CONSUMPTION AND ASSET PRICES WITH RECURSIVE PREFERENCES

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ABSTRACT. We analyze consumption and asset pricing with recursive preferences given by Kreps–Porteus stochastic differential utility (K–P SDU). We show that utility depends on two state variables: current consumption and a second variable (related to the wealth–consumption ratio) that captures all information about future opportunities. This representation of utility reduces the internal consistency condition for K–P SDU to a restriction on the second variable in terms of the dynamics of a forcing process (consumption, the state–price deflator, or the return on the market portfolio). Solving the model for (*i*) optimal consumption, (*ii*) the optimal portfolio, and (*iii*) asset prices in general equilibrium amounts to finding the process for the second variable that satisfies this restriction. We show that the wealth–consumption ratio is the value of an annuity when the numeraire is changed from units of the consumption good to units of the consumption process, and we characterize certain features of the solution in a non-Markovian setting. In a Markovian setting, we provide a solution method that it quite general and can be used to produce fast, accurate numerical solutions that converge to the Taylor expansion.

1. INTRODUCTION

We solve for the dynamics of consumption, investment, and asset prices in a general-equilibrium, continuous-time stochastic model with a representative agent who has recursive preferences. The setting varies and determines what the main problem is. In an endowment economy, the dynamics of consumption is given and we solve for asset prices (the *exchange problem*); in a partial equilibrium setting, prices are given and we solve for the optimal consumption and investment plan (the *planning problem*); in a production economy, a set of linear technologies is given and we solve for consumption, investment and asset prices (the *production problem*). By focusing on the consumption–wealth ratio, we find that these three problems are essentially equivalent, and we solve them all at once.

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The recursive utility framework generalizes the standard time-separable power utility model, allowing the separation of risk aversion and intertemporal substitution. This framework was introduced by Epstein and Zin (1989), who analyze recursive preferences in a discrete-time setting, and Duffie and Epstein (1992b), who develop a continuous-time formulation of Epstein and Zin's class of recursive utility called stochastic differential utility. We use a martingale approach to solve for the equilibrium, along the lines of Duffie and Skiadas (1994), who show that the first-order condition for optimality is equivalent to the absence-of-arbitrage conditions for asset prices—namely, that asset prices deflated by the state-price deflator are martingales. In addition, they provide a representation for the state-price deflator for the Kreps–Porteus stochastic differential utility (K–P SDU) that we adopt here.

We find that utility depends on two state variables: current consumption and a second variable (the *growth variable*) that captures all information about future opportunities. This representation of utility depends on the homotheticity of K–P SDU, and holds for the exchange problem as well as in economies with linear investment opportunities (covering both the case of the planning problem and that of the production problem). Equilibrium in the model reduces to a central restriction on the growth variable in terms of the dynamics of a forcing process. This forcing process can be either consumption (for the exchange problem), the real state–price deflator (for the planning problem), the return on the market portfolio (for the production problem), or something entirely different (for example, the state-price deflator expressed in an arbitrary numeraire). Solving the model for (*i*) optimal consumption, (*ii*) the optimal portfolio, and (*iii*) asset prices amounts to finding the process for the growth variable that satisfies this restriction.

Unless the elasticity of intertemporal substitution is unity, we can replace the growth variable with the wealth–consumption ratio. The homogeneity properties of the representative agent's planning problem (homothetic preferences and linear technology) ensure that optimal consumption is proportional to wealth. We show that the optimal wealth–consumption ratio is the value of an annuity when the numeraire has been changed from units of the consumption good to shares in the consumption process. Thus, the wealth-consumption ratio is the value of an asset. As such, it must obey a standard absence-of-arbitrage condition.

As a practical matter, the model is solved when we know how to obtain, analytically or numerically, an expression for the consumption-wealth ratio that satisfies this condition. It is then straightforward to obtain expressions for the rate of interest and the price of risk—determined by the dynamics of the so-called *state-price deflator*—and other variables of interest. In order to focus on the role of preferences, it is convenient, in the spirit of Lucas (1978) (as well as Mehra and Prescott (1985) and Weil (1989)), to start with the exchange problem, in which the forcing process is consumption and we solve for the supporting prices, *i.e.*, the state-price deflator. For the planning problem, we reverse the process, solving for the optimal consumption and investment plans using the state-price deflator as the forcing process. Finally, in the spirit of Cox, Ingersoll, Jr., and Ross (1985a) and Campbell (1993), we model technology, which we interpret as the return on the optimally invested wealth of the representative consumer. For this production problem, then, we solve for consumption and prices using the return on the market portfolio as the forcing process.

The choice of forcing process is not limited to the three already mentioned, namely consumption, the state-price deflator and the return on optimally invested wealth. It turns out, for example, that with the forcing process chosen as the product of the state-price deflator and wealth (which can be interpreted as the state-price deflator when the numeraire is shares in the wealth process) the model can be solved algebraically. Although this choice does not correspond to any natural setting, it opens the path to generating examples with arbitrarily complex dynamics.

With a natural forcing process, it is not possible in general to find a consumption-wealth ratio that satisfies the no-arbitrage condition, but progress is achieved by modeling the dynamics of the forcing process as driven by a finite set of Markovian state variables. In such a Markovian setting the no-arbitrage condition becomes a partial differential equation (PDE) that we wish to solve for the wealth-consumption ratio as a function of the state variables (and time). Because this ratio is an annuity, its value is that of an integral of bond prices. In some circumstances, standard methods deliver exact solutions (numerically at least and sometimes even analytically) to the bond pricing problem, and we get the wealth-consumption ratio by numerical integration. In all other cases, we attack the annuity PDE directly and provide an approximate solution method that is quite general and can be used to produce fast, accurate numerical solutions that converge to the Taylor expansion of the exact solution.

Much like standard bond pricing methods, our general solution method transforms the PDE into a set of simultaneous ordinary differential equations (ODE) when the horizon is finite. A unique solution is guaranteed to exist, but only for horizons that are sufficiently short. We solve the infinite-horizon problem by extending the finite horizon and taking a limit. Such a limit does not necessarily exist, but when it does, it is the solution of a set of algebraic equations.

Related work. As noted above, Duffie and Epstein (1992b) and Duffie and Skiadas (1994) lay the groundwork for continuous-time modeling of recursive preferences. Schroder and Skiadas (1997) extend the earlier work in a number of important ways. They prove existence and uniqueness of solutions and address the relation between the first-order conditions and optimality in a more general non-Markovian setting than has been treated previously, and we refer the reader to their paper regarding these issues.[1] They also provide some closed-form solutions to the planning problem in special cases that we also consider below.

Duffie and Epstein (1992a) derive the representation for risk premia in the setting we adopt here. Both Duffie and Epstein (1992a) and Duffie, Schroder, and Skiadas (1997) solve one-factor models of the term structure in the special case where the

[1]By contrast, we propose a method that delivers candidate solutions for continuation utility and consumption. A martingale property must be checked for our proposed continuation utility to be valid, and the consumption process is only guaranteed to satisfy the first-order condition for optimality. Schroder and Skiadas (1997) establish the sufficiency of the first-order condition for some, but not all, parameter values.

dynamics of the state variable are introduced through the growth rate of consumption. Among other things, these papers address the how a change in the coefficient of relative risk aversion affects the shape of the yield curve.

Campbell (1993) linearizes the discrete-time model of Epstein and Zin (1991), and derives an approximate solution to the model in the homoskedastic case that is exact in the special case. We derive more general conditions under which important aspects of Campbell's solution are essentially exact, providing insight into the performance of his approximate solutions. In addition, we examine the approximate relations Campbell describes between the volatility of a perpetuity and the price of risk. Campbell's model is used by Campbell and Viceira (1996) to study the planning problem.

Outline. In Section 2, we adopt a non-Markovian setting to analyze the structure of the model. We introduce the utility function (K–P SDU), for which we derive a two-state-variable representation in the context of the exchange problem, thereby simplifying the model's central restriction. The state variables independently capture the level and growth features of the endowment process. The wealth–consumption ratio depends in a simple way on the growth variable; the relation is one-to-one except in the case of unit elasticity of intertemporal substitution. We demonstrate that this ratio is the value of an annuity after the numeraire has been changed from units of consumption good to shares in the endowment itself. We then address the planning problem, for which the dynamics of the state–price deflator are given, and we show that after changing the level variable from consumption to wealth and presenting the problem as an exercise in dynamic programming, our representation for utility satisfies the envelop condition identically. Next we turn to general equilibrium where we model technology and derive the restriction on the growth variable with respect to those dynamics. Finally we address the finite-horizon problem explicitly.

In Section 3 we investigate the features of solutions to the model that can be inferred in the non-Markovian setting of Section 2. We unify the three restrictions on the growth variable in terms of the dynamics of a generic forcing variable. In the case of unit elasticity of intertemporal substitution combined with homoskedasticity, we show that the growth variable is a weighted average of expected future growth rates of the forcing variable. For other elasticities, we rely on the fact that the wealth–consumption ratio is an asset price (the value of an endowment annuity) to investigate the model. We show that the weak form of the expectations hypothesis as applied to the endowment term structure delivers useful results. In addition, we examine a number of limiting cases regarding the preference parameters.

In Section 4, we present the PDEs that characterize the solution to the model in a Markovian setting. We solve by integration of bond prices when bond pricing methods deliver exact solutions, and we describe our generic power series solution method through a sequence of examples. We also provide an example that illustrates that (i) that a solution may fail to exist for an arbitrary finite horizon and (ii) that, even when a solution exists for all finite horizons, it does not convergence as the horizon goes to infinity.

In Section 5 we summarize briefly the contribution of this paper, and we discuss how the tools we provide can be brought to bear on asset pricing puzzles in future research. In a future version of this paper, we intend to include a numerical investigation to illustrate our method. In the meantime, we have included in an Appendix a complete *Mathematica* package that implements all aspects of our method.

2. The structure of the model

Tastes: Stochastic differential utility. We now introduce the preferences of the representative agent, for which we adopt Kreps–Porteus stochastic differential utility (SDU). We present a value function for Kreps–Porteus SDU that is valid for the entire parameter space. Using this value function and the general representation for the SDU gradient given by Duffie and Skiadas (1994), we obtain an explicit representation for the state-price deflator. We derive expressions for the interest rate and the price of risk in terms of this representation.

As explained by Duffie and Epstein (1992a) and Duffie and Epstein (1992b), SDU (not just the Kreps–Porteus specification) can be represented by a pair of functions $(\bar{f}, \bar{A})$ called an *aggregator*.[2] The functions $\bar{f}$ and $\bar{A}$ can be interpreted as capturing separately attitudes toward intertemporal substitution and attitudes toward risk. Hypothetical experiments can be conducted, for example, by fixing $\bar{f}$ and varying $\bar{A}$ to study the effect of increasing risk aversion. Associated with the aggregator, there is a process $\bar{\mathcal{V}}(t)$, called *continuation utility*, such that the value of the consumption plan $\{c(t) \mid t \geq 0\}$ is $\bar{\mathcal{V}}(0)$. When SDU is well-defined, the process for $\bar{\mathcal{V}}$ is uniquely given by

$$d\bar{\mathcal{V}}(t) = \left(-\bar{f}(c(t), \bar{\mathcal{V}}(t)) - \frac{1}{2}\,\bar{A}(\bar{\mathcal{V}}(t))\,\|\sigma_{\bar{\mathcal{V}}}(t)\|^2\right) dt + \sigma_{\bar{\mathcal{V}}}(t)^\top dW(t),$$

for some $\sigma_{\bar{\mathcal{V}}}(t)$.

To represent a given SDU, the aggregator is not unique. Importantly, there exists a normalized form (f, A) where $A \equiv 0$. Significant analytical simplification is achieved by using the normalized aggregator, although the convenient separation referred to above is lost since both aspects of preferences are combined in f. Suppose that we define $\mathcal{V}(t) := \Upsilon(\bar{\mathcal{V}}(t))$, where $\Upsilon(v)$ is a twice-continuously differentiable and strictly increasing transformation. Since only the ordinal properties of utility are of interest, the change of variables has no effect on choices, but it changes the form of the aggregator (through Itô's lemma). If we choose Υ to satisfy $\Upsilon''(v) - \bar{A}(v)\,\Upsilon'(v) = 0$, then the new aggregator is (f, A), where $A = 0$ and $f(c, v)$ is defined implicitly in $f(c, \Upsilon(z)) = \Upsilon'(z)\,\bar{f}(c, z)$. With the new aggregator, we have

$$d\mathcal{V}(t) = \mu_{\mathcal{V}}(t)\,dt + \sigma_{\mathcal{V}}(t)^\top dW(t), \tag{2.1}$$

where

$$\mu_{\mathcal{V}}(t) = -f(c(t), \mathcal{V}(t)), \tag{2.2}$$

[2] Our notation reverses the roles of (f, A) and $(\bar{f}, \bar{A})$.

