{\partial h(u)} = -w_1 e^{-h(u)} \leq 0$. Our set-up is guided by the belief that financial distress can diminish the ability of the borrower to pay its creditors in the event of default (see also equation (8)). The recovery rate assumption (32) is not only economically appealing but also affords technical tractability. For instance, it leads to a closed-form expression for the characteristic function in (19) and also permits the analytical valuation of the third (recovery) term in equations (21) and (22).

Following Duffie and Singleton (1997, 1999), Duffee (1999) and Bakshi, Madan, and Zhang (2000), we make the convenient assumption that the hazard rate is linear in the short interest rate (under the risk-neutral measure):

$$h(t) \quad = \quad \Lambda_0 + \Lambda_1 \, r(t), \tag{33}$$

$$dr(t) \quad = \quad \kappa \, (\, \theta - r(t) \,) \, dt \, + \, \sigma \, \sqrt{r(t)} \, dw(t), \tag{34}$$

where $\Lambda_0 > 0$. The parameter Λ_1 reflects the correlation between the interest rate and the hazard rate. Although firm-specific variables can be incorporated into (33), the empirical investigation of Bakshi, Madan, and Zhang (2000) has shown that this class of hazard rate functions captures the first-order effect of default. The single-factor interest rate process is of the Cox-Ingersoll-Ross (1985) type.

Since the hazard rate and the recovery rate are both a function of the short interest rate, we specialize the characteristic function in equation (19) to the following:

$$J(t,u;\phi,\upsilon) \quad \equiv \quad E_t^Q\left\{\exp\left(i\,\phi\int_t^u r(s)\,ds + i\,\upsilon\,r(u)\right)\right\}. \tag{35}$$

Solving this conditional expectation results in the analytical solution

$$J(t,u;\phi,\upsilon) = \exp\left[\,\mathcal{Y}(t,u;\phi,\upsilon) - \mathcal{Z}(t,u;\phi,\upsilon)\times r(t)\,\right], \tag{36}$$

where defining $\gamma(\phi) \equiv \sqrt{\kappa^2 - 2\,i\,\phi\,\sigma^2}$, and noting that

$$\mathcal{Y}(t,u;\phi,\upsilon) \quad = \quad \frac{2\,\kappa\,\theta}{\sigma^2}\,\log\left(\frac{\gamma\,\exp\left(\frac{\kappa\,(u-t)}{2}\right)}{\gamma\,\cosh(\frac{\gamma\,(u-t)}{2}) + (\kappa - i\,\upsilon\,\sigma^2)\sinh(\frac{\gamma\,(u-t)}{2})}\right), \quad \text{and,} \tag{37}$$

$$\mathcal{Z}(t,u;\phi,\upsilon) \quad = \quad \frac{i\,\upsilon\,\gamma\,\coth(\frac{\gamma\,(u-t)}{2}) - i\,\upsilon\,\kappa - 2i\,\phi}{\gamma\,\coth(\frac{\gamma\,(u-t)}{2}) + \kappa - i\,\upsilon\,\sigma^2}. \tag{38}$$

Given the solution to the characteristic function (35), the zero-coupon bond price in (16) is then $B(u,T) = J(u,T;i,0)$. We are now ready to provide the valuation expressions for the defaultable coupon bonds for both the RFV model and the RT model.

Proposition 3 *Let the recovery rate be as displayed in (32) and the hazard rate obey the structure given in (33). Moreover, let the interest rate follow the square-root process (34). The following pricing results are then obtained by combining the characteristic function (35) with equations (21) and (22) of Proposition 2:*

1. *For the* **RFV model** *with recovery convention (15), the defaultable coupon bond price is:*

$$\begin{aligned} P(t,T) \quad = \quad & \int_t^T e^{-\Lambda_0(u-t)}\times J(t,u;i(1+\Lambda_1),0)\times c(u)\,du + e^{-\Lambda_0(T-t)}\,J(t,T;i(1+\Lambda_1),0)\times F \\ & + F\,w_0\,\Lambda_0\int_t^T e^{-\Lambda_0(u-t)}\times J(t,u;i(1+\Lambda_1),0)\,du \\ & - i\,F\,w_0\,\Lambda_1\int_t^T e^{-\Lambda_0(u-t)}\times J_\upsilon(t,u;i(1+\Lambda_1),0)\,du \\ & + F\,w_1\,\Lambda_0\,e^{-\Lambda_0}\int_t^T e^{-\Lambda_0(u-t)}\times J(t,u;i(1+\Lambda_1),i\,\Lambda_1)\,du \\ & - i\,F\,w_1\,\Lambda_1\,e^{-\Lambda_0}\int_t^T e^{-\Lambda_0(u-t)}\times J_\upsilon(t,u;i(1+\Lambda_1),i\,\Lambda_1)\,du, \end{aligned} \tag{39}$$

where $J_\upsilon(t,u;\phi,\upsilon)$ represents the partial derivative of the characteristic function (36) with respect to υ.

2. For the **RT model** *with recovery convention (16), the defaultable coupon bond price is:*

$$
\begin{aligned}
P(t,T) \;=\; & \int_t^T e^{-\Lambda_0(u-t)} \times J(t,u;i(1+\Lambda_1),0) \times c(u)\,du + e^{-\Lambda_0(T-t)}\,J(t,T;i(1+\Lambda_1),0) \times F \\
& +F\,w_0\,\Lambda_0 \int_t^T e^{-\Lambda_0(u-t)} \times J(t,u;i(1+\Lambda_1),0) \times J(u,T;i,0)\,du \\
& -i\,F\,w_0\,\Lambda_1 \int_t^T e^{-\Lambda_0(u-t)} \times J_v(t,u;i(1+\Lambda_1),0) \times J(u,T;i,0)\,du \\
& +F\,w_1\,\Lambda_0\,e^{-\Lambda_0} \int_t^T e^{-\Lambda_0(u-t)} \times J(t,u;i(1+\Lambda_1),i\,\Lambda_1) \times J(u,T;i,0)\,du \\
& -i\,F\,w_1\,\Lambda_1\,e^{-\Lambda_0} \int_t^T e^{-\Lambda_0(u-t)} \times J_v(t,u;i(1+\Lambda_1),i\,\Lambda_1) \times J(u,T;i,0)\,du. \qquad (40)
\end{aligned}
$$

In (39) and (40), the partial derivative $J_v(t,u;\phi,v)$ can be analytically computed as:

$$
J_v(t,u;\phi,v) = \exp\left[\mathcal{Y}(t,u;\phi,v) - \mathcal{Z}(t,u;\phi,v) \times r(t)\right]\left\{\frac{\partial\mathcal{Y}(t,u;\phi,v)}{\partial v} - \frac{\partial\mathcal{Z}(t,u;\phi,v)}{\partial v} \times r(t)\right\}. \qquad (41)
$$

The exact expressions for the partial derivatives $\frac{\partial\mathcal{Y}(t,u;\phi,v)}{\partial v}$ and $\frac{\partial\mathcal{Z}(t,u;\phi,v)}{\partial v}$ are presented in (48) and (49) of the Appendix.

Proposition 3 derives the solution for defaultable coupon bonds under stochastic recovery rates, respectively for the RFV model and the RT model. As equations (39) and (40) demonstrate, the pricing solution can be complex and needs the evaluation of several terms even for the single-factor interest rate case. For this reason, we have not empirically pursued a multi-factor extension although its characteristic function can be derived for a large family of multi-factor interest rate (hazard rate) models. In the Appendix, we present an example where the hazard rate has a three-factor structure and the interest rate is driven by a two-factor structure. The details are available in (50)-(59) of the Appendix. To stay focussed on the modeling issues and the corresponding empirical work, the interest rate is assumed to follow the single-factor interest rate process in equation (34).

Each bond model incorporates a flexible correlation structure between the hazard rate, the interest rate, and the recovery rate. As stressed in Duffie and Singleton (1999), the burden of analytically computing defaultable bond prices can be substantial especially for the RT model. By presenting the characteristic function of the remaining uncertainty, we have resolved this analytical intractability mentioned by Duffie-Singleton. For instance, we do not have to restrict recovery rates to be a constant or assume that the evolution of the hazard rates is independent of the evolution of the risk-free interest rate. In fact, we can introduce additional complexities such as orthogonal jumps and still arrive at closed-form formulations of defaultable debt prices, both for the RFV model and the RT model.

In each model, the role of recovery rate is captured by the pricing terms involving w_0 and w_1. Observe that the constant recovery rate assumption can be obtained by setting $w_1 \equiv 0$ in (32). Accordingly, the last two terms in the pricing formula (39) and (40) are set equal to zero. In applications involving the general model (39) and (40), the structural parameters can be estimated by fitting the model determined prices to the market determined prices. Provided the valuation model is not overly misspecified, the model can be employed to reverse-engineer recovery rates from the market debt prices. Although tedious, the partial derivative of the defaultable bond price with respect to the interest rate and the recovery rate parameters can be determined analytically for a large parametric model class. These sensitivities can be used in hedging interest rate and recovery rate risks.

With the convenience of the solutions (39) and (40), the yield-to-maturity on the defaultable coupon bond, $R(t,T)$, can be determined by solving the non-linear equation:

$$P(t,T) - \int_t^T c(s)\, \exp\left[-R(t,T)\,(s-t)\right]\, ds - F\, \exp\left[-R(t,T)\,(T-t)\right] = 0, \tag{42}$$

which now potentially takes into account both recovery and default considerations.

In the remainder of the paper, we implement parametric defaultable bond models using a sample of BBB-rated corporate bonds. Our empirical investigation provides several new perspectives about recovery and default:

1. Which recovery model is better supported by the market debt data? That is, we evaluate the out-of-sample pricing implications of the RFV, the RT, and the RMV models;[3]

2. What is the quality of market implied recovery rates embedded in individual bond prices? In particular, we assess the stability of the implied recovery rates.

In discussing possible model misspecifications and related recovery estimates, we throughout take the stand that the market fairly prices defaultable coupon bonds.

4 Description of BBB-Rated Corporate Bond Sample

Based on several considerations, we investigate model implications using a cross-section of coupon bonds rated BBB by Standard and Poor's. First, the impact of recovery payout is relatively small

[3]Duffie and Singleton (1999) use a four-factor model in comparing RMV and RFV models. It is important to note that there is no empirical or market test involved in this exercise but a mere comparison of analytical models with one another. Our intent is to provide empirical test that rigorously compare recovery concepts.

on higher-rated corporate bonds. For example, given the low likelihood of default for AA-rated bonds, the recovery-in-default is of a secondary concern. On the other hand, creditors care much more about recovery payout for lower-rated and speculative-grade (high-yield) bonds. However, most high-yield bonds are callable. To match our theoretical developments we have eliminated bonds with option features. We are therefore led to test our defaultable bond models using BBB-rated straight bonds. This data set provides a reasonable environment for judging the performance of defaultable bond models.

For model implementation purposes, the cross-section of treasury prices is also required. To achieve our empirical objectives, the corporate coupon bond data and the treasury STRIPS prices are obtained from Lehman Brothers Fixed Income Database. For each issuing firm, the database contains entries on the month-end flat price, the accrued interest, the maturity date, the annual coupon rate, and the yield-to-maturity. Trader bid quotes on treasury STRIPS are used to estimate the risk-neutralized parameters of the interest rate process. The STRIPS data is comprehensive with 20,173 quotes. Since there are few straight corporate bonds prior to 1989, our corporate bond (and treasury STRIPS) sample spans the nine-year period from January 1989 through March 1998.

Several empirical issues must be addressed at the outset. One, matrix quotes are discarded – only trader bid quotes are employed in our empirical exercises. Two, bond issues with maturity longer than one-year are retained due to their higher liquidity. To facilitate parameter estimation, we only consider firms that have at least four bond issues outstanding per month. We also confine our empirical analysis to bonds that pay semi-annual coupons (only a small number of bonds pay coupons at frequencies other than semi-annual). Lastly, we focus on a sample of 25 BBB-rated straight bonds (to economize on space). The resulting bond sample has 12,228 observations.

Three-month treasury bill rate is adopted as the proxy for the short interest rate $r(t)$. Corporate bonds with remaining maturity between 1 and 5 years are classified as short-term. Similarly bonds with maturity between 5 and 10 (higher than 10) years are classified as medium-term (long-term). None of the debt issues in our sample are secured by any collateral.

For a sample of individual bonds (Delta Airlines, Enron, Federal Express, K-Mart, and Wells Fargo), Table 1 reports the summary statistics for (i) the yield-to-maturity, (ii) the coupon rate, (iii) the bond maturity, and (iv) the number of bonds outstanding. According to this table, Delta Airlines has an average yield-to-maturity of 8.68% with an average coupon rate of 9.59%. Across the entire sample of 25 bonds, the average yield and coupon rate is 7.70% and 8.73%, respectively. As seen the average bond maturity in the sample varies between 5.19 years to 17.86 years, with a sample average of 9.71 years (average standard deviation of 5.76 years). The number of corporate bonds outstanding can vary from a minimum of 4 to a maximum of 13 each month with an average of 6.68 issues across all bonds. This dataset forms the basis for our empirical inquiry on the role of recovery in default modeling.

5 Estimation of Debt Models and Results

Each defaultable bond pricing model requires three sets of parameter estimates: (1) the (risk-neutral) interest rate parameters (κ, θ and σ), (2) the hazard rate parameters (Λ_0 and Λ_1) and (3) the recovery rate parameters (w_0 and w_1). It must be recognized that the hazard and recovery rate parameters are unique to each firm. However, the interest rate parameters are economy-wide. For this reason, we estimate the interest rate parameters using the cross-section of treasury STRIPS (as in Bakshi, Madan, and Zhang (2000)). The same interest rate parameters are employed for discounting cash-flows in the entire cross-section of corporate bonds.