Whenever two processes $c(t)$ and $\mathcal{V}(t)$ satisfy (2.2), then $\mathcal{V}(t)$ is the process for continuation utility corresponding to the consumption plan $c(t)$.[3] Using (2.1) and (2.2), we can also express recursive utility as

$$\mathcal{V}(t) = E_t\left[\int_t^T f(c(s), \mathcal{V}(s))\, ds\right]. \tag{2.3}$$

We are interested in both finite- and infinity-horizon settings. For the infinite horizon, we take the limit of (2.3) as $T \to \infty$. The Riesz representation of the utility gradient for such preferences is given by:[4]

$$\mathcal{G}(t) := \exp\left\{\int_{s=0}^t f_v(c(s), \mathcal{V}(s))\, ds\right\} f_c(c(t), \mathcal{V}(t)), \tag{2.4}$$

where f_c and f_v are the partial derivatives of f.[5] Optimality of consumption requires that the utility gradient be proportional to the state–price deflator. (See Appendix A for a discussion of the state–price deflator.) Note that the relative dynamics of the utility gradient are given by $d\mathcal{G}/\mathcal{G} = df_c/f_c + f_v\, dt$.

One of the aggregators for Kreps–Porteus SDU is[6]

$$\bar{f}(c, v) = \frac{\theta\, v\,((c/v)^{1-1/\eta} - 1)}{1 - 1/\eta} \quad \text{and} \quad \bar{A}(v) = \frac{-\gamma}{v}, \tag{2.5}$$

where θ, η, and γ are constant parameters. As shown by Duffie and Epstein (1992a), these preferences allow a disentangling of attitudes toward risk from attitudes toward intertemporal substitution. In our parameterization, $\theta > 0$ is the rate of time preference, $\eta > 0$ is the elasticity of intertemporal substitution and $\gamma \geq 0$ is the coefficient of relative risk aversion. When $\gamma\,\eta = 1$, Kreps–Porteus SDU specializes to standard time-separable preferences with power utility, characterized by indifference toward the timing of resolution of uncertainty. (With $\eta\,\gamma > 1$, the consumer prefers early resolution and with $\eta\,\gamma < 1$, late resolution). That $\gamma\,\eta = 1$ reduces to the case of standard preferences is more easily seen in terms of the utility gradient using the normalized aggregator, which we derive below.

Duffie and Epstein (1992b) show that preferences are homothetic if and only if there is an ordinally equivalent aggregator (f, A) satisfying (i) f is homogeneous of degree 1 and (ii) A is linearly homogeneous of degree -1. The aggregator $(\bar{f}, \bar{A})$ given in (2.5) clearly satisfies these conditions. To normalize the aggregator into the canonical form $(f, 0)$, use the transformation

$$\Upsilon(v) = \frac{v^{1-\gamma} - 1}{1 - \gamma}, \tag{2.6}$$

[3]Duffie and Lions (1992) address the existence and uniqueness of $\mathcal{V}$ when c is modeled in terms of state variables (where c itself may be a state variable).

[4]Duffie and Epstein (1992a) derive a Markovian version of (2.4) using the Bellman equation, while Duffie and Skiadas (1994) derive (2.4) in a more general non-Markovian semimartingale setting.

[5]Marginal utility in the direction of the consumption process q is given by $E_t\left[\int_{s=t}^T \mathcal{G}(s)\, q(s)\, ds\right]$. See Duffie (1996) for a discussion of the utility gradient.

[6]The functional form of f comes from Duffie and Epstein (1992a), p. 418. Their ρ is $1 - 1/\eta$, their α is $1 - \gamma$, and their β is θ.

producing

$$f(c,v) = \frac{\theta\, V \left(\left(\frac{c}{V^{1/(1-\gamma)}} \right)^{1-1/\eta} - 1 \right)}{1 - 1/\eta}, \quad V := 1 + (1-\gamma)\, v. \tag{2.7}$$

For each of the cases $\gamma = 1$ and $\eta = 1$, the aggregator follows from taking a limit in (2.7).[7] Duffie and Epstein (1992a) use $\Upsilon(v) = v^{1-\gamma}/(1-\gamma)$. As a result we have $1 + (1-\gamma)\, v$ where they have $(1-\gamma)\, v$ in their normalized aggregator (p. 420). The advantage of our formulation is that we get the correct limit for $\gamma = 1$. Although $\Upsilon(0) \neq 0$ for our transformation, it will turn out that V is always positive for all positive values of γ and η.

Continuation utility, the utility gradient, and the exchange problem. The first problem we face is that of finding a representation for continuation utility and the utility gradient of an agent with K-P SDU and a given consumption process. This is the usual setting of an endowment economy. In this setting, the famous tree metaphor clarifies some discussion, so we imagine for the time being that the endowment grows on identical trees, so normalized that in the current period each tree produces one unit of the good. In the current period, then, c denotes both current consumption and the number of trees.

We now establish an important result. For a consumer with K-P SDU preferences, we can represent the state of the world with only two state variables: the level of current consumption, $c(t)$, and another variable, $\psi(t)$, that captures—conditional on current consumption—all relevant information about future growth opportunities for consumption, *i.e.*, about the productivity of a tree. As such, $\psi(t)$ depends on the dynamics of log consumption but not its current value. We can write the value of continuation utility as a deterministic function of the two state variables: $\mathcal{V}(t) = g(c(t), \psi(t))$. Note that we have two ways to measure the increase in utility from an increase in consumption: $f_c(c, g(c,\psi))$ and $g_c(c,\psi)$. The first expression indicates the marginal utility of a unit of current consumption (holding the future value of continuation utility fixed), while the second indicates the marginal utility of a tree. The ratio $g_c(c,\psi)/f_c(c, g(c,\psi))$ is the marginal rate of substitution between trees and current consumption. Given the homotheticity of preferences, this ratio must be independent of current consumption.

Define the function

$$h(\psi) := \frac{g_c(c,\psi)}{f_c(c, g(c,\psi))}. \tag{2.9}$$

Since $\pi(t) := h(\psi(t))$ is the value of a tree expressed in terms of consumption good and $c(t)$ is the number of trees (as normalized in period t), we may interpret $k(t) := \pi(t)\, c(t)$ as the evaluation the consumer would give of his wealth, given his endowment, even in the absence of a market. Given the form of the normalized

[7] We take a particular interest in the case $\eta = 1$, which yields

$$f(c,v) = \theta\, V \left(\log(c) - \frac{1}{1-\gamma} \log(V) \right). \tag{2.8}$$

aggregator, $h(\psi)$ is independent of c if and only if $1+(1-\gamma)\,g(c,\psi)$ is homogeneous of degree $1-\gamma$ in c, that is $g(c,\psi) = (c^{1-\gamma}\,\widehat{g}(\psi) - 1)/(1-\gamma)$. Now, we have considerable freedom in the definition of ψ, because, at the cost of changing the form of $\widehat{g}$, we can replace ψ by any other variable that is in one-to-one relation with ψ. The simplest choice is to define ψ so that $\widehat{g}(\psi) = \psi$, but this choice complicates future expressions and is hard to interpret. Choosing $\widehat{g}(\psi) = \psi^{1-\gamma}$ achieves the greatest simplification of future expressions, but our choice is slightly different and easier to interpret. By setting $\widehat{g}(\psi) = (\psi/\theta)^{\eta\,(1-\gamma)}$, it will turn out that ψ is the marginal utility of wealth corresponding to the unnormalized aggregator $(\bar{f}, \bar{A})$ (this aggregator is characterized by constant returns to scale, so that marginal utility of wealth is independent of wealth or current consumption). As a result, we have[8]

$$g(c,\psi) = \frac{(c\,(\psi/\theta)^{\eta})^{1-\gamma} - 1}{1-\gamma} \tag{2.10a}$$

$$\pi = h(\psi) = \theta^{-\eta}\,\psi^{\eta-1}. \tag{2.10b}$$

Using (2.10a), we can write

$$f(c, g(c,\psi)) = -\,(c\,(\psi/\theta)^{\eta})^{1-\gamma}\left\{\frac{\eta\,\theta\,\left((\psi/\theta)^{1-\eta} - 1\right)}{1-\eta}\right\}. \tag{2.11}$$

Note that

$$\lim_{\eta\to 1}\frac{\eta\,\theta\,\left((\psi/\theta)^{1-\eta} - 1\right)}{1-\eta} = \theta\,\log(\psi/\theta).$$

The partial derivatives $f_c(c,v)$ and $f_v(c,v)$ evaluated at $v = g(c,\psi)$ are given by

$$f_c(c, g(c,\psi)) = \theta^{\gamma\,\eta}\,\psi^{1-\gamma\,\eta}\,c^{-\gamma} \tag{2.12a}$$

$$f_v(c, g(c,\psi)) = -\theta - \left(\frac{1}{\eta} - \gamma\right)\left\{\frac{\eta\,\theta\,\left((\psi/\theta)^{1-\eta} - 1\right)}{1-\eta}\right\}. \tag{2.12b}$$

We obtain the Riesz representation of the utility gradient by inserting (2.12) into (2.4). With standard preferences, $\eta = 1/\gamma$, the utility gradient specializes to $\mathcal{G}(t) = \theta\,e^{-\theta\,t}\,c(t)^{-\gamma}$ as expected. Another benchmark case is $\gamma = 1$, for which $f_c(c, g(c,\psi)) = \theta^{\eta}\,\psi^{1-\eta}\,c^{-1}$ and $f_v(c, g(c,\psi)) = \theta^{\eta}\,\psi^{1-\eta} = \pi^{-1}$.

Thus far in our discussion of g and ψ, we have not distinguished between infinite- and finite-horizon problems. The finite horizon imposes the boundary condition $\pi(T) = 0$. This condition cannot be met when $\eta = 1$ with the normalization adopted. At the end of this section, we will explicitly address the finite-horizon problem. For the present, we treat only the infinite-horizon case.

A necessary condition for $g(c(t),\psi(t))$ to be the continuation utility corresponding to the endowment, equation (2.2) requires that the drift of $g(c(t),\psi(t))$ equal $-f(c(t), g(c(t),\psi(t)))$. This requirement produces the consistency condition that ψ

[8]As asserted above, $V = 1 + (1-\gamma)\,g(c,\psi) = (c\,(\psi/\theta)^{\eta})^{1-\gamma}$ is always positive.

must satisfy in order that $g(c(t), \psi(t))$ be continuation utility:

$$\frac{\eta\,\theta\,\big((\psi(t)/\theta)^{1-\eta}-1\big)}{1-\eta} = \widetilde{\mu}_c(t) + \eta\,\widetilde{\mu}_\psi(t) + (1-\gamma)\,\frac{1}{2}\,\|\sigma_c(t)+\eta\,\sigma_\psi(t)\|^2, \qquad (2.13)$$

where all the terms are implicitly defined by[9]

$$\begin{aligned} d\log(c(t)) &= \widetilde{\mu}_c(t)\,dt + \sigma_c(t)^\top dW(t) \\ d\log(\psi(t)) &= \widetilde{\mu}_\psi(t)\,dt + \sigma_\psi(t)^\top dW(t). \end{aligned}$$

Our choice of g has allowed us to cancel $\big(c(t)\,(\psi(t)/\theta)^\eta\big)^{1-\gamma}$ from both sides of (2.13). If ψ solves equation (2.13), then g as given in (2.10a) is continuation utility, provided $\int_{s=0}^t \sigma_{\mathcal{V}}(s)^\top dW(s)$ is a martingale, where

$$\sigma_{\mathcal{V}}(t) = \big(c(t)\,(\psi(t)/\theta)^\eta\big)^{1-\gamma}\,\big\{\sigma_c(t) + \eta\,\sigma_\psi(t)\big\}.$$

(See Proposition 3 in Schroder and Skiadas (1997) for a proof.) Thus, whenever a solution to the underlying problem exists, the solution to (2.13) provides it. However, the solution to (2.13) does not provide the solution to the underly problem unless the volatility of g is well behaved.

We now seek the interest rate and price of risk that support the endowment. To support a consumption plan, prices must be aligned with marginal rates of substitution, which in the present context means that the state-price deflator $m(t)$ must be colinear with the utility gradient $\mathcal{G}(t)$. In accord with (A.1), we note (from applying Itô's lemma to $\mathcal{G}(t)$) that in this case the short rate r and the price of risk λ are given by

$$r(t) = \theta + \frac{1}{\eta}\,\widetilde{\mu}_c(t) + \frac{(1-\gamma)\,(1-\gamma\,\eta)}{\eta}\,\frac{1}{2}\,\|\sigma_c(t)+\eta\,\sigma_\psi(t)\|^2 - \frac{1}{2}\,\|\lambda(t)\|^2 \qquad (2.14a)$$

$$\lambda(t) = \gamma\,\sigma_c(t) + (\gamma\,\eta - 1)\,\sigma_\psi(t), \qquad (2.14b)$$

where we have used (2.13) to eliminate the term in curly brackets from $f_v(c, g(c,\psi))$ in (2.12b). As a by-product, $\widetilde{\mu}_\psi$ has been eliminated as well from the expression for the interest rate. Consequently, σ_ψ is the only aspect of ψ we will need for asset pricing. Note that σ_ψ enters the price of risk with a sign that depends on whether early or late resolution of uncertainty is preferred. Again, with $\eta = 1/\gamma$, (2.14) specializes to the expected expressions for r and λ under the C–CAPM: $r(t) = \theta + \gamma\,\widetilde{\mu}_c(t) - \frac{1}{2}\,\gamma^2\,\|\sigma_c(t)\|^2$ and $\lambda(t) = \gamma\,\sigma_c(t)$. This is consistent with Theorem 2(a) (under condition I) in Schroder and Skiadas (1997).