Consider the estimation of the interest rate parameters. In this estimation procedure, we first solve for the parameters that minimize the root-mean-squared percentage pricing errors (one for each month t). The objective function is:

$$\min_{\kappa,\theta,\sigma} \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\overline{B}(t,T_n)-B(t,T_n)}{\overline{B}(t,T_n)}\right)^2} \qquad \forall t, \tag{43}$$

where $\overline{B}(t,T)$ is the market price of the treasury STRIPS and $B(t,T)$ is the corresponding model price (given by the Cox-Ingersoll-Ross model). To account for time-variation in the risk premia, we have allowed the interest rate parameters to vary from one month to the next. This means that we have estimated the parameter time-series $\{\kappa(t),\theta(t),\sigma(t)\,:\,t=1,\ldots,T\}$. Our estimation procedure results in the following average parameter values for the interest rate process: $\kappa=0.48$, $\theta=9.4\%$ and $\sigma=0.31$. These estimates imply a long-run interest rate mean of 9.4% and a long-run interest rate volatility of about 9.5%. Furthermore, the parameter values are in-line with such well-known counterparts as Chan, Karolyi, Longstaff, and Sanders (1992) and Pearson and Sun (1994). Suggestive of parameter stability, the coefficients of variation are each estimated to be lower than 0.5. With an average root-mean-squared error of 1.68%, the square-root model (34) provides reasonably low in-sample errors when fitted to the term structure of interest rates.

Since it must be conducted separately for each issuing firm, the estimation of Λ_0, Λ_1, w_0 and w_1 is admittedly more cumbersome. To be concrete, consider the recovery convention (16) which posits recovery payout to be a fraction of the discounted face value. Let $P(t,T)$ be the theoretical RT model bond price (as displayed in (40)) and let $\overline{P}(t,T)$ be the corresponding market price. Substituting the estimated interest rate parameters into the RT model price, we solve for the remaining structural parameters that fit the individual bond prices (the implicit market valuation process) as closely as the model structure would allow:

$$\text{RMSE}(t) \equiv \min_{\Lambda_0,\Lambda_1,w_0,w_1} \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\overline{P}(t,T_n)-P(t,T_n)}{\overline{P}(t,T_n)}\right)^2} \qquad \forall t. \tag{44}$$

For greater parameter accuracy, the minimization (44) is done only once each quarter by combining all bonds within that quarter (the available monthly cross-section is not sufficiently large). For instance, the first quarterly estimation employs all corporate coupon bonds available in April, May and June of 1989. Finally, the above optimization procedure is repeated for every firm, thereby yielding a (firm-specific) quarterly time-series of $\left\{\Lambda_0(t), \Lambda_1(t), w_0(t), w_1(t) \,:\, t = 1, \ldots, \frac{T}{4}\right\}$. In order to afford each model an equal treatment, the same procedure is applied in estimating the RFV model (and the RMV model).

Table 2 reports the parameter estimates for the RFV model and the RT model (in Panels A and B respectively). An interesting empirical finding is that the estimate of the hazard rate parameter Λ_1 is negative (across all the 25 firms). Specifically from Panel B it can be observed that the mean Λ_1 estimate is -0.145 for Delta Airlines, -0.132 for K-Mart and -0.14 across all the 25 firms (see also Duffee (1999)). Economically a $\Lambda_1 < 0$ implies a negative co-movement between the hazard rate and the interest rate. In a market in which interest rates are trending upwards, one would expect Λ_1 to be negative as higher interest rates tend to increase the earning capacity relative to the cost of debt. The reverse logic applies to periods of declining interest rates: falling interest rates would cause higher default probabilities on account of lower earnings capacity relative to debt cost.

Regardless of the model, Panels A and B of Table 2 indicate that the sample estimate of Λ_0 are positive. This is essential to guarantee positive hazard rates especially since Λ_1 is negative. For the RT model, its reported value ranges between 1.6% to 3.9% with an average Λ_0 of 2.6%. The linear specification of hazard rates is accordingly positive for interest rates below 18.57%. Additionally, for each firm and each model, the estimated hazard rates are strictly positive. Comparing the hazard rate estimates, we can see that Λ_0 is systematically higher for the RFV model, while the Λ_1 estimate from the RT model is slightly more negative. This means that the hazard rate estimate for the RFV model is generally higher than that for the RT model. The documented difference in the hazard rate estimates is a potential source of the differential pricing between the two models.

The recovery rate parameters of equation (32) are also consistent with theoretical predictions. Each for the RFV model and the RT model, the parameters w_0 and w_1 are strictly positive. Furthermore, as hypothesized, $w_0 + w_1$ is strictly less than unity. Overall, Table 2 suggests considerable cross-sectional variations in w_0 and w_1. This finding may suggest that bond prices contain information about expected recovery levels. For the RFV model, the estimate of w_0 varies between 0.135 in the case of K-Mart to 0.248 for Enron. The respective average w_0 and w_1 is 0.24 and 0.245. These estimates imply that a 4% worsening of the hazard rate is associated with a 1% decline in the recovery rates. Irrespective of the firm, the estimated w_0 and w_1 appear higher for the RT model. The discrepancy in the recovery rate parameters are also suggestive of differential pricing and hedging implications between the two recovery conventions. The cross-sectional properties of the implied recovery rates $w(u) = w_0 + w_1 e^{-h(u)}$ will be discussed in subsection 3.4.

Based on the RMSE and the in-sample absolute dollar pricing errors (denoted DPE), we may draw the conclusion that the recovery of treasury model fits the bond data better than its undiscounted counterpart. To illustrate this point, take Delta Airlines as an example. The RFV model has an average RMSE of 2.05% and a DPE of \$1.93 (per \$100 par value), which is larger than the RMSE of 1.57% and \$1.41 obtained with the RT model. Table 2 verifies that in-sample error measures are larger for the RFV model than for the RT model for virtually every firm.

In summary, this subsection points to three principal empirical findings. First, our single-factor parameterization of the hazard rate, the interest rate, and the recovery rates result in theoretical defaultable bond prices that reasonably match market bond prices. Second, the recovery assumption (16) appears to provide a better in-sample fit than assumption (15). Finally, the estimates of recovery and hazard rates are not at odds with theory. Our empirical methods appear sufficiently robust in capturing the first-order effect of recovery embedded in market bond prices.

6 What Concept of Recovery Does the Data Support?

While the in-sample comparison among models is an important guideline for selecting valuation models for marking-to-market purposes, a more stringent test is whether one model outperforms another based on out-of-sample pricing-error measures. To construct such measures, we rely on parameters estimated from bond prices in the previous three months and use them to compute the model determined prices in the following three months. More precisely, the out-of-sample prices in month t (for the RFV and the RT models) are obtained by substituting the prior-month treasury and prior-quarter default parameters and the current market interest rate, in the bond formulas (39) and (40). Finally, subtracting the model determined bond price from the market price produces the dollar pricing error series, or the market price normalized counterpart: the percentage pricing error series. Yield basis point errors are similarly constructed as the difference between the market and the model determined yields (see equation (42)). For each out-of-sample yardstick, we report the mean absolute error, the mean error and the standard deviation of the errors.