Using the utility gradient as state-price deflator, we can price assets. For example, an asset that pays a continuous dividend at the rate of one unit of good per year (a

[9] We use the following notational convention. If $z(t)$ is explicitly strictly positive, then μ_z, $\widetilde{\mu}_z$ and σ_z refer to the quantities implicitly defined in $dz(t)/z(t) = \mu_z(t)\,dt + \sigma_z(t)^\top dW(t)$ and $d\log(z(t)) = \widetilde{\mu}_z(t)\,dt + \sigma_z(t)^\top dW(t)$, implying $\widetilde{\mu}_z(t) := \mu_z(t) - \frac{1}{2}\,\|\sigma_z(t)\|^2$. The state variables $X(t)$ in section 4 are not necessarily positive. For these variables, we write $dX(t) = \mu_X(t)\,dt + \sigma_X(t)^\top dW(t)$, so that $\mu_X(t)$ refers to the drift of X (in level) and σ_X to its volatility.

real consol) is valued at

$$E_t\left[\int_{s=t}^{\infty}\frac{\mathcal{G}(s)}{\mathcal{G}(t)}\,ds\right]=\int_{s=t}^{\infty}p(t,s)\,ds, \tag{2.15}$$

where $p(t,s)=E_t[\mathcal{G}(s)/\mathcal{G}(t)]$ is the value at time t of a zero-coupon bond that pays one unit of consumption at time s. We can also find the value of the endowment, which can be interpreted as the consumer's wealth:

$$k(t)=E_t\left[\int_{s=t}^{\infty}\frac{\mathcal{G}(s)}{\mathcal{G}(t)}\,c(s)\,ds\right]. \tag{2.16}$$

We note that the right-hand side of (2.16) is marginal utility in the direction of the endowment. Earlier, we claimed that wealth is given by $k(t)=h(\psi(t))\,c(t)=\theta^{-\eta}\,\psi(t)^{\eta-1}\,c(t)$, where $h(\psi):=g_c(c,\psi)/f_c(c,g(c,\psi))$. The two notions of wealth are, of course, the same. To see this, let $\nabla\mathcal{V}(c,c',t)$ be the Gateaux derivative of $\mathcal{V}(t)$ evaluated at the endowment and in the direction of process $c'(t)$. Then, our definition of a tree implies that $g_c(c(t),\psi(t))=\nabla\mathcal{V}(c,c,t)$, and the desired result follows from the Riesz representation of $\nabla\mathcal{V}(c,c,t)$ given in Duffie and Skiadas (1994) (note that with equation (2.12) substituted in, (2.4) is the Riesz representation of $\nabla\mathcal{V}(c,c,0)$).

It is instructive to examine this result from a slightly different angle. To do this, it is convenient to define $\pi(t):=k(t)/c(t)$, where $k(t)$ is defined by (2.16), so that we need to show that $\pi(t)=h(\psi(t))$. Dividing both sides of (2.16) by $c(t)$ produces

$$\pi(t)=E_t\left[\int_{s=t}^{\infty}\frac{\mathcal{G}_e(s)}{\mathcal{G}_e(t)}\,ds\right], \tag{2.17}$$

where $\mathcal{G}_e(t)=\mathcal{G}(t)\,c(t)$. Equation (2.17) shows that $\pi(t)$ is the value of a consol (an *endowment consol*) where the state–price deflator is given by $\nabla\mathcal{G}_e(t)$. Formally, $\mathcal{G}_e(t)$ is the state–price deflator where the numeraire has been changed from units of the consumption good to units of the endowment process. (See Appendix A for a discussion of changing numeraires.) Given (2.17) we can write $\pi(t)=\int_{s=t}^{T}p_e(t,s)\,ds$, where $p_e(t,s)=E_t[\mathcal{G}_e(s)/\mathcal{G}_e(t)]$ is the value at time t of a zero-coupon bond that pays one unit of the endowment at time s.

Let the dynamics of $\pi(t)$ be given by $d\pi(t)/\pi(t)=\mu_\pi(t)\,dt+\sigma_\pi(t)^\top dW(t)$. Because $\pi(t)$ is the value of an asset (when measured in endowment units), the drift of π will be determined by the martingale property of deflated gains:

$$\mu_\pi(t)+\frac{1}{\pi(t)}=r_e(t)+\lambda_e(t)^\top\sigma_\pi(t), \tag{2.18}$$

where $r_e(t)$ and $\lambda_e(t)$ follow from applying Itô's lemma to $\mathcal{G}_e(t)$:

$$r_e(t)=r(t)-\left(\widetilde{\mu}_c(t)+\frac{1}{2}\,\|\sigma_c(t)\|^2\right)+\lambda(t)^\top\sigma_c(t)\,, \tag{2.19a}$$

$$\lambda_e(t)=\lambda(t)-\sigma_c(t). \tag{2.19b}$$

Substituting (2.14) into (2.19) produces

$$r_e(t) = \theta + \frac{1-\eta}{\eta}\left\{\widetilde{\mu}_c(t) + (1-\gamma)\,\frac{1}{2}\,\|\sigma_c(t)\|^2 - \eta\,(1-\gamma\,\eta)\,\frac{1}{2}\,\|\sigma_\psi(t)\|^2\right\} \tag{2.20a}$$

$$\lambda_e(t) = (\gamma-1)\,\sigma_c(t) + (\gamma\,\eta-1)\,\sigma_\psi(t). \tag{2.20b}$$

For $\eta = 1$, $r_e(t) = \theta$. With this constant interest rate, (2.18) implies $\pi(t) = 1/\theta$ as asserted in (2.10b). For $\eta \neq 1$, the assertion in (2.10b) that $\pi(t) = \theta^{-\eta}\,\psi(t)^{\eta-1}$ implies

$$\mu_\pi(t) = (\eta-1)\,\widetilde{\mu}_\psi(t) + \frac{1}{2}\,\|(\eta-1)\,\sigma_\psi(t)\|^2 \tag{2.21a}$$

$$\sigma_\pi(t) = (\eta-1)\,\sigma_\psi(t). \tag{2.21b}$$

Substituting (2.20) and (2.21) into (2.18) produces (2.13), which establishes the internal consistency of our assertion.

Terminology. For lack of better terms, we will refer to r_e and λ_e as the endowment interest rate and the endowment price of risk, respectively, to distinguish them from the real interest rate and real price of risk, r and λ. As will become evident, these constructs have important applications beyond an endowment economy.

Optimal consumption and portfolio choice: The planning problem. In this section, we consider the problem of the optimal investment of wealth. When the agent has the recursive preferences assumed here, this problem is analyzed by Campbell and Viceira (1996) in a discrete-time setting, and by Duffie and Epstein (1992a) and Schroder and Skiadas (1997) in a continuous-time setting.

In the previous section, the consumption process was given, and so there was no question of the optimality of consumption. Current opportunities were given by current consumption and future opportunities were determined by the dynamics of consumption. Nevertheless, we were able to find wealth—the value of the endowment process. In this section, we change perspective: Consumption is no longer given exogenously. Current opportunities are given by current wealth, $k(t)$, and future opportunities are determined by the state–price deflator (as reflected in asset prices) and current consumption which decreases the amount to invest. The second state variable, ψ, summarizes all relevant information about future opportunities as reflected in the dynamics of the state–price deflator, namely the interest rate, $r(t)$, and the price of risk, $\lambda(t)$. In this setting, we will solve for the optimal consumption and investment plans.

The investment opportunity set can be characterized by n risky securities with dynamics of the form

$$\frac{d\phi_i(t)}{\phi_i(t)} = \mu_{\phi_i}(t)\,dt + \sigma_{\phi_i}(t)^\top dW(t). \tag{2.22}$$

The expected return on security i is determined by the absence-of-arbitrage condition

$$\mu_{\phi_i}(t) = r(t) + \sigma_{\phi_i}(t)^\top \lambda(t). \tag{2.23}$$

(The dynamics of the risky assets reflect the reinvestment of any dividends paid.) In addition there is the money-market account (MMA):

$$\frac{d\beta(t)}{\beta(t)} = r(t)\, dt.$$

A portfolio can be characterized by a vector of weights, $\alpha(t)$, for the risky securities and a weight $\alpha_0(t)$ for the MMA, such that $\sum_{i=0}^{n} \alpha_i = 1$. Let Σ_ϕ be the matrix whose i-th column is σ_{ϕ_i} and define $M_\phi := (\mu_{\phi_1}, \ldots, \mu_{\phi_n})^\top$ and $\alpha := (\alpha_1, \ldots, \alpha_n)^\top$. The value of a portfolio evolves as follows:

$$\frac{d\phi(t)}{\phi(t)} = \alpha_0(t)\, \frac{d\beta(t)}{\beta(t)} + \sum_{i=1}^{n} \alpha_i(t)\, \frac{d\phi_i(t)}{\phi_i(t)} = \mu_\phi(t)\, dt + \sigma_\phi(t)^\top dW(t),$$

where (using (2.23))

$$\mu_\phi(t) = \alpha_0(t)\, r(t) + M_\phi(t)^\top \alpha(t) = r(t) + \lambda(t)^\top \sigma_\phi(t)$$

and

$$\sigma_\phi(t) = \Sigma_\phi(t)\, \alpha(t). \tag{2.24}$$

Wealth evolves according to

$$dk(t) = k(t)\, \frac{d\phi(t)}{\phi(t)} - c(t)\, dt. \tag{2.25}$$

On the optimal path, continuation utility is given by (2.10a) and the utility gradient by (2.12a), where ψ must satisfy (2.13). Optimality of the consumption process requires that the utility gradient be aligned with the state–price deflator: $\mathcal{G}(t) = \alpha\, m(t)$, for some positive constant α, so that equations (2.14) are satisfied. We assume here that this first-order condition is also sufficient for the optimality of the solution. This is an important caveat, because sufficiency has not been proved for all parameter values. The most complete results so far can be found in Schroder and Skiadas (1997). Using these equations to eliminate $\widetilde{\mu}_c(t)$ and $\sigma_c(t)$ from (2.13), we can reduce these optimality conditions to the following restriction:

$$\begin{aligned} \theta + \frac{\theta\left((\psi(t)/\theta)^{1-\eta} - 1\right)}{1-\eta} = \left\{ r(t) + \frac{1}{2}\, \|\lambda(t)\|^2 \right\} + \widetilde{\mu}_\psi(t) \\ + \left(\frac{1-\gamma}{\gamma}\right) \frac{1}{2}\, \|\lambda(t) + \sigma_\psi(t)\|^2. \end{aligned} \tag{2.26}$$

The solution to (2.26) for ψ given r and λ can then be used in (2.14) to solve for the dynamics of optimal consumption, $\widetilde{\mu}_c(t)$ and $\sigma_c(t)$. Hereafter, we identify the utility gradient with the state–price deflator (ignoring the constant of proportionality).

The next step is to show that the dynamics of optimal consumption imply the dynamics of the optimal portfolio. Since $\phi(t)$ is the value of an asset, its drift is determined by its volatility (conditional on r and λ):

$$\mu_\phi(t) = r(t) + \lambda(t)^\top \sigma_\phi(t). \tag{2.27}$$

Thus the portfolio problem is reduced to solving for $\sigma_\phi(t)$. To establish the link between ϕ and c we can use (2.25) and $k(t) = \pi(t)\, c(t)$ to produce

$$\phi(t) = c(t)\,\pi(t)\,\exp\left(\int_{s=0}^{t} \pi(s)^{-1}\, ds\right). \tag{2.28}$$

Applying Itô's lemma to (2.28) and matching drifts and diffusions yields[10]

$$\widetilde{\mu}_\phi(t) = \widetilde{\mu}_c(t) + \widetilde{\mu}_\pi(t) - \frac{1}{\pi(t)} \tag{2.29a}$$

$$\sigma_\phi(t) = \sigma_c(t) + \sigma_\pi(t). \tag{2.29b}$$

Recall that $\pi(t) = \theta^{-\eta}\,\psi(t)^{\eta-1}$, so that $\sigma_\pi(t) = (\eta-1)\,\sigma_\psi(t)$. Together with (2.14b), we have established[11]

$$\sigma_\phi(t) = \left(\frac{1}{\gamma}\right)\lambda(t) + \left(\frac{1-\gamma}{\gamma}\right)\sigma_\psi(t). \tag{2.30}$$

Any solution α to $\sigma_\phi(t) = \Sigma_\phi(t)\,\alpha(t)$ is a solution to the portfolio problem. The matrix Σ_ϕ is $l \times n$, where l is the number of Brownian motions that determine the state of information and n is the number of linear activities. For simplicity, we assume that Σ_ϕ is of full rank. If $l > n$, it is impossible to hedge against all sources of risk in the economy. In this case, if necessary through a Choleski decomposition of the Brownian motions that redefines the matrix Σ_ϕ and all volatilities but keeps all covariances unchanged, one can assume without loss of generality that $\Sigma_\phi^\top = \left(\Sigma_\phi^{*\top}\; 0\right)$, where Σ_ϕ^* is an $n \times n$ invertible matrix. In terms of the new Brownian motions, we have $\sigma_\phi^\top = \left(\sigma_\phi^{*\top}\; {\sigma'_\phi}^\top\right)$. If $\sigma'_\phi \neq 0$, the optimal investment problem has no solution. So we assume $\sigma'_\phi = 0$. If $n > l$, then it is possible to drop activities that are not needed in the optimal portfolio, keeping only l such activities. By renumbering activities if necessary, we can write $\Sigma_\phi = \left(\Sigma_\phi^*\; 0\right)$ and we set $\sigma_\phi^* = \sigma_\phi$.[12] We can now write the solution in the general case as

$$\alpha^*(t) = \Sigma_\phi^{*-1}(t)\,\sigma_\phi^*(t) = \left(\frac{1}{\gamma}\right)\Sigma_\phi^{*-1}(t)\,\lambda^*(t) + \left(\frac{1-\gamma}{\gamma}\right)\Sigma_\phi^{*-1}(t)\,\sigma_\psi(t) \tag{2.31}$$

using (2.30). In this equation, λ^* corresponds to the n first components of λ (under the new Brownian motions) if $l > n$ and is equal to λ otherwise.[13] This result is consistent with Theorem 2(c) (under conditions I and II) in Schroder and Skiadas (1997).