To put our empirical results into perspective, Table 3 first contrasts the out-of-sample valuation implications of the RFV model versus the RT model. Essentially our results affirm that the RT model has better out-of-sample performance relative to the RFV model. There are several reasons supporting this conclusion. Let us start with the pricing quality implicit in the yield basis point errors. The first matter of note is that the average absolute yield basis point errors for the RT model is 24.10 bps compared to 28.88 bps for the RFV model (when averaged across all the 25 firms). The second matter of note is the lower overall dispersion of the yield basis point errors for the RT model. As the table documents, the average across-firm standard deviation of the yield basis point errors is 27.03 bps with the RT model compared to 29.60 with the RFV model. In fact, the standard deviation of the yield basis point errors for the RT model is lower across most firms. Similar conclusions can be drawn on the basis of mean yield basis point errors.

Dollar pricing errors confirm the superiority of the RT model from a different angle. Notice that the database scales bond prices to have a par value of \$100, so the reported errors must be interpreted accordingly. Based on the absolute dollar errors, the model difference is also economically significant: the pricing difference translates into \$2200 for every \$1 million notional. In view of the fact that it has lower absolute dollar pricing errors for 23 out of 25 firms, the RT model also consistently outperforms the RFV model. The lower standard deviation of the dollar pricing errors again supports the broader argument that the RT model better tracks market prices and is closer to the market's valuation process. A final inspection of the table reveals that the percentage pricing errors do not materially alter the performance ranking between the two recovery models.

The outperformance of the RT model over the RFV model is also statistically significant. To conduct statistical tests, a representative error series is created by pooling the absolute errors across our bond sample each month. The mean of the difference between the RFV model and the RT model errors, divided by the standard error of the difference of the error series is t-distributed. The null hypothesis that the RFV model's absolute yield basis point errors are equivalent to those from the RT model is rejected with a t-statistic of 7.49. When absolute percentage pricing errors are adopted as the benchmark, we find an equally strong rejection with a t-statistic of 9.27. Our evaluation establishes that the RT and the RFV models are significantly different on both statistical and economic grounds.

Are our conclusions also robust controlling for bond maturity? To address this concern, we sort our bond sample into short-term, medium-term and long-term. Fixing maturity, we aggregate pricing errors across available firms at each month and then in the time-series. Our assertion that the RT model is less misspecified is still valid. For instance, our results verify that for *short-term* *bonds* the average absolute yield basis point errors for the RT model is 26.69 bps, compared to 34.83 bps with the RFV model (not reported in the table). In the case of *medium-term bonds*, a likewise divergence is documented: the RT (RFV) model provides an average absolute yield basis point errors of 27.19 (32.55) bps. According to the t-tests discussed earlier, the RFV model is still rejected in favor of the RT model (the minimum t-statistic across maturities is 7.56). Consequently the recovery of treasury model provides superior valuation irrespective of the bond maturity.

We now compare the pricing quality of a version of the RMV model (Duffie-Singleton (1999)) with the RT and the RFV models. Before we present these results, the assumptions behind the RMV model need some clarifications. In the RMV model price (23), we suppose that $R(t) \equiv \Lambda_0^* + \Lambda_1^* r(t)$, where $r(t)$ obeys the square-root process displayed in (34). In our testing framework, the RFV, the RT and the RMV models all share the single-factor interest rate feature. Estimating the parameters Λ_0^* and Λ_1^* for each firm, we compute the out-of-sample pricing errors. When consolidated across all the firms, we obtained the following average **absolute** pricing errors for the RMV model:

Yield Basis Point Errors (in bps):	All=28.60,	Short=35.81,	Medium=30.34.
Dollar Pricing Errors (in \$):	All=1.67,	Short=0.97,	Medium=1.70.

Comparing these numbers with the corresponding entries in Table 3, we can observe that the single-factor version of the Duffie-Singleton model does worse than the RT model (the t-statistic for the yield basis point errors is higher than 8). In addition, the RMV model is virtually indistinguishable from the RFV model (with t-statistics below 2). The largest discrepancy between the various models emerges for short-term bonds: the yield basis point errors for the RT model, the RFV model, and the RMV model are respectively 26.99, 34.83, and 35.81 bps. For medium-term bonds, the yield basis point errors for the RFV model and the RMV model are 32.55 and 30.34 bps, which are also higher than the RT model's 27.19 bps (and statistically significant). In sum, the RT model is still a first-place performer.

Two additional exercises are carried out to examine the general robustness of the results. When we divided the full sample into two subsamples, (1989:03-1993:12 subsample and the 1994:01-1998:03 subsample) a similar performance ranking was documented. That the RT model has better performance is not an artifact of the sample period considered.[4]

[4]The possible sources of the divergence between the RT and the RFV models warrants some remarks. First, we calculated the model-implied default probabilities and found it to be systematically higher for the RFV model. For example, the average default probability is 2.55% with the RFV model versus 1.94% for the RT model. If the RT model is closer to the market's valuation, then the RFV model appears to overestimate the default probability. Another difference arises due to recovery, which is also overestimated with the RFV model. To be exact, the contribution of the final model pricing term in (14) is 3.53% (of the market price) for the RT model compared to 5.74% with the RFV model. The better performance of the RT model can be possibly attributable to a more accurate assessment of hazard and recovery rates.

To further understand the differences and similarities between different recovery models, we implemented the RFV and RT valuation models under the constant recovery case. Constraining $w_1 \equiv 0$ in the RT model (40), we find that the RT model continues to perform better than the RFV (with $w_1 \equiv 0$ in (39)) and the RMV models.

However, a somewhat surprising result that emerges is that the pricing errors obtained from the stochastic recovery rate are quantitatively similar to those obtained with w_1 set to zero. This seemingly contradictory result can be explained in two ways. First, since the models are estimated each quarter, it is possible that by allowing w_0 to fluctuate from one quarter to the next it implicitly accounts for the time-variation in the recovery rates. In other words, the remaining structural parameters adjust internally so as to bring the model price closer to the market's valuation process as reflected in the market prices. The next reason may be that the variation in individual recovery rates may not be so strongly correlated with default probabilities for BBB-rated bonds. Stronger pricing implications may surface when the time-varying recovery rate model is applied to high-yield bonds.

To summarize, a significant finding from this part of the investigation is that debt data is more supportive of the RT model. This may be because the RT model recognizes time-value of money considerations; its supposition that full recovery is made upon the payment of the face value of debt may be more closer to truth in market valuation. Perhaps less plausibly so, the RFV model asserts that full recovery necessitates the immediate payment of the face value at the default time. Therefore, one interpretation of the lesser misspecification of the RT model is that the defaultable bonds are priced in such a way that the market explicitly incorporates this type of recovery uncertainty in their intrinsic value calculations. As also argued in Longstaff and Schwartz (1995), the recovery in the RT model may be consistent with typical reorganizations in which security holders receive new securities rather than cash for their original claims. Our results validate that the market is not optimistic about immediate cash recovery in default. Time-value considerations are an important determinant of recovery modeling.[5]

[5] One may additionally argue that if the underlying bond to be priced was a par bond and the market sought compensation for all future coupons in present value terms for full recovery, then RFV model is the most appropriate candidate for modeling recovery. This suggests another recovery definition

$$y(u) = w(u) \times B(u,T) \times F \, + \, w(u) \int_u^T B(u,s) \times c(s)\, ds,$$

which can be interpreted as *recovery of outstanding value*. The superiority of the RT model over the RFV model may just be a reflection of low coupons in the sample for which RFV is clearly incorrect when recovery of outstanding value is the right answer, and RT is the better approximation. From the perspective of the recovery of outstanding value model, both models may thus be misspecified. As the pricing formula now needs a double numerical integration for the term involving remaining coupons, this model is considerably more involved for actual implementation purposes. At a heuristic level, none of the bonds in our sample have coupon rates that are small enough to be ignored.

7 Market Implied Recovery Rates

Even when the RFV, the RT, and the RMV, models are comparable along the pricing dimension, understanding recovery rate estimates obtained from the RFV and the RT models is potentially important. As already reasoned, model-specific recovery rates inverted from debt prices can serve as a benchmark for writing claims contingent on recoveries (just like options on VIX, the implied volatility index). Therefore, this subsection addresses two related empirical issues: Are quantitative estimates of (risk-neutral) expected recovery rates in-line with the historical evidence (adjusted for risk aversion)? Which recovery convention is better suited for writing contingent claims? We investigate the stability of implied recovery rates.

Let us first elaborate on the contractual aspects of recovery associated with the RFV, the RT and the RMV models. To do so, consider a generic bond with a \$100 par value. Suppose only \$40 is recovered in the event of default (i.e., y(u)=\$40 in equation (14)). According to the RFV model, the recovery rate, $w(u)$, can be measured in a straightforward manner and is 40%. In contrast, in the Duffie-Singleton definition of recovery, the recovery rate requires the knowledge of the pre-default debt value either from market sources or the model. Even when the RMV model is perfectly specified, the reliance on the pre-default debt value renders this notion of recovery rate unattractive from contractual standpoints. The measurement of the recovery rate in the RT model is relatively less cumbersome but still requires the specification of the yield curve. Therefore, despite the relatively worse out-of-sample performance of the RFV model, it may yet be an attractive choice to measure and manage recovery related risks.

To nonetheless provide an empirical comparison between the RFV and the RT models (the recovery rate is unidentified in the RMV model), the expected recovery rates are backed-out from market bond prices as follows. Focus on the RFV model where $w(t) = w_0 + w_1 \, e^{-h(t)}$ and $h(t) = \Lambda_0 + \Lambda_1 \, r(t)$. The out-of-sample implied recovery rate is then constructed by taking the prior-quarter estimates of w_0, w_1, Λ_0 and Λ_1, in conjunction with the current market interest rate. Table 4 presents the mean, the standard deviation, the maximum and the minimum expected recovery rates. The first point to note is that the RFV model and the RT model provide plausible (risk-neutral) recovery rates that lie between zero and one. The second point that deserves emphasis is that the two models imply different recovery rates. Take the case of Delta, where the implicit average recovery rate is 44% with the RFV model; this departs substantially from 58.5% with the RT model. Table 4 confirms that the average recovery rate is 48.5% for the RFV model compared to 56.7% with the RT model. Our empirical methodology is informative about expected recovery rates.

In particular, these implied estimates are broadly comparable to actual recovery rates. Even though not directly comparable due to the impacts of risk aversion, the calculations of Altman (2001) shows that recoveries for *high-yield bonds* for all seniorities has a weighted-average of \$36.88 per \$100 face value (\$43.77 for senior secured). Similarly, the recovery rates have averaged between 40% and 60% in Weiss (1990) for unsecured debt. Potentially useful in managing recovery risk, our recovery estimates can be employed, among others, to gauge which bonds are subject to lower recovery (within the same credit class). Moreover, these estimates can be adopted as signals for changes in credit rating or setting capital adequacy requirements.

Albeit not stressed earlier, it must be recognized from (15) and (16) that the differences in the recovery rates for the RFV and the RT model have information content on the arrival of default. For the same dollar recovery, $y(u)$, notice that ratio of the RFV recovery rate to the RT recovery rate is the present value of face divided by the face value. At a heuristic level, one may think of this ratio as $\exp(-r(T-u))$ where u represents the default time. For a fixed T, a higher (lower) ratio suggests earlier (distant) default time. Certainly one would expect the recovery rate from the RFV model to be below those from the RT model and this is consistently observed. From the recovery rate estimates in Table 4 and the average maturity displayed in Table 1, a rough approximation indicates an expected arrival of default at 6% interest rate of around 10.97 years for Delta Airlines (i.e., 15.72 - $\frac{1}{0.06}\log(0.440/0.585)$) versus an early default time of 3.09 years for K-Mart. For the entire sample, the relative recovery rate comparison is indicating a default-time of about 7.10 years.

Third, the estimates of the recovery rates appear more time-stable for the RFV model. We can draw this inference by comparing the standard deviation of the recovery rates between the RFV model and the RT model. One possible implication of the higher volatility of the RT model determined recovery rate is that it may be a less desirable contracting choice for writing recovery rate related contingent claims. The RFV convention may be preferable on grounds of simplicity and more reliable recovery estimates.

We should emphasize that market-implied recovery rates are impacted by risk aversion considerations in the change of measure to risk-neutral, and physical assessments of recovery (as elaborated in Proposition 1). Risk-neutral recoveries cross-sectionally are affected by possibly firm-specific differences in counterparty risk aversion and firm-specific moments of the physical recovery density. For instance, it is possible that firms with more left-skewed (physical) recovery densities may have their expected recoveries discounted further. A properly specified risk-neutral recovery rate model may be the key to understanding cross-sectional and time-series variations in credit spreads and market implied recovery rates. More research is therefore needed in the theoretical and empirical modeling of the recovery rates.

8 Conclusions

Under the recommendations set forth by the Basle committee, market participants subject to credit risk should not just evaluate the risk of default but also assess recovery if default occurs. Despite substantial progress in modeling credit risk, existing debt models have only provided a minimal parameterization of recovery. Even less appreciated is which recovery concept is appropriate in default modeling. The relative pros and cons of a recovery concept versus another still remains a less than understood phenomenon at the empirical level. Realizing the potential significance of jointly modeling recovery and default probability, this paper has presented a framework for pricing (and hedging) defaultable securities under alternative recovery payout definitions. Expected recovery is a crucial modeling element in applications ranging from high-yield bonds to sovereign bonds. If the recovery rates are misspecified then the default probabilities can get distorted. In a default context, our theoretical analysis shows that risk-neutral expected recovery rates are related to risk aversion and firm-specific moments of the physical recovery density.