[10]Note that $\mu_\phi(t) = \widetilde{\mu}_\phi(t) + \frac{1}{2}\,\|\sigma_\phi(t)\|^2$.

[11]This expression has appeared in Campbell and his citations.

[12]Some activities may slip in and out of the optimal portfolio. At the expense of some notation for keeping track of which activities are in the optimal portfolio at time t, we do not need to assume that the set of activities in the portfolio never changes.

[13]Defining $\lambda^*(t) = \Sigma_\phi^{*-1}(t)^\top (M_\phi(t) - r(t))$, we can eliminate λ^* from (2.31):

$$\alpha^*(t) = \left(\frac{1}{\gamma}\right)\Sigma_\phi^{*-1}(t)\,\Sigma_\phi^{*-1}(t)^\top (M_\phi(t) - r(t)) + \left(\frac{1-\gamma}{\gamma}\right)\Sigma_\phi^{*-1}(t)\,\sigma_\psi(t).$$

We see that when $\gamma = 1$, $\alpha(t) = \Sigma_\phi^{*\,-1}(t)\,\lambda^*(t)/\gamma$. This component is the so-called "myopic" component of portfolio demand, in the terminology of Campbell and Viceira (1996). The other component constitutes a hedge against changes in investment opportunities. The first component is easily found without knowledge of the consumption plan, but evaluation of the second component requires such knowledge (through $\psi(t)$).

The production problem. In a representative-agent general equilibrium, we interpret $k(t)$ as the value of the capital stock and $d\phi(t)/\phi(t)$ as the return on the aggregate investment portfolio—*i.e.*, the return on the market portfolio. At this level of analysis, we ignore the portfolio allocation problem, except to require zero net investment in the money-market account, treating this as an economy with a single investment opportunity. We can think of $\phi(t)$ itself as the value of a portfolio where the consumption "dividends" are continuously reinvested. In this role, we refer to ϕ as the *capital account.* If we wish, we may think of this economy as a production economy, where the return on the capital account is the result of linear production technology subject to random shocks as in Cox, Ingersoll, Jr., and Ross (1985a).

The state variables in this case are $k(t)$ and $\psi(t)$, where $\psi(t)$ impounds information regarding the dynamics of technology. In this case, we need to ensure that the interest rate and the price of risk be properly related to the dynamics of technology. We can achieve this by using (2.27) and (2.30) to eliminate r and λ from (2.26):

$$\theta + \frac{\theta\left((\psi(t)/\theta)^{1-\eta} - 1\right)}{1-\eta} = \widetilde{\mu}_\phi(t) + \widetilde{\mu}_\psi(t) + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\phi(t) + \sigma_\psi(t)\|^2. \qquad (2.32)$$

Having solved (2.32) for ψ given $\widetilde{\mu}_\phi$ and σ_ϕ we can use (2.27) and (2.30) to solve for r and λ and then use (2.14) to solve for $\widetilde{\mu}_c$ and σ_c.

Relation to stochastic control. The traditional approach to solving the consumption–investment problem is to apply the stochastic control method. In this approach, we assume we can write optimal consumption (the policy function) and optimized utility (the value function) in terms of the state variables: $c(t) = \mathcal{C}(k(t), \psi(t))$ and $\mathcal{V}(t) = j(k(t), \psi(t))$, where $j(k, \psi) := g(\mathcal{C}(k, \psi), \psi)$. In our case, the envelop condition, $j_k = f_c$, delivers the form of the policy function. Given the definition of j, we have $j_k = \mathcal{C}_k\, g_c$, and so the envelop condition implies $\mathcal{C}_k = f_c/g_c = 1/h$ as established in (2.9). We conclude that $\mathcal{C}(k, \psi) = k/h(\psi)$. Therefore we can write[14]

$$\mathcal{C}(k, \psi) = \theta^\eta\, \psi^{1-\eta}\, k \quad \text{and} \quad j(k, \psi) = \frac{(k\,\psi)^{1-\gamma} - 1}{1-\gamma}. \qquad (2.33)$$

Once we have the form of the policy function, Bellman's *principle of optimality* in essence turns the stochastic control problem into a "recursive utility" problem for the value function:

$$\mu_{\mathcal{V}}(t) = -f^*(k(t), \psi(t)), \quad \text{where } f^*(k, \psi) := f(\mathcal{C}(k, \psi), g(\mathcal{C}(k, \psi), \psi)) \qquad (2.34)$$

[14]Closely related to $\mathcal{C}$ and j in (2.33) are the functions (A.1) and (A.2) in Giovannini and Weil (1989).

and $\mu_{\mathcal{V}}(t)$ is the drift of $j(k(t), \psi(t))$. If we take ϕ as the forcing variable (for example), then (2.34) is equivalent to (2.32).

Finite horizon and $\eta \neq 1$. When the horizon is finite, the wealth–consumption ratio, $\pi(t)$, is the value of an annuity, which goes to zero as the horizon goes to zero. We can rewrite the absence-of-arbitrage condition for $\pi(t)$ given in (2.18) as

$$\bar{\mu}_\pi(t) + 1 = r_e(t)\,\pi(t) + \lambda_e(t)^\top \bar{\sigma}_\pi(t), \qquad \text{subject to } \pi(T) = 0. \tag{2.35}$$

We have expressed the dynamics of the π in (2.35) in *absolute* terms: $d\pi(t) = \bar{\mu}_\pi(t)\,dt + \bar{\sigma}_\pi(t)^\top dW(t)$. Given (2.10b), the boundary condition requires ψ to depend on the horizon:

$$\lim_{t \to T} \psi(t) = \begin{cases} 0 & \text{if } \eta > 1 \\ \infty & \text{if } \eta < 1. \end{cases}$$

Even though ψ behaves badly as the horizon approaches, π itself is well-behaved and, as long as $\eta \neq 1$, we can use (2.35) as the restriction to solve for π and then we can find ψ by inverting $\pi = \theta^{-\eta}\,\psi^{\eta-1}$. Therefore, for $\eta \neq 1$, we can simply reinterpret the equations from our previous analysis of the infinite horizon problem. The case $\eta = 1$ is more complicated.

Finite horizon and $\eta = 1$. For $\eta = 1$, the boundary condition $\pi(T) = 0$ cannot be satisfied without changing the way c enters $g(c, \psi)$. For this case we define[15]

$$G(c, \psi, \tau) := g\left(c^{q(\tau)}, \psi\right), \quad \text{where } q(\tau) := 1 - e^{-\theta\,\tau}.$$

Note that $G(c, \psi, \infty) = g(c, \psi)$. Using G, we have

$$\pi = H(\tau) = \frac{G_c(c, \psi, \tau)}{f_c(c, G(c, \psi, \tau))} = \frac{q(\tau)}{\theta}, \tag{2.36}$$

which agrees with Theorem 2(b) (under condition II) in Schroder and Skiadas (1997). Using $G(c(t), \psi(t), T - t)$ in place of $g(c(t), \psi(t))$, the restriction $\mu_{\mathcal{V}} + f = 0$ becomes

$$\theta\,\log(\psi(t)/\theta) = q(T-t)\,\widetilde{\mu}_c(t) + \widetilde{\mu}_\psi(t) + (1-\gamma)\,\frac{1}{2}\,\|q(T-t)\,\sigma_c(t) + \sigma_\psi(t)\|^2, \tag{2.37}$$

subject to $\psi(T) = \theta$. Note that (2.13) as $\eta \to 1$ and (2.37) as $T \to \infty$ converge to the same restriction. The partial derivatives of $f(c, v)$ evaluated at $v = G(c, \psi, \tau)$ are

$$f_c(c, G(c, \psi, \tau)) = \theta^\gamma\,\psi^{1-\gamma}\,c^{-Q(\tau)} \tag{2.38a}$$

$$f_v(c, G(c, \psi, \tau)) = -\theta - \theta\,(1-\gamma)\left\{\log(\psi/\theta) - e^{-\theta\,\tau}\,\log(c)\right\}, \tag{2.38b}$$

[15] The expression c^q can be seen in the term structure example in Duffie and Epstein (1992a) for example.

where $Q(\tau) = \gamma + (1-\gamma)\, e^{-\theta\,\tau}$. With standard preferences, $\gamma = 1$, the utility gradient specializes to $\mathcal{G}(t) = \theta\, e^{-\theta\, t}\, c(t)^{-\gamma}$ as expected. In the general case, the interest rate and price of risk are given by

$$r(t) = \theta + \widetilde{\mu}_c(t) + \left(\frac{1}{2} - Q(T-t)\right) \|\sigma_c(t)\|^2 + (1-\gamma)\, \sigma_c(t)^\top \sigma_\psi(t) \tag{2.39a}$$

$$\lambda(t) = Q(T-t)\, \sigma_c(t) + (\gamma - 1)\, \sigma_\psi(t), \tag{2.39b}$$

where (as before) we can have used $\mu_\mathcal{V} + f = 0$ to eliminate both ψ and $\widetilde{\mu}_\psi$ from the interest rate. Again, with $\gamma = 1$, (2.39) specializes to the expected expressions for r and λ under the C–CAPM: $r(t) = \theta + \widetilde{\mu}_c(t) - \frac{1}{2}\, \|\sigma_c(t)\|^2$ and $\lambda(t) = \sigma_c(t)$. To reaffirm the internal consistency, note that substituting (2.39) into (2.19) produces $r_e(t) = \theta$. With this constant interest rate, (2.35) implies $\pi(t) = q(T-t)/\theta$ as asserted by (2.36).

Comparing (2.14) with (2.39), a discontinuity is evident for the finite-horizon case. Suppose $\widetilde{\mu}_c$ and σ_c are constant, so that $\sigma_\psi = 0$. For $\eta \neq 1$ the price of risk is constant at $\gamma\, \sigma_c$, while for $\eta = 1$ the price of risk depends on the horizon, moving toward σ_c as $t \to T$. As we show below in the context of a specific planning problem, the consumption process (viewed as a function of the preference parameters) is discontinuous at $\eta = 1$ whenever the data are continuous, but only in the finite-horizon case. In the infinite-horizon, the discontinuity disappears and in any case, the wealth-consumption ratio $\pi(t)$ is continuous.

Turning to optimal consumption and optimal portfolio, we can use (2.39) to eliminate $\widetilde{\mu}_c$ and σ_c from (2.37) produces

$$\theta\, \log(\psi(t)/\theta) = \widetilde{\mu}_\psi(t) + q(T-t) \left(r(t) + \frac{1}{2}\, \|\lambda(t)\|^2 - \theta\right) + \frac{1-\gamma}{Q(T-t)}\, \frac{1}{2}\, \|q(T-t)\, \lambda(t) + \sigma_\psi(t)\|^2. \tag{2.40}$$

In this case, the optimal portfolio weights must satisfy

$$\sigma_\phi(t) = \left(\frac{1}{Q(T-t)}\right) \lambda(t) + \left(\frac{1-\gamma}{Q(T-t)}\right) \sigma_\psi(t). \tag{2.41}$$

Finally, using (2.27) and (2.41) to eliminate r and λ from (2.40), we have

$$\theta\, \log(\psi(t)/\theta) = \widetilde{\mu}_\phi(t) + q(T-t)\, (\widetilde{\mu}_\psi(t) - \theta) + (1-\gamma)\, \frac{1}{2}\, \|q(T-t)\, \sigma_\phi(t) + \sigma_\psi(t)\|^2. \tag{2.42}$$

3. Analysis of the model in a non-Markovian setting

Thus far, we have considered three forcing processes: consumption, the state–price deflator, and technology, each within a specific context: endowment economy, optimal consumption in a partial equilibrium, and production economy. Having established the equilibrium relationships among all three processes, we are now free to model whichever forcing process we find convenient and (based on the solution for ψ related to that process) infer the dynamics of the other two processes. For example, we can choose consumption as the forcing process and infer the dynamics of technology that would generate that consumption process.

It is useful to recognize that there are other ways in which one can introduce dynamics into the model (in addition to the three choices listed above). We may in fact start by choosing ψ and it dynamics. By itself, this is insufficient to solve the model (in the sense of being able to determine the dynamics of consumption, etc.). Examining (2.32), for example, we see that if we also model σ_ϕ, then (2.32) determines $\widetilde{\mu}_\phi$ and we can in fact solve the model for the other processes of interest. This approach is tantamount to modeling the deflated value of wealth, $m_k(t) = m(t)\,k(t)$, which can be interpreted as a state–price deflator after a change of numeraire:[16] Indeed, from the dynamics of m and k given in (2.14) and (2.25), we see that

$$\frac{dm_k(t)}{m_k(t)} = -\theta^\eta\,\psi(t)^{1-\eta}\,dt + (1-\gamma)\,(\sigma_\psi(t) + \sigma_\phi(t))^\top dW(t),$$

where we have used (2.30) to eliminate λ from the volatility of m_k. Instead of modeling σ_ϕ in conjunction with ψ, we can model either λ or σ_c. The corresponding restriction can then be solved for r or $\widetilde{\mu}_c$. Although this approach to modeling the dynamics may seem unnatural, it has the advantage of offering immediate solutions for equilibrium relationships among variables of interest. Nevertheless, one may choose to model one of the three forcing processes, in which case a solution for ψ must be found.