Reconciling prevailing modeling frameworks, we show how to price defaultable coupon bonds when recovery is an exogenous proportion of the face value, or when recovery is a proportion of the discounted face value. The apparent strength of the methodology comes from three modeling traits. First, our theoretical characterizations allow distinct roles for recovery rates and default probabilities. In our parameterizations, recovery rates can depend on default probabilities and other state variables. Second, the models are amenable to analytical solutions for defaultable debt for a wide class of stochastic processes describing the recovery rates, the default probabilities and the risk-free interest rate. Third, unlike Duffie-Singleton (1999) where default-adjusted discount rates subsume the effects of recovery, recovery rates now have identifiable impacts on credit spreads. Hence some of these models can be employed to infer market expectation of recovery levels embedded in the prices of debt instruments. These recovery conventions are also attractive from the perspective of writing recovery contingent claims. Such claims can assist in the efficient allocation of default risks across the economy.

When our parsimoniously parameterized defaultable debt models are tested using a sample of BBB-rated bonds, we find that the recovery specification that relies on discounted face value provides a better fit to the data. Introducing dimensions of time-value in the recovery payout to creditors is found to reduce out-of-sample pricing errors relative to both a version of the Duffie-Singleton (1999) model and the recovery of face value model. Finally, the average recovery rates implicit in defaultable coupon bonds are broadly consistent with empirical studies on unsecured debt. For the purpose of designing recovery contingent claims, we note that defining recovery as a proportion of face has substantially lower standard deviation for market implied recovery rates.

This paper has made a special attempt to model recovery and quantify its impact on defaultable debt.

Appendix: Proof of Proposition 3

We wish to prove the defaultable coupon bond valuation models (39)-(40). Guided by Proposition 2, consider the characteristic function

$$\begin{aligned} J(t,u;\phi,\upsilon) &\equiv E_t^Q\left\{\exp\left(i\,\phi\int_t^u r(s)\,ds + i\,\upsilon\,r(u)\right)\right\}, \\ &= \int \exp\left(i\,\phi\int_t^u r(s)\,ds + i\upsilon\,r(u)\right) q(\nu)\,d\nu, \end{aligned}$$

where $q(\nu)$ is the risk-neutral density $\nu \equiv (\int_t^u r(s)\,ds, r(u))$. With interest rate dynamics (34), the characteristic function satisfies the partial differential equation:

$$\frac{1}{2}\,J_{rr}\,\sigma^2\,r + J_r\,\kappa\,(\theta - r) + J_t + i\,\phi\,r\,J = 0 \tag{45}$$

subject to the boundary condition $J(u,u) = \exp\left(i\,\upsilon\,r(u)\right)$. By direct substitution it can be verified that the solution to (45) is given by $\exp[\mathcal{Y}(t,u) - \mathcal{Z}(t,u)\,\times r(t)]$. Then solving the resulting Ricatti equations produces the closed-form expressions for $\mathcal{Y}(t,u;\phi,\upsilon)$ and $\mathcal{Z}(t,u;\phi,\upsilon)$ displayed in (37) and (38) of the text.

Inserting $h(u) = \Lambda_0 + \Lambda_1\,r(u)$ and $y(u) = \left(w_0 + w_1\,e^{-h(u)}\right) \times F$ into the pricing equation (14) results in (40). In deriving each term in equation (40), we note that

$$\frac{\partial J(t,u;\phi,\upsilon)}{\partial \upsilon} = \int \exp\left(i\,\phi\int_t^u r(s)\,ds + i\upsilon\,r(u)\right)\,i\,r(u)\,q(\nu)\,d\nu. \tag{46}$$

Therefore,

$$\frac{\partial J(t,u;\phi,\upsilon)}{\partial \upsilon} = \exp\left[\mathcal{Y}(t,u;\phi,\upsilon) - \mathcal{Z}(t,u;\phi,\upsilon)\,r(t)\right]\left\{\frac{\partial \mathcal{Y}(t,u;\phi,\upsilon)}{\partial \upsilon} - \frac{\partial \mathcal{Z}(t,u;\phi,\upsilon)}{\partial \upsilon}\,r(t)\right\}. \tag{47}$$

The required partial derivatives can be determined analytically as:

$$\frac{\partial \mathcal{Y}(t,u;\phi,\upsilon)}{\partial \upsilon} = \frac{2\,i\,\kappa\,\theta\,\sinh(\frac{\gamma\,(u-t)}{2})}{\gamma\,\cosh(\frac{\gamma\,(u-t)}{2}) + (\kappa - i\,\upsilon\,\sigma^2)\,\sinh(\frac{\gamma\,(u-t)}{2})}, \tag{48}$$

and,

$$\frac{\partial \mathcal{Z}(t,u;\phi,\upsilon)}{\partial \upsilon} = \frac{2\,\phi\,\sigma^2 + i\,\gamma^2\,\coth^2(\frac{\gamma\,(u-t)}{2}) - i\,\kappa^2}{\left(\gamma\,\coth(\frac{\gamma\,(u-t)}{2}) + \kappa - i\,\upsilon\,\sigma^2\right)^2}. \tag{49}$$

Evaluating (47)-(49) at $\phi = i\,(1 + \Lambda_1)$, $\upsilon = 0$ and $\upsilon = i\,\Lambda_1$ leads to the desired closed-form expressions. All the terms in the defaultable coupon bond models (39)-(40) are therefore in explicit closed-form. □.

An Illustrative Multi-factor Defaultable Bond Valuation Model

In this example setting, we assume that (i) the short interest rate obeys a two-factor process and the hazard rate obeys a three-factor process. We continue to maintain that recovery is of the type (32). For the interest rate dynamics assume that

$$dr(t) = \kappa_r\,(\,m(t) - r(t)\,)\,dt + \sigma_r\,d\omega_r(t), \tag{50}$$

$$dm(t) = \kappa_m\,(\,\mu_m - m(t)\,)\,dt + \sigma_m\,d\omega_m(t), \tag{51}$$

where $m(t)$ represents the long-run stochastic mean interest rate and ω_r and ω_m are correlated standard Brownian motions (Bakshi, Madan, and Zhang (2000) and Collin-Dufresne and Solnik (2001)). Let $\rho_{j,k} \equiv \mathrm{Cov}_t\,(\omega_j, \omega_k)$ for any j and k. For the hazard rate we make the simplifying assumption that

$$h(t) = \Lambda_0\, +\, \Lambda_1\, r(t)\, +\Lambda_2\, m(t) +\, \Lambda_3\, g(t), \tag{52}$$

where $g(t)$ is some firm-specific variable that is governed by an autonomous process, as in:

$$dg(t) = \kappa_g\,(\,\mu_g - g(t)\,)\,dt + \sigma_g\,d\omega_g(t). \tag{53}$$

In this higher-dimensional setting, the characteristic function of the remaining uncertainty is given by:

$$J(t, u-t; \phi, v) \equiv E_t^Q \left\{ \exp\left(-\int_t^u [r(s) + h(s)]\,ds + i\,\phi\, r(u) + i\, v\, g(u) \right) \right\}. \tag{54}$$

We assert that

$$J(t, u-t; \phi, v) = \exp\left[-\mathcal{Y}(t,u) - \mathcal{Z}(t,u) \times r(t) - \mathcal{U}(t,u) \times m(t) - \mathcal{V}(t,u) \times g(t) \right], \tag{55}$$

where:

$$\mathcal{Z}(t,u;\phi,v) \equiv -(i\,\phi + i\,v\,\Lambda_1)\,e^{-\kappa_r\,(u-t)} + \frac{(1+\Lambda_1)}{\kappa_r}\left(1 - e^{-\kappa_r(u-t)}\right), \tag{56}$$

$$\begin{aligned} \mathcal{U}(t,u;\phi,v) \equiv\; & -i\,v\,\Lambda_2\,e^{-\kappa_m(u-t)} + \frac{(1+\Lambda_1+\Lambda_2)}{\kappa_m}\left(1 - e^{-\kappa_m(u-t)}\right) \\ & - \frac{(1+\Lambda_1) + \kappa_r\,(i\,\phi + i\,v\Lambda_1)}{(\kappa_m - \kappa_r)}\left(e^{-\kappa_r(u-t)} - e^{-\kappa_m(u-t)}\right), \end{aligned} \tag{57}$$

$$\mathcal{V}(t,u;v) \equiv -i\,v\,\Lambda_3\,e^{-\kappa_g\,(u-t)} + \frac{\Lambda_3}{\kappa_g}\left(1 - e^{-\kappa_g\,(u-t)}\right), \tag{58}$$

and finally

$$
\begin{aligned}
\mathcal{Y}(t,u;\phi,\upsilon) \;\equiv\;& -i\upsilon\Lambda_0 + \Lambda_0(u-t) - \frac{1}{2}\,\sigma_r^2\int_t^u \mathcal{Z}^2(s)\,ds + \kappa_m\,\mu_m\int_t^u \mathcal{U}(s)\,ds + \kappa_g\,\mu_g\int_t^u \mathcal{V}(s)\,ds \\
& -\frac{1}{2}\,\sigma_m^2\int_t^u \mathcal{U}^2(s)\,ds - \frac{1}{2}\,\sigma_g^2\int_t^u \mathcal{V}^2(s)\,ds - \rho_{g,r}\,\sigma_g\,\sigma_r\int_t^u \mathcal{Z}(s)\,\mathcal{V}(s)\,ds \\
& -\rho_{m,r}\,\sigma_m\,\sigma_r\int_t^u \mathcal{Z}(s)\,\mathcal{U}(s)\,ds. \qquad (59)
\end{aligned}
$$

The bond prices for the RFV and the RT model can now be obtained via Proposition 2 and Proposition 3, and are in analytical closed-form. □

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Table 1: BBB-rated Corporate Bond Sample

This table reports the yield-to-maturity, the coupon, the maturity structure, and the number of bonds outstanding for a sample of 25 BBB-rated corporate bonds. For each variable we display the mean, the standard deviation, the maximum, and the minimum. Included in this bond sample are Amr Corp, Arizona Public, Boise Cascade Corp, Champ International, Csx Corp, Delta Airlines, Enron Corp, Fedex Corp, First Energy, First Interstate, Georgia Pacific, K-Mart, Niagara Mohawk Hld., Oryx Energy, Paine Webber, PECO Energy, Safeway, Tele-Communications, Tenneco, Time Warner, Unicom Corp Hld., United Airlines, Unocal, Usx Corp, and Wells Fargo. These bonds are straight bonds with S&P determined credit rating of BBB or lower. The sample period is March 1989 through March 1998. This sample contains 12,228 observations. The data is collected from Lehman Brothers Fixed Income Database.

		Delta	**Enron**	**Fedex**	**K-Mart**	**Wells Fargo**	**All 25 Firms**
Yield (in %)	**Mean**	8.68	7.41	7.73	8.59	6.92	7.70
	Std.	1.15	1.10	1.43	1.77	0.88	1.12
	Max.	11.02	10.79	10.94	17.53	8.81	8.68
	Min.	5.73	4.10	4.68	3.95	4.10	6.81
Coupon (in %)	**Mean**	9.59	8.65	9.26	9.16	7.70	8.73
	Std.	0.66	1.21	1.17	1.93	0.91	1.00
	Max.	10.37	10.00	10.00	12.50	8.75	9.90
	Min.	8.25	6.45	6.25	7.75	6.12	6.94
Maturity (in years)	**Mean**	15.72	8.43	6.54	17.35	6.23	9.71
	Std.	9.17	4.19	5.23	8.35	2.74	5.76
	Max.	29.98	19.97	19.97	30.02	10.01	17.86
	Min.	1.00	1.00	1.04	1.00	1.00	5.19
No. of Bonds	**Mean**	7.81	7.85	4.50	5.11	5.17	6.68
	Std.	1.99	2.16	0.65	0.52	0.61	1.46
	Max.	10.00	14.00	6.00	6.00	6.00	13.09
	Min.	4.00	4.00	4.00	4.00	4.00	4.10

Table 2: Parameter Estimates for the RFV Model and the RT Model

All parameters are obtained by minimizing the sum-of-squares percentage error between the market price and the model determined price. The criterion function is: RMSE$(t) \equiv \min \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left|\frac{\overline{P}(t,T_n)-P(t,T_n)}{\overline{P}(t,T_n)}\right|^2}$. To ensure a sufficiently large bond cross-section for the estimation, the minimization procedure is performed once every three months. Therefore, we have a time-series of $\{\Lambda_0(t), \Lambda_1(t), w_0(t), w_1(t) : t = 1, \cdots, T\}$. Reported for each structural parameter is the mean, the standard deviation, the maximum and the minimum. The respective averages across all 25 firms are reported under the heading "All 25 firms." The average and standard deviation of the mimimized root-mean-squared error is displayed in the row marked "RMSE." In-sample absolute dollar pricing errors are reported in the row marked DPE. The estimation is done for the **RFV model** (where the recovery is $y(u) = w(u) \times F$), and the **RT model** (where the recovery is $y(u) = w(u) \times B(u,T) \times F$). The model price for the RFV model and the RT model are respectively displayed in equations (37) and (38). Sample period for all estimations is March 1989 through March 1998. The bond sample is described in the note to Table 1.