The challenge to using K–P SDU preferences is simply finding the process ψ that solves the proper restriction identically given a forcing process. In this section, we will go as far as we can in a non-Markovian setting. Below we will have more to say on this subject after we adopt a Markovian structure. Before proceeding, though, it is convenient to unify all three solutions. To that end, we denote the forcing variable y and its dynamics $d\log(y(t)) = \widetilde{\mu}_y(t)\,dt + \sigma_y(t)^\top dW(t)$, where y is either consumption (c), the capital account (ϕ), and the inverse of the state–price deflator ($1/m$).

$y(t)$	$\widetilde{\mu}_y(t)$	$\sigma_y(t)$	d_0	d_1	d_2	d_3	$d_4(\tau)$	$\varepsilon = d_1 + d_2$
$c(t)$	$\widetilde{\mu}_c(t)$	$\sigma_c(t)$	θ	$\frac{1}{\eta} - 1$	$1-\gamma$	0	1	$\frac{1}{\eta} - \gamma$
$m(t)^{-1}$	$r(t) + \frac{1}{2}\,\|\lambda(t)\|^2$	$\lambda(t)$	$\eta\,\theta$	$1-\eta$	$\frac{1}{\gamma} - 1$	-1	$Q(\tau)$	$\frac{1}{\gamma} - \eta$
$\phi(t)$	$\widetilde{\mu}_\phi(t)$	$\sigma_\phi(t)$	$\eta\,\theta$	$1-\eta$	$1-\gamma$	-1	1	$2-\eta-\gamma$

TABLE 1. The coefficients of Equations (3.2) and (3.11) in terms of the preference parameters. $Q(\tau) = \gamma + (1-\gamma)\,e^{-\theta\,\tau}$.

[16] The interest rate—measured in units of wealth—is the consumption–wealth ratio, $1/\pi$.

Unit elasticity of intertemporal substitution and infinite horizon. Given $\eta = 1$, each of (2.37), (2.40), and (2.42) can be written, for finite T, as

$$\widetilde{\mu}_\psi(t) = \theta \log(\psi(t)/\theta) - q(T-t)\,(\widetilde{\mu}_y(t) + d_3\,\theta) - \frac{(1-\gamma)}{2\,d_4(T-t)}\,\|q(T-t)\,\sigma_y(t) + \sigma_\psi(t)\|^2, \quad (3.1)$$

where d_3 and $d_4(\tau)$ are given in Table 1. For $T = \infty$, (3.1) specializes to

$$\widetilde{\mu}_\psi(t) = \theta \log(\psi(t)/\theta) - \widetilde{\mu}_y(t) - d_2\,\frac{1}{2}\,\|\sigma_y(t) + \sigma_\psi(t)\|^2 - d_3\,\theta, \quad (3.2)$$

where d_2 is given in Table 1. In this case, as long as $C = -d_2\,\frac{1}{2}\,\|\sigma_y(t)+\sigma_\psi(t)\|^2 - d_3\,\theta$ is constant, the solution to (3.2) is

$$\psi(t) = \theta\,\exp(\zeta(t) - C/\theta), \quad (3.3)$$

from which it follows that

$$\widetilde{\mu}_\psi(t) = \mu_\zeta(t) \quad \text{and} \quad \sigma_\psi(t) = \sigma_\zeta(t), \quad (3.4)$$

where $d\zeta(t) = \mu_\zeta(t)\,dt + \sigma_\zeta(t)^\top dW(t)$. Substituting (3.3) and (3.4) into (3.2) produces the restriction that ζ must satisfy:

$$\mu_\zeta(t) = \theta\,\zeta(t) - \widetilde{\mu}_y(t). \quad (3.5)$$

Now define $\zeta(t)$ as a weighted average of expected growth rates of the forcing process y:

$$\zeta(t) := \int_{s=t}^{\infty} \theta\,e^{-\theta\,(s-t)} \left(\int_{u=t}^{s} E_t[\widetilde{\mu}_y(u)]\,du \right) ds. \quad (3.6)$$

Itô's lemma applied to (3.6) delivers (3.5) and

$$\sigma_\psi(t) = \sigma_\zeta(t) = \int_{s=t}^{\infty} \theta\,e^{-\theta\,(s-t)} \left(\int_{u=t}^{s} \widehat{\sigma}_{\widetilde{\mu}_y}(t,u)\,du \right) ds, \quad (3.7)$$

where $\widehat{\sigma}_{\widetilde{\mu}_y}(t,s)$ is the volatility of $E_t[\widetilde{\mu}_y(s)]$.

C is constant under either of two conditions. The first is $\gamma = 1$, which (with $\eta = 1$) produces log utility. The second condition obtains when (i) $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$ is a deterministic function of $u - t$, so that $\sigma_\zeta(t)$ and, therefore, $\sigma_\psi(t)$ is constant, and (ii) $\sigma_y(t)$ is constant. The most important feature of the solution is the expression for $\sigma_\psi(t) = \sigma_\zeta(t)$ given in (3.7), because it is $\sigma_\psi(t)$ that contributes to risk premia.[17]

Non-unit elasticity of intertemporal substitution. We turn to analyzing the model when $\eta \neq 1$.

[17] Campbell (1993) derives a similar result for the case of $y = \phi$ in a discrete-time version of this model, where he refers to $\|\sigma_\psi\|^2$ as "news about the discounted value of all future market returns."

The utility gradient in terms of observables. As long as $\eta \neq 1$, we can replace the unobservable ψ with the observable ϕ in the utility gradient, following Epstein and Zin (1991). Using (2.28) with $\pi(t) = \theta^{-\eta}\,\psi(t)^{\eta-1}$, we can solve for

$$\psi(t) = \exp\left(\int_{s=t}^{t} \frac{\theta^\eta\,\psi(s)^{1-\eta}}{1-\eta}\,ds\right)\theta^{-\eta/(1-\eta)}\,c(t)^{1/(1-\eta)}\,\phi(t)^{-1/(1-\eta)}.$$

Substituting this expression for $\psi(t)$ into (2.12a) produces

$$m(t) = \mathcal{G}(t) = \theta^\delta\,e^{-\theta\,\delta\,t}\,c(t)^{-\delta/\eta}\,\phi(t)^{\delta-1}, \tag{3.8}$$

where the parameter δ is defined by

$$\delta := \frac{1-\gamma}{1-1/\eta}.$$

Applying Itô's lemma directly to (3.8) yields a convenient representation for the short rate $r(t)$ and the price of risk $\lambda(t)$ in terms of the observable dynamics of the growth rate of consumption and the return on the market portfolio ϕ:

$$r(t) = (\delta/\eta)\,\widetilde{\mu}_c(t) + (1-\delta)\,\widetilde{\mu}_\phi(t) + \theta\,\delta - \frac{1}{2}\,\|\lambda(t)\|^2, \tag{3.9a}$$

$$\lambda(t) = (\delta/\eta)\,\sigma_c(t) + (1-\delta)\,\sigma_\phi(t). \tag{3.9b}$$

Note that $\delta = 1$ (*i.e.*, $\eta\,\gamma = 1$) delivers standard preferences and the C–CAPM. By contrast $\delta = 0$ (*i.e.*, $\gamma = 1$) delivers an intertemporal CAPM, where risk premia are determined by the covariance with the market portfolio. These loci are plotted in Figure 1 (where γ is plotted on the vertical axis against η on the horizontal axis), along with $\eta = 1$ and a fourth locus that we will discuss after some discussion of the consol equation.

The annuity equation. When $\eta \neq 1$, we can make progress by focusing on solving (2.35), which describes $\pi(t)$ as the price of an annuity and which we repeat here as

$$\bar{\mu}_\pi(t) + 1 = r_e(t)\,\pi(t) + \lambda_e(t)^\top\bar{\sigma}_\pi(t), \tag{3.10}$$

subject to $\pi(T) = 0$, where $r_e(t)$ and $\lambda_e(t)$ are given in (2.19).

By modeling r_e and λ_e directly, we can solve for "endowment bond" prices and hence π. On the other hand, we may choose instead to model the dynamics of either ϕ or c or m directly. Based on equations (2.19) and (3.9), and using (2.28) and $k(t) = \pi(t)\,c(t)$, we can write[18]

$$r_e(t) = d_0 + d_1\,\widetilde{\mu}_y(t) + d_1\,d_2\,\frac{1}{2}\,\|\sigma_y(t)\|^2 + (\varepsilon/d_1)\frac{1}{2}\left\|\frac{\bar{\sigma}_\pi(t)}{\pi(t)}\right\|^2 \tag{3.11a}$$

$$\lambda_e(t) = -d_2\,\sigma_y(t) - (\varepsilon/d_1)\frac{\bar{\sigma}_\pi(t)}{\pi(t)}, \tag{3.11b}$$

[18]Note that $\lim_{t\to T}\ \|\bar{\sigma}_\pi(t)/\pi(t)\|^2 = 0$, since

$$\bar{\sigma}_\pi(t) = \int_{s=t}^{T} p_e(t,s)\,\sigma_{p_e}(t,s)\,ds \quad\text{and}\quad \pi(t) = \int_{s=t}^{T} p_e(t,s)\,ds,$$

where $\sigma_{p_e}(T,T) = 0$ and $p_e(T,T) = 1$.

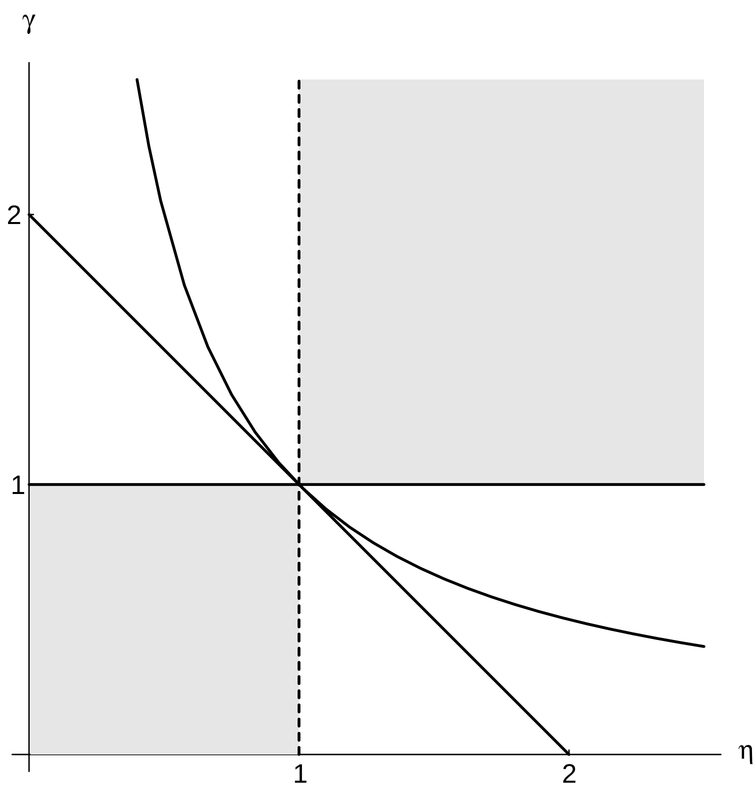

FIGURE 1. The coefficient of relative risk aversion, γ, versus the elasticity of intertemporal substitution, η. The shaded areas show where $\delta = (1-\gamma)/(1-1/\eta) < 0$.

where d_1 depends only on η and d_2 depends only on γ. (See Table 1 for the particular values.) Given $\widetilde{\mu}_y$ and σ_y, if we can solve (3.10) for π, the remaining remaining drifts and diffusions can be found via the relations established above.[19]

Recall that the endowment deflator, which has r_e as interest rate and λ_e as price of risk, is given by $m_e(t) = m(t)\, c(t)$, which, in light of (3.8), is

$$m_e(t) = \theta^\delta\, e^{-\theta\,\delta\,t}\, c(t)^{1-\delta/\eta}\, \phi(t)^{\delta-1}. \tag{3.12}$$

Note that the conditions for $\varepsilon = 0$ in Table 1 are the same as $1 - \delta/\eta = 0$ and $\delta - 1 = 0$, which lead one or the other of $c(t)$ and $\phi(t)$ to be absent from $m_e(t)$. We discuss these two cases.

Case 1: $\delta = 1$, *i.e.*, $\gamma = 1/\eta$.

This locus is plotted in Figure 1 as the rectangular hyperbola of standard preferences. (Recall that this is the dividing line between preference for early versus late resolution of uncertainty.) In this case, the endowment deflator depends only on consumption: $m_e(t) = \theta\, e^{-\theta\,t}\, c(t)^{1-\gamma}$. As a consequence, an

[19] It turns out that other drifts and diffusions can always be expressed in terms of $\widetilde{\mu}_y$, σ_y, and σ_π without $\widetilde{\mu}_\pi$. See (2.14).

exact solution to the model is possible when the forcing process is c. Moreover, (2.19) can be used to replace $\widetilde{\mu}_c(t)$ and $\sigma_c(t)$ with $r(t)$ and $\lambda(t)$, so that an exact solution is also possible when the forcing process is $1/m$. In both cases, $\varepsilon = 0$ in (3.11).