		Panel A: RFV Model Recovery is $y(u) = w(u) \times F$						**Panel B: RT Model** Recovery is $y(u) = w(u) \times B(u,T) \times F$					
		Delta	**Enron**	**Fedex**	**K-Mart**	**Wells Fargo**	**All 25 Firms**	**Delta**	**Enron**	**Fedex**	**K-Mart**	**Wells Fargo**	**All 25 Firms**
Λ_0	**Mean**	0.039	0.019	0.029	0.034	0.019	0.031	0.031	0.018	0.027	0.031	0.019	0.026
	Std.	0.018	0.007	0.011	0.032	0.007	0.017	0.007	0.006	0.008	0.024	0.005	0.009
	Max.	0.070	0.034	0.051	0.147	0.037	0.067	0.043	0.034	0.042	0.122	0.032	0.039
	Min.	0.005	0.005	0.005	0.005	0.005	0.017	0.020	0.005	0.014	0.005	0.008	0.016
Λ_1	**Mean**	-0.159	-0.134	-0.087	-0.108	-0.128	-0.107	-0.145	-0.115	-0.132	-0.132	-0.132	-0.140
	Std.	0.274	0.131	0.139	0.192	0.058	0.148	0.017	0.075	0.030	0.174	0.047	0.035
	Max.	0.512	0.207	0.416	0.183	0.158	0.081	-0.107	0.152	-0.047	0.052	0.099	-0.115
	Min.	-0.900	-0.594	-0.143	-0.849	-0.143	-0.191	-0.184	-0.162	-0.224	-0.900	-0.148	-0.176
w_0	**Mean**	0.217	0.248	0.255	0.135	0.235	0.240	0.289	0.311	0.291	0.289	0.266	0.279
	Std.	0.080	0.024	0.050	0.068	0.010	0.057	0.120	0.092	0.094	0.148	0.044	0.074
	Max.	0.448	0.366	0.433	0.266	0.244	0.316	0.479	0.482	0.480	0.478	0.479	0.325
	Min.	0.075	0.234	0.235	0.075	0.195	0.135	0.070	0.251	0.075	0.070	0.250	0.237
w_1	**Mean**	0.221	0.254	0.260	0.128	0.246	0.245	0.295	0.318	0.298	0.295	0.273	0.286
	Std.	0.078	0.024	0.048	0.066	0.009	0.056	0.118	0.090	0.092	0.147	0.043	0.072
	Max.	0.447	0.369	0.431	0.270	0.254	0.319	0.482	0.481	0.481	0.481	0.483	0.331
	Min.	0.075	0.240	0.240	0.075	0.211	0.128	0.070	0.259	0.088	0.070	0.258	0.244
RMSE (in %)	**Mean**	2.05	1.46	1.24	2.64	1.06	1.55	1.57	1.28	1.00	2.39	1.00	1.35
DPE (in $)	**Mean**	1.93	1.26	1.01	2.06	0.94	1.31	1.41	1.03	0.82	1.80	0.88	1.11

Table 3: Out-of-Sample Pricing Errors for the RFV and the RT Models

For each bond, we compute the (i) dollar pricing errors, (ii) the percentage pricing errors and (iii) the yield-basis point errors. The dollar pricing errors are computed as the market price minus the model determined price; the percentage pricing errors are dollar pricing errors normalized by the market price. Yield-to-maturity on each bond is computed by solving $0 = P(t,T) - \int_t^T c(s)\, \exp\left[-R(t,T)\,(s-t)\right] ds - F\, \exp\left[-R(t,T)\,(T-t)\right]$, and the yield basis point errors represent the difference between the market-determined-yield and the model-determined-yield. Displayed for each error series is the (i) average absolute errors, (ii) the standard deviation, and (iii) the average error. The respective entire sample averages are reported under "All 25 Firms." The **RFV** model assumes that recovery is a fraction of face; the **RT** model assumes recovery as a fraction of face paid in risk-free bond. In both models, the recovery rate is modeled as $w(u) = w_0 + w_1\, e^{-h(u)}$. The model prices for the RFV model and the RT model are respectively displayed in equations (37) and (38) of proposition 2.

		Panel A: RFV Model $y(u) = w(u) \times F$						Panel B: RT Model $y(u) = w(u) \times B(u,T) \times F$					
		Delta	**Enron**	**Fedex**	**K-Mart**	**Wells Fargo**	**All 25 Firms**	**Delta**	**Enron**	**Fedex**	**K-Mart**	**Wells Fargo**	**All 25 Firms**
Yield Basis Point Errors (in bps)	**Mean Abs.**	37.16	24.29	33.98	27.02	22.02	28.88	29.47	18.29	27.93	24.02	20.70	24.10
	Std.	37.54	27.45	35.23	27.81	24.82	29.60	35.44	20.24	33.85	28.91	24.67	27.03
	Mean	5.33	1.50	12.99	7.44	5.92	8.17	2.24	2.14	4.61	0.67	4.60	3.95
Dollar Pricing Errors (in \$ for \$100 par)	**Mean Abs.**	2.64	1.49	1.45	2.18	1.03	1.65	2.19	1.16	1.23	1.99	0.98	1.43
	Std.	2.88	1.68	1.50	2.32	1.25	1.76	2.71	1.34	1.46	2.36	1.20	1.65
	Mean	-0.04	0.11	-0.30	-0.23	-0.20	-0.19	0.01	-0.03	-0.19	0.09	-0.18	-0.10
Percentage Pricing Errors (in %)	**Mean Abs.**	2.46	1.39	1.32	2.20	1.00	1.55	2.05	1.08	1.12	2.01	0.95	1.35
	Std.	2.71	1.57	1.38	2.43	1.20	1.67	2.56	1.25	1.35	2.43	1.16	1.55
	Mean	0.02	0.11	-0.26	-0.20	-0.19	-0.17	0.03	-0.04	-0.15	0.05	-0.17	-0.09

Table 4: Market Implied Recovery Rates

This table reports the recovery rates by backing-out the relevant recovery rate parameters from the market price of bonds. Out-of-sample recovery rates are computed (each quarter) by computing $w(t) = w_0 + w_1\, e^{-h(t)}$, where $h(t) = \Lambda_0 + \Lambda_1\, r(t)$. The estimates of Λ_0, Λ_1, w_0, and w_1 are taken from estimations done in quarter t-1. Reported is the average, the standard deviation, the maximum, and the minimum recovery rate. The respective averages across all 25 firms is shown under "All 25 Firms." The **RFV** model assumes that dollar recovery, y, is $y(u) = w(u) \times F$, while the **RT model** assumes that $y(u) = w(u) \times B(u, T) \times F$. The estimation methodology is as described in table 2.

		Delta	**Enron**	**Fedex**	**K-Mart**	**Wells Fargo**	**All 25 Firms**
RFV Model	**Mean**	0.440	0.505	0.516	0.253	0.482	0.485
	Std.	0.156	0.048	0.097	0.124	0.019	0.109
	Max.	0.893	0.733	0.862	0.538	0.500	0.629
	Min.	0.150	0.476	0.476	0.150	0.405	0.253
RT Model	**Mean**	0.585	0.634	0.591	0.595	0.541	0.567
	Std.	0.239	0.184	0.186	0.297	0.088	0.148
	Max.	0.964	0.966	0.961	0.963	0.960	0.654
	Min.	0.140	0.512	0.164	0.140	0.509	0.482