Case 2: $\delta = \eta$, *i.e.*, $\gamma + \eta = 2$.

This locus is plotted in Figure 1 as the diagonal line. In this case, the endowment deflator depends only on the capital account: $m_e(t) = \theta\,\phi(t)^{\eta-1}$. As a consequence an exact solution to the model is possible when the forcing process is ϕ. In this case, $\varepsilon = 0$ in (3.11).

Deterministic endowment interest rates. The terms in (3.11) involving σ_π present challenges to solving the model, and as such we refer to them as "nuisance" terms. However, there are two circumstances under which the nuisance terms will not be present: (i) when r_e is deterministic, $\sigma_\pi \equiv 0$, and (ii) when $\varepsilon = 0$. We deal with the first case immediately and the second case later in this section and in the next section where we introduce Markovian state variables. We treat the case where the nuisance terms are present in the next section as well.

There are two ways for $r_e(t)$ to be deterministic. First, if $\eta = 1$ (*i.e.*, $d_1 = 0$), then $r_e(t) = \theta$, and (3.10) devolves to $\pi(t) = q(T-t)/\theta$. This equation provides no other information and we solve (3.1) for $\psi(t)$ as above. Second, if $\widetilde{\mu}_y(t) + \frac{d_2}{2}\,\|\sigma_y(t)\|^2$ is deterministic, then $\sigma_\pi(t) = 0$ so that $r_e(t)$ is deterministic, even when $y(t)$ is stochastic (that is, $\sigma_y \neq 0$). Below we show an analytical solution to (3.10) for a special case where the drift and volatility of $\log(y(t))$ are each deterministic but nontrivial; the next example shows the solution in the constant case.

Constant investment opportunity set with $\eta \neq 1$. When $\widetilde{\mu}_y$ and σ_y are constant r_e is constant as well. In this case, the solution to (3.10) is

$$\pi(t) = \frac{1 - e^{-r_e\,(T-t)}}{r_e}, \tag{3.13}$$

so that $\sigma_\pi = 0$. Setting $y(t) = 1/m(t)$, for example, so that r and λ are constant, we see that (3.11) implies

$$r_e = \eta\,\theta + (1-\eta)\left(r + \frac{1}{2\,\gamma}\,\|\lambda\|^2\right) \quad \text{and} \quad \lambda_e = \left(1 - \frac{1}{\gamma}\right)\lambda\,.$$

Now, from (2.19), we get

$$\mu_c(t) = \eta\,(r-\theta) + (1+\eta)\,\frac{\|\lambda\|^2}{2\,\gamma} \quad \text{and} \quad \sigma_c = \frac{\lambda}{\gamma}.$$

Schroder and Skiadas (1997) call this case the constant investment opportunity set, and derive the solution through a different approach in their Theorem 2.[20] Note that when $\pi(t)$ is deterministic, ψ is deterministic as well, from (2.10b).

We have seen in the context of the exchange problem (when the consumption process is fixed) that the equilibrium interest rate and price of risk exhibits a discontinuity at $\eta = 1$ in the finite horizon case. We now show that in the context

[20]Their β is θ, their γ is $1 - 1/\eta$, and their α is $\delta - 1$ if $\eta \neq 1$ and $1 - \gamma$ otherwise.

of the planning problem with a constant opportunity set, the consumption process exhibits a discontinuity at $\eta = 1$.

The discontinuity of consumption at $\eta = 1$. When $\eta = 1$, and r and λ are constant, the solution to (2.40) is

$$\theta \log(\psi(t)/\theta) = q(\tau)\left(r - \theta + \frac{\|\lambda\|^2}{2\gamma}\right) \\ - e^{-\theta\tau}\left\{\tau\,\theta\left(r - \theta + \frac{\|\lambda\|^2}{2\gamma^2}\right) + \log(Q(\tau))\,\frac{\|\lambda\|^2}{2\gamma^2}\right\},$$

where $\tau := T - t$. This result, then, implies that $\sigma_\psi(t) = 0$, and we obtain the dynamics of consumption from (2.39):

$$\mu_c(t) = r - \theta + \frac{\|\lambda\|^2}{Q(\tau)} \quad \text{and} \quad \sigma_c = \frac{\lambda}{Q(\tau)}.$$

Since $Q(\tau)$ differs from γ except when the horizon is infinite, contrasting this case with that of $\eta \neq 1$ reveals a discontinuity in the dynamics of consumption. This discontinuity disappears when the horizon is infinite. Note, however, that $\pi(t)$ is given by (3.13) even when $\eta = 1$, in which case $r_e = \theta$, so that the wealth consumption ratio is continuous in the preference parameters even when the consumption process is not.

The relation between the volatility of ψ and the volatility of forecast revisions. For the infinite-horizon case when $\eta = 1$, we showed that if $\sigma_y(t)$ is constant and $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$ is a deterministic function of $u - t$, then $\log(\psi(t))$ equals a weighted average of expected growth rates of y (plus a constant), where the weights are exponentially declining. As a consequence, the volatility of ψ equals the weighted average of revisions to expected growth rates of y. In this section, we show that essentially the same relation between the volatilities holds when $\eta \neq 1$ (under the same conditions for $\sigma_y(t)$ and $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$) as long as $\varepsilon = 0$. In particular, for $y = \phi$ or $y = 1/m$, the volatility of ψ still equals the volatility of a weighted average of revisions to expected growth rates of y (where the weights are roughly exponentially declining). For $y = c$, the volatility of ψ is proportional to the average of the revisions, with proportionality constant η. These results are a consequence of the weak form of the expectations hypothesis of the term structure of interest rates where the numeraire is taken to be units of the endowment process.

Let $p(t,s)$ be the value at time t of a zero-coupon bond that pays one unit of an arbitrary numeraire at time s. Define the yield to maturity as follows: $y(t,s) := -\log(p(t,s))/(s-t)$. Without loss of generality we can write

$$y(t,s) = \frac{\int_{u=t}^{s} E_t[r(u)]\,du - \beta(t,s)}{s-t}, \tag{3.14}$$

for some process $\beta(t,s)$ such that $y(t,t) \equiv r(t)$. Let the dynamics of $\beta(t,s)$, $r(t)$, and $E_t[r(u)]$ be given by

$$\begin{aligned} d\beta(t,s) &= \mu_\beta(t,s)\,dt + \sigma_\beta(t,s)^\top dW(t) \\ dr(t) &= \mu_r(t)\,dt + \sigma_r(t)^\top dW(t) \\ dE_t[r(u)] &= \widehat{\sigma}_r(t,u)^\top dW(t). \end{aligned}$$

The strong form of the expectations hypothesis implies $\beta(t,s) \equiv 0$, while the weak form implies (i) $\sigma_\beta(t,s) \equiv 0$ and (ii) $\mu_\beta(t,s)$ is a deterministic function of $s-t$. Sufficient conditions for the weak form to hold are σ_r and λ constant. Given the weak form, we have

$$d\log(p(t,s)) = \big(r(t) + \mu_\beta(t,s)\big)\,dt - \left(\int_{u=t}^{s} \widehat{\sigma}_r(t,u)\,du\right)^\top dW(t).$$

Let $\varpi(t) := \int_{s=t}^{\infty} p(t,s)\,ds$ be the value of the perpetuity. Applying Itô's lemma to this definition of $\varpi(t)$ produces $d\varpi(t)/\varpi(t) = \mu_\varpi(t)\,dt + \sigma_\varpi(t)^\top dW(t)$, where

$$\mu_\varpi(t) = r(t) - \frac{1}{\varpi(t)} + \int_{s=t}^{\infty} w(t,s)\left\{\mu_\beta(t,s) + \frac{1}{2}\left\|\int_{u=t}^{s}\widehat{\sigma}_r(t,u)\,du\right\|^2\right\}ds \quad (3.15a)$$

$$\sigma_\varpi(t) = -\int_{s=t}^{\infty} w(t,s)\left(\int_{u=t}^{s}\widehat{\sigma}_r(t,u)\,du\right)ds, \quad (3.15b)$$

and where $w(t,s) := p(t,s)/\varpi(t)$.

At this point, we apply the foregoing to the endowment term structure. We assume (i) $\varepsilon = 0$, (ii) $\sigma_y(t)$ is constant, and (iii) $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$ is a deterministic function of $u-t$. These conditions are sufficient to ensure that σ_{r_e} and λ_e are constant and the weak form holds. Given these conditions, we can write (3.15b) for $r(t) = r_e(t)$ as

$$\sigma_\pi(t) = -\int_{s=t}^{\infty} w_e(t,s)\left(\int_{u=t}^{s}\widehat{\sigma}_{r_e}(t,u)\,du\right)ds, \quad (3.16)$$

where $w_e(t,s) := p_e(t,s)/\pi(t)$.[21] The preceding conditions also ensure that $\widehat{\sigma}_{r_e}(t,u) = d_1\,\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$, where $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$ is the volatility of $E_t[\widetilde{\mu}_y(u)]$. Therefore, given $\sigma_\psi(t) = \sigma_\pi(t)/(\eta-1)$, we can write (3.16) as

$$\sigma_\psi(t) = \left(\frac{d_1}{1-\eta}\right)\int_{s=t}^{\infty} w_e(t,s)\left(\int_{u=t}^{s}\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)\,du\right)ds. \quad (3.17)$$

For $y = \phi$ and $y = 1/m$, $d_1/(1-\eta) = 1$, while for $y = c$, $d_1/(1-\eta) = \eta$. Comparing (3.17) with (3.7) (in which $\sigma_\zeta(t) = \sigma_\psi(t)$), we see there is a close relationship between $\sigma_\psi(t)$ and $\sigma_\zeta(t)$ even when $\eta \neq 1$ as long as the expectations hypothesis holds.

[21] Note that if the term premium were stochastic, we would have

$$\sigma_\pi(t) = \int_{s=t}^{\infty} w_e(t,s)\left(\int_{u=t}^{s}\widehat{\sigma}_{r_e}(t,u) + \sigma_{\beta_e}(t,u)\,du\right)ds.$$

Recall that when $y = 1/m$, $\widetilde{\mu}_y(t) = r(t) + \frac{1}{2}\|\lambda(t)\|^2$ and $\sigma_y(t) = \lambda(t)$, where r is the real interest rate. Our assumption that σ_y is constant implies that $d\widetilde{\mu}_y(t) = dr(t)$, so that $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u) = \widehat{\sigma}_r(t,u)$. Thus for $y = 1/m$, (3.17) is

$$\sigma_\psi(t) = \int_{s=t}^{\infty} w_e(t,s)\left(\int_{u=t}^{s} \widehat{\sigma}_r(t,u)\,du\right) ds. \tag{3.18}$$

On the other hand, the volatility of a real perpetuity is given by (3.15b) where r is interpreted as the real interest rate. Therefore we have established a close relationship between σ_ψ and $-\sigma_\varpi$. They differ only by the weights: $w_e(t,s)$ versus $w(t,s)$. However, even when $\eta = 1$, the weights do not converge: When $\eta = 1$, the endowment perpetuity weights are $w_e(t,s) = \theta\, e^{-(s-t)\,\theta}$ (see (3.7)), while the real perpetuity weights are determined by r and λ and need not bear any particular relation to $w_e(t,s)$. Nevertheless, whenever consumption remains constant we have m_e proportional to m, so that the real and endowment term structures are identical. This situation occurs with $\eta = 0$ and $\gamma = \infty$ and does not require homoskedasticity. (See the discussion on limit cases below.)[22]

When $y = \phi$, (3.17) becomes

$$\sigma_\psi(t) = \int_{s=t}^{\infty} w_e(t,s)\left(\int_{u=t}^{s} \widehat{\sigma}_{\widetilde{\mu}_\phi}(t,u)\,du\right) ds. \tag{3.19}$$

This expression sheds some light on an approximation in Campbell (1993). Relying on the relation between ψ and ζ that holds when $\eta = 1$ (see (3.7)), Campbell approximates σ_ϕ with a weighted average of forecast revisions of $y = \phi$ in a homoskedastic model. He compares the solutions based on this approximation with full numerical solutions to the discrete-time version of our model. Equation (3.19) shows that Campbell's approximation is essentially exact not just for $\eta = 1$, but also for $\gamma + \eta = 2$: Along both lines the nuisance terms are absent. The fact that the result holds on more than one line in the parameter space helps explain why Campbell's approximation is as good as it is over such a wide range of parameter values.

Limit cases. In this section we analyze the model at the limit values of the parameter space for intertemporal substitution and risk aversion. We consider the cases where η and γ are zero or infinity. We can identify certain features of the solution even in cases where we cannot solve the model entirely. We must take care, however: For a given limit, there may be no equilibrium for arbitrarily chosen dynamics of a given forcing variable. By examining the limiting values of d_1 and d_2 in Table 1, we can rule out certain combinations.

Case 1: $\eta = 0$.

This corresponds to extreme aversion toward intertemporal substitution. As indicated in Table 1, this is consistent with solving for equilibrium when the forcing process is either ϕ or $1/m$; but it is inconsistent with solving for equilibrium in an endowment economy (d_1 and ε are both infinite). In this case, (2.13) provides a restriction on the consumption process that must be

[22]Campbell (1993) discusses the relation between the real and endowment perpetuities.

satisfied. One might think that $\eta = 0$ would lead to choose a constant rate of consumption. Instead, we see that $\widetilde{\mu}_c = (\gamma - 1)\,\frac{1}{2}\,\|\sigma_c\|^2$. It is interesting to note that if in addition $\gamma = 1$, the CAPM case, then log consumption is a martingale.

Case 2: $\eta = 0$ and $\gamma = \infty$.

Suppose the agent is extremely risk averse as well as unwilling to substitute. Table 1 indicates that this is possible only if $y = 1/m$ (there is no equilibrium with arbitrary technology or endowment). In this case, (2.13) can only be satisfied if $\widetilde{\mu}_c = \widetilde{\sigma}_c = 0$, which means that $c(t)$ is a constant determined by the initial wealth. Not surprisingly (2.30) shows that the optimal portfolio is determined entirely by the hedging component: $\sigma_\phi(t) = -\sigma_\psi(t)$. Since consumption is constant, $r_e = r$ and $\lambda_e = \lambda$, so $\pi(t)$ is the value of a real annuity as well as that of an endowment annuity. Finally, given $\pi(t) = 1/\psi(t)$ when $\eta = 0$, we have $-\sigma_\psi(t) = \sigma_\pi(t)$. Therefore the optimal portfolio is a real annuity in this case (a real perpetuity in the infinite-horizon case).[23]

Case 3: $\eta = 0$ and $\gamma = 0$.

Now, the forcing process must be $y = \phi$. The agent is unwilling to substitute consumption across periods but perfectly willing to substitute across states of nature (risk neutral). The state-price deflator is $m(t) = c(t)/\phi(t)$, and from (2.13), we infer that $c(t)$ is a martingale, a fact that has far-reaching implications in the infinite-horizon case. Because $c(t)$ is a positive martingale, it converges. If technology is such that a constant and strictly positive asymptotic consumption flow is feasible, then consumption might converge to such a value. Otherwise, the consumer asymptotically exhausts his wealth and consumption converges to zero, for any value of the rate of time preference θ. For example, suppose that μ_ϕ is constant (even if $\widetilde{\mu}_\phi$ and σ_ϕ are not). Then the solution is $1/\pi(t) = \psi(t) = r(t) = \mu_\phi$, $\lambda(t) = 0$, and $\sigma_c(t) = \sigma_k(t) = \sigma_\phi(t)$. If $\sigma_\phi(t)$ does not go to zero, then $c(t)$ and $k(t)$ both do almost surely (though, of course, not in the L_1 norm).

Case 4: $\gamma = 0$.

This is the risk neutrality case, and Table 1 suggests that there is no solution when $y = 1/m$. In fact, there is a solution, but in general it involves setting $c(t) = 0$ often, a corner solution that does not satisfy our system of equations (our solution method assumes positivity of log consumption, and thus rules out corner solutions). An important point to make, however, is that risk neutrality of agents does not imply that the price of risk, $\lambda(t)$, is zero. In light of equation (2.14b), $\lambda = -\sigma_\psi$ and thus the condition $\sigma_\psi = 0$ must hold for the price of risk to vanish. But $\sigma_\psi = 0$ requires $\eta = \infty$ in addition to $\gamma = 0$. With standard preferences, of course, $\gamma = 0$ implies $\eta = \infty$, so that in such models risk neutrality indeed implies $\lambda(t) = 0$. (See Case 6 below.)

For example, let $y = \phi$. If $\eta = 2$ then $\varepsilon = 0$, in which case, $r_e(t) = 2\,\theta - \mu_\phi(t)$ and $\lambda_e(t) = -\sigma_\phi(t)$. Therefore, as long as μ_ϕ is not deterministic (and the

[23] This point is made by Campbell and Viceira (1996).

average value of $\widetilde{\mu}_\phi$ is less than $2\,\theta$), $\sigma_\pi \neq 0$. With $\eta = 2$, equation (2.21) shows that $\sigma_\pi(t) = \sigma_\psi(t)$, so that (2.14b) implies $\lambda(t) = -\sigma_\pi(t) \neq 0$.

Case 5: $\eta = \infty$.

The forcing variable must be consumption. In this case $\psi = \theta$, since

$$\lim_{\eta\to\infty} \psi(t) = \lim_{\eta\to\infty} \theta^{\eta/(\eta-1)}\,\pi(t)^{1/(\eta-1)} = \theta$$

regardless of the value of $\pi(t)$. Since $\varepsilon = \gamma$, we have, equating (3.11) with (2.19), $\lambda(t) = \gamma\,(\sigma_c(t) - \sigma_\pi(t))$.

Case 6: $\eta = \infty$ and $\gamma = 0$.

This case inherits all of the properties of Case 5. With $\gamma = 0$, $\varepsilon = 0$, the price of risk vanishes, $\lambda(t) = 0$, and, from (3.9a), the interest rate is determined by the rate of time preference, $r(t) = \theta$. As a result, the expected rate of return on any asset is equal to θ, and the yield curve is flat at that level.

4. A Markovian setting

In this section we introduce state variables and provide explicit solutions to the model.

Modeling the dynamics of the forcing process. We suppose there are d Markovian state variables X driving the exogenous process y, where y is either ϕ, c, or $1/m$. The joint dynamics of X and y are given by

$$\begin{pmatrix} dX(t) \\ d\log(y(t)) \end{pmatrix} = \begin{pmatrix} \mu_X(X(t)) \\ \widetilde{\mu}_Y(X(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma_X(X(t))^\top \\ \sigma_Y(X(t))^\top \end{pmatrix} dW(t),$$

where $W^\top = \big(W_x^\top, W_y\big)$ is a l-dimensional vector of orthonormal Brownian motions, with W_x $(l-1)$-dimensional and W_y scalar. The dimensions of $\sigma_X(x)$ and $\sigma_Y(x)$ are respectively $l \times d$ and $l \times 1$. We assume that the last column of $\sigma_X(x)^\top$ is a vector of zeros, so that the state variables are not affected by W_y: $\sigma_X(x)^\top = \big(\Sigma_X(x)^\top\ 0\big)$, where $\Sigma_X(x)$ is $(l-1) \times d$.[24] Note that even if the state variables are deterministic ($\sigma_X(x) \equiv 0$), y can be stochastic; a completely deterministic economy would require $\sigma_Y(x) \equiv 0$ as well.

The PDE. We assume that the horizon is finite, treating the case of an infinite horizon as a limit. We distinguish two cases.

First case: $\eta \neq 1$. We transform the annuity equation (3.10) into a PDE in terms of a Markovian annuity function $\mathit{\Pi}(x,\tau)$ such that where $\pi(t) = \mathit{\Pi}(X(t), T-t)$. Let $\mathit{\Pi}_x$, $\mathit{\Pi}_{xx}$, and $\mathit{\Pi}_\tau$ denote the obvious partial derivatives, and define an operator

[24] This assumption is without loss of generality. It simply allows for the possibility that there exists a shock, W_y, that affects y but not X.

$\mathcal{F}(x, \tau, \partial\cdot)$ operating on the space of candidate solutions so that[25]

$$\mathcal{F}(x,\tau,\partial\Pi) := 1 + \bar{\mu}_\Pi(x,\tau) - \left(d_0 + d_1\,\widetilde{\mu}_Y(x) + d_1\,d_2\,\frac{1}{2}\,\|\sigma_Y(x)\|^2\right)\Pi(x,\tau) \\ + d_2\,\sigma_Y(x)^\top\bar{\sigma}_\Pi(x,\tau) + (\varepsilon/d_1)\frac{1}{2}\,\frac{\|\bar{\sigma}_\Pi(x,\tau)\|^2}{\Pi(x,\tau)}, \quad (4.1)$$

where

$$\bar{\mu}_\Pi(x,\tau) = \Pi_x(x,t)\,\mu_X(x) + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X(x)^\top\sigma_X(x)\,\Pi_{xx}(x,\tau)\right] - \Pi_\tau(x,\tau)$$
$$\bar{\sigma}_\Pi(x,\tau) = \Pi_x(x,\tau)\,\sigma_X(x).$$

The PDE to solve is

$$\mathcal{F}(x,\tau,\partial\Pi) = 0 \text{ subject to } \Pi(x,0) = 0, \quad (4.2)$$

which is a linear PDE if the nuisance term is zero and quasi-linear otherwise.

Second case: $\eta = 1$. We Markovianize (3.1) after changing variable from ψ to Ξ through $\psi(t) = \exp(\Xi(X(t), T-t))$. For this case, the definition of $\mathcal{F}$ in (4.1) is replaced by

$$\mathcal{F}(x,\tau,\partial\Xi) := \theta\,(\Xi(x,\tau) - \log(\theta)) - \bar{\mu}_\Xi(x,\tau) - q(\tau)\,(\widetilde{\mu}_Y(x) + d_3\,\theta) \\ - \frac{(1-\gamma)}{2\,d_4(\tau)}\,\|q(\tau)\,\sigma_Y(x) + \bar{\sigma}_\Xi(x,\tau)\|^2, \quad (4.3)$$

where

$$\bar{\mu}_\Xi(x,\tau) = \mu_X(x)^\top\,\Xi_x(x,\tau) + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X(x)^\top\sigma_X(x)\,\Xi_{xx}(x,\tau)\right] - \Xi_\tau(x,\tau)$$
$$\bar{\sigma}_\Xi(x,\tau) = \sigma_X(x)\,\Xi_x(x,\tau).$$

In this case, the PDE is

$$\mathcal{F}(x,\tau,\partial\Xi) = 0 \text{ subject to } \Xi(x,0) = \log(\theta), \quad (4.4)$$

which is a linear PDE.

A general solution method. Except in a few special cases, closed-form analytic expressions for the functions Π and Ξ will not be available. (We will discuss some of those special cases below.) We describe a solution method that is quite general. The method generates an approximation to the solution that can be made arbitrarily accurate. In the limit, the method generates the Taylor series representation of the solution. The method applies to both the $\eta \neq 1$ and $\eta = 1$ cases with any number of state variables. For expositional simplicity, however, we describe the method with a single state variable. The generalization is obvious.

Markovian bond-pricing techniques that decompose a PDE into a system of ODEs can be applied to (4.3), as in Duffie and Epstein (1992a), Duffie, Schroder, and Skiadas (1997), and Schroder and Skiadas (1997). These techniques can also be applied indirectly to (4.1) through the PDE for endowment bond prices when the nuisance

[25] The notation $\partial\cdot$ occurring in $\mathcal{F}$ is short for all the partial derivative operators that appear in the PDE.

term is absent (*i.e.*, when the last term in (4.1) is identically zero). These techniques decompose a PDE into a system of ODEs by expressing the log of bond prices as a Taylor polynomial.[26] They have typically been applied where the Taylor expansion of the solution is finite-order (it is first-order in the Duffie-Kan exponential-affine class). Our method extends these techniques to the case where the nuisance term is present in (4.1), producing an infinite system of first-order ODEs. Truncating this system produces an approximation to the solution. But by including more equations in the truncated system, the solution can be made arbitrarily accurate.

Bond-pricing techniques, including our extension, owe their tractability to the fact that the boundary condition is independent of the state variables. For bond prices the condition is $P^e(x,0)=1$, while for the value of an annuity the condition is $\Pi(x,0)=0$. The implicit boundary condition for the value of a perpetuity is $\lim_{\tau\to\infty} P^e(x,\tau)=0$ which again is independent of the state variables.

To apply our method, the following functions of the state variable, called collectively *the data*, must be real analytic

$$\mu_X(x), \quad \sigma_X(x)^\top\sigma_X(x), \quad \sigma_X(x)\,\sigma_Y(x), \quad \widetilde{\mu}_Y(x), \quad \text{and} \quad \sigma_Y(x)^\top\sigma_Y(x). \tag{4.5}$$

This assumption is sufficient to guarantee that there exists a unique real analytic solution $\Pi(x,\tau)$ for any x in the domain of the functions in the data and any τ in a neighborhood of 0 (Cauchy–Kowaleskaya theorem, see Rauch (1991, Chapter 1)). Therefore, treating τ as a parameter, we can write the solution as a Taylor series

$$\Pi(x,\tau)=\sum_{i=0}^{\infty}\mathcal{A}_i(\tau)\,(x-x_0)^i, \quad \text{where } \mathcal{A}_i(\tau)\equiv\frac{\Pi^{(i)}(x_0,\tau)}{i!}, \tag{4.6}$$

where each $\mathcal{A}_i$ is analytic in τ. This representation is guaranteed to exist in a neighborhood of $\tau=0$, that is, when the horizon is short enough. The derivatives $\Pi_x(x,\tau)$ and $\Pi_{xx}(x,\tau)$ can be computed term-by-term in (4.6). In addition

$$\Pi_\tau(x,\tau)=\sum_{i=0}^{\infty}\mathcal{A}_i'(\tau)\,(x-x_0)^i,$$

where $\mathcal{A}_i'(\tau)$ denote the derivative with respect to τ, and the boundary condition can be written

$$\Pi(x,0)=0 \implies \mathcal{A}_i(0)=0, \quad \text{for } i=0,1,2,\ldots.$$

With the solution Π expanded as in (4.6), define $F(x,\tau):=\mathcal{F}(x,\tau,\partial\Pi)$. Then, F is itself real analytic, so that we can also write it as a Taylor series:

$$F(x,\tau)=\sum_{i=0}^{\infty}\frac{F^{(i)}(x_0,\tau)}{i!}\,(x-x_0)^i, \tag{4.7}$$

where $F^{(i)}(x_0,\tau)$ denotes the i-th derivative of F with respect to x. The unique solution $\Pi(x,\tau)$ is characterized by $\Pi(x,0)=0$ and $F(x,\tau)\equiv 0$, so that $F^{(i)}(x_0,\tau)=0$

[26]See Duffie and Kan (1996) for a discussion of this technique as applied to the exponential-affine class of term-structure models. Other models such as those in Constantinides (1992) and Rogers (1997) can be solved using the same techniques.

for $i = 0, 1, 2, \ldots$. These conditions produce a system of first-order ODEs in the coefficient functions that (along with the initial conditions) uniquely determine them. For our approximate solution method, we fix n and we find a polynomial approximation of order n to $\varPi$, written as

$$\varPi_n(x,\tau) = \sum_{i=0}^{n} a_{ni}(\tau)\,(x-x_0)^i, \quad \text{where } a_{ni}(0) = 0 \text{ for } i = 0, 1, \ldots, n.$$

We do not claim that $a_{ni} = \mathcal{A}_i$, so $\varPi_n$ is not necessarily the truncation of the Taylor series of $\varPi$. Defining $a_{ni}(\tau) := 0$ for $i > n$, the coefficients a_{ni} (for $i \le n$) are found by replacing $\mathcal{A}_i$ by a_{ni} for $i = 0, 1, \ldots, \infty$ in $F(x,\tau)$, and then solving the system of equations $F^{(i)}(x_0,\tau) = 0$ for $i = 0, 1, \ldots, n$. Nevertheless, as a consequence of the assumed analyticity, $\lim_{n\to\infty} a_{ni} = \mathcal{A}_i$ for all i.

When our method is applied to (4.4), it delivers exact solutions with a finite number of terms when the data in (4.5) are polynomial. We illustrate our method with a sequence of examples.

Bond pricing examples. We begin with the case $\eta \neq 1$ and $\varepsilon = 0$. In this case we can write

$$\mathcal{F}(x,\tau,\partial\varPi) = 1 + \varPi_x(x,\tau)^\top \widehat{\mu}_X(x) - \varPi_\tau(x,\tau) - \mathcal{R}(x)\,\varPi(x,\tau) + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X(x)^\top \sigma_X(x)\,\varPi_{xx}(x,\tau)\right], \tag{4.8}$$

where

$$\mathcal{R}(x) = d_0 + d_1\,\widetilde{\mu}_Y(x) + d_1\,d_2\,\frac{1}{2}\,\|\sigma_Y(x)\|^2 \tag{4.9a}$$

$$\widehat{\mu}_X(x) = \mu_X(x) + d_2\,\sigma_Y(x)^\top \sigma_X(x). \tag{4.9b}$$

To evaluate our solution method, it is useful to have an alternative method. In the present case, one approach to solving (4.2) is to solve the related PDE for endowment zero-coupon bond prices. Let $P^e(x,\tau)$ be the price of an endowment bond with maturity τ, (so $p_e(t,s) = P^e(X(t), s-t)$, in the notation of section 2). The PDE for bond prices is

$$\mathcal{R}(x) = \left(\frac{P^e_x(x,\tau)}{P^e(x,\tau)}\right)^\top \widehat{\mu}_X(x) + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X(x)^\top \sigma_X(x)\left(\frac{P^e_{xx}(x,\tau)}{P^e(x,\tau)}\right)\right] - \frac{P^e_\tau(x,\tau)}{P^e(x,\tau)}, \tag{4.10}$$

subject to the boundary condition $P^e(x,0) = 1$. The solution for $\varPi$ then is

$$\varPi(x,\tau) = \int_{s=0}^{\tau} P^e(x,s)\,ds. \tag{4.11}$$

Note that $\varPi_\tau(x,\tau) = P^e(x,\tau)$. A necessary condition for $\lim_{\tau\to\infty} \varPi(x,\tau)$ to be well-defined is $\lim_{\tau\to\infty} P^e(x,\tau) = 0$.

The Taylor expansion for the bond around $x = x_0$ is given by

$$P^e(x,\tau) = \sum_{i=0}^{\infty} \frac{P^{e,(i)}(x_0,\tau)}{i!}\,(x-x_0)^i.$$

where $P^{e\,(i)}(x_0, \tau)$ denotes the i-th derivative with respect to x. Given (4.11) we have

$$\mathcal{A}_i(\tau) = \int_{s=0}^{\tau} \frac{P^{e\,(i)}(x_0, s)}{i!}\, ds.$$

Viewing the annuity as a flow of zero-coupon bonds, then, provides a solution method. In the present case, this method produces all the coefficients of the Taylor expansion of $\mathit{\Pi}$. Many Markovian models of bond prices are of the following form:

$$\log(P^e(x, \tau)) = -\sum_{i=0}^{N} B_i(\tau)\,(x - x_0)^i,$$

where N is finite. For the exponential-affine class, where $N = 1$, the Taylor expansion for the price of a bond is

$$P^e(x, \tau) = e^{-B_0(\tau)} \sum_{i=0}^{\infty} \frac{(-1)^i\, B_1(\tau)^i}{i!}\,(x - x_0)^i.$$

Even though $\log(P^e(x, \tau))$ has a Taylor expansion of finite order, $P^e(x, \tau)$ does not. In this case, the Taylor coefficients for the annuity are given by

$$\mathcal{A}_i(\tau) = \int_{s=0}^{\tau} e^{-B_0(s)} \frac{(-1)^i\, B_1(s)^i}{i!}\, ds. \tag{4.12}$$

Equation (4.12) will serve as a benchmark in the examples that follow.

Deterministic state variables, $\eta \neq 1$. Consider the following example. The forcing variable y is stochastic, driven by a single Brownian motion, and its drift and diffusion depend on a single deterministic state variable, X, in such a way that we can write

$$\mathcal{R}(x) = \mathcal{R}_0 + \mathcal{R}_1\,(x - \bar{X}).$$

The "risk-adjusted" dynamics of X are given by:

$$\widehat{\mu}_X(x) = \kappa\,(\bar{X} - x) \quad \text{and} \quad \sigma_X(x) = 0.$$

This is an exponential-affine model where, choosing to expand around $x_0 = \bar{X}$,

$$B_0(\tau) = \mathcal{R}_0\,\tau \quad \text{and} \quad B_1(\tau) = (\mathcal{R}_1/\kappa)\left(1 - e^{-\kappa\,\tau}\right). \tag{4.13}$$

In this case we have

$$\mathcal{A}_i(\tau) = \frac{(-1)^i\,(\mathcal{R}_1/\kappa)^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{1 - e^{-(j\,\kappa + \mathcal{R}_0)\,\tau}}{j\,\kappa + \mathcal{R}_0}, \tag{4.14}$$

where $\binom{i}{j}$ is the binomial coefficient. This provides an exact solution for Π.

Turning to our power series approximation method, write $\mathcal{F}(x, \tau, \partial\mathit{\Pi}) = 0$ as

$$\mathit{\Pi}_\tau(x, \tau) = 1 + \kappa\,(\bar{X} - x)\,\mathit{\Pi}_x(x, \tau) - \mathcal{R}(x)\,\mathit{\Pi}(x, \tau) \tag{4.15}$$

Substituting into (4.15) the Taylor expansions of Π and its partial derivatives, we then can write $F(x,\tau) = 0$ as

$$\mathcal{A}_0'(\tau) + \sum_{i=1}^{\infty} \mathcal{A}_i'(\tau)\,(x - \bar{X})^i = 1 - \mathcal{R}_0\,\mathcal{A}_0(\tau) - \sum_{i=1}^{\infty} \Big\{ \mathcal{R}_1\,\mathcal{A}_{i-1}(\tau) + (i\,\kappa + \mathcal{R}_0)\,\mathcal{A}_i(\tau) \Big\} (x - \bar{X})^i.$$

We see that the PDE decomposes into a system of linear ODEs with constant coefficients, each corresponding to a condition of the form $F^{(i)} = 0$:

$$\begin{aligned} \mathcal{A}_0'(\tau) &= 1 - \mathcal{R}_0\,\mathcal{A}_0(\tau) \\ \mathcal{A}_i'(\tau) &= -\mathcal{R}_1\,\mathcal{A}_{i-1}(\tau) - (i\,\kappa + \mathcal{R}_0)\,\mathcal{A}_i(\tau) \quad \text{for } i = 1, 2, \ldots. \end{aligned} \tag{4.16}$$

Subject to $\mathcal{A}_i(0) = 0$ for $i = 0, 1, 2, \ldots$, system (4.16) has the unique solution given in (4.14).

Following our solution method, we choose n to specify $\Pi_n(x,\tau)$ and set $F^{(i)}(x,\tau) = 0$ for $i = 0, 1, \ldots, n$ and $a_{ni} = 0$ for $i > n$. This produces the following system of $n+1$ ODEs:

$$\begin{aligned} a_{n0}'(\tau) &= 1 - \mathcal{R}_0\,a_0(\tau) \\ a_{ni}'(\tau) &= -\mathcal{R}_1\,a_{n,i-1}(\tau) - (i\,\kappa + \mathcal{R}_0)\,a_{ni}(\tau) \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{4.17}$$

System (4.17) has a unique solution subject to $a_{ni}(0) = 0$ for $i = 0, 1, 2, \ldots, n$. Comparing (4.16) with (4.17) we see that $a_{ni}(\tau) \equiv \mathcal{A}_i(\tau)$ for $i = 0, 1, 2, \ldots, n$ for all n. This occurs because a_{ni}' does not depend on $a_{n,j}$ for $j > i$, and is not the general case as we show below.

Increasing the horizon. The Cauchy–Kowaleskaya theorem guarantees the existence of a solution, but only in a neighborhood of $\tau = 0$. As the horizon increases, the value of the annuity converges to the value of a perpetuity, but only if the latter is well-defined. A sufficient condition for the value of a perpetuity to be well-defined is that the infinite-horizon asymptotic forward rate exists and is positive. The forward rate function is given by

$$f(x,\tau) := -\frac{\partial \log(P^e(x,\tau))}{\partial \tau} = \mathcal{R}_0 + e^{-\kappa\,\tau}\,\mathcal{R}_1\,(x - \bar{X}).$$

For $\lim_{\tau\to\infty} f(x,\tau)$ to be well-defined and positive, we need $\mathcal{R}_0 > 0$ and $\kappa > 0$.

When the value of a perpetuity is well-defined, the time derivative in the PDE vanishes as $\tau \to \infty$, in which case

$$\lim_{\tau\to\infty} \mathcal{A}_i'(\tau) = 0 \quad \text{for } i = 0, 1, 2, \ldots.$$

Therefore in the limit (4.16) becomes a system of linear algebraic equations:

$$0 = 1 - \mathcal{R}_0\,\mathcal{A}_0(\infty) \tag{4.18a}$$

$$0 = \mathcal{R}_1\,\mathcal{A}_{i-1}(\infty) + (i\,\kappa + \mathcal{R}_0)\,\mathcal{A}_i(\infty) \quad \text{for } i = 1, 2, \ldots. \tag{4.18b}$$

In the infinite-horizon case, then, there is no differential equation to solve; the solution is found by solving a set of algebraic equations. In the present case, the form of the general term can be found directly from (4.18) or by specializing (4.14):

$$\mathcal{A}_i(\infty) = \frac{(-1)^i \, \mathcal{R}_1^i}{\prod_{j=0}^{i} j\,\kappa + \mathcal{R}_0}. \tag{4.19}$$

The power series for the perpetuity can be written as

$$\Pi(x, \infty) = \frac{e^{\beta(x)}\, \Gamma(\alpha, 0, \beta(x))}{\kappa\, \beta(x)^{\alpha}}, \tag{4.20}$$

where

$$\alpha := \mathcal{R}_0/\kappa, \quad \beta(x) := -\mathcal{R}_1\,(x - \bar{X})/\kappa, \quad \text{and} \quad \Gamma(\alpha, \chi, \beta) := \int_{t=\chi}^{\beta} t^{\alpha-1}\, e^{-t}\, dt,$$

where $\Gamma(\alpha, \chi, \beta)$ is the generalized incomplete gamma function.[27]

Stochastic state variable. We extend the previous example by allowing a non-zero volatility for the state variable, though we still assume the nuisance terms are zero. We let $\sigma_X(x) = s_X\,\sqrt{\varsigma(x)}$, where $\varsigma(x) := (1-\alpha) + \alpha\, x$. This volatility function encompasses both the Gaussian ($\alpha = 0$) and square-root ($\alpha = 1$) models. To simplify notation, we keep the same risk adjusted drift for X. In this case, the series representation for $\Pi(x,\tau)$ in $\mathcal{F}(x, \tau, \partial\,\Pi)$ produces the following infinite system of first-order linear ODEs with constant coefficients:

$$\begin{aligned} \mathcal{A}_0'(\tau) &= 1 - \mathcal{R}_0\,\mathcal{A}_0(\tau) + s_X^2\,\varsigma(\bar{X})\,\mathcal{A}_2(\tau) \\ \mathcal{A}_i'(\tau) &= -\mathcal{R}_1\,\mathcal{A}_{i-1}(\tau) - (i\,\kappa + \mathcal{R}_0)\,\mathcal{A}_i(\tau) \\ &\qquad + s_X^2\big(\alpha\, c_i\,\mathcal{A}_{i+1}(\tau) + \varsigma(\bar{X})\, c_{i+1}\,\mathcal{A}_{i+2}(\tau)\big) \quad \text{for } i = 1, 2, \ldots \end{aligned} \tag{4.21}$$

where $c_i = \sum_{j=1}^{i} j$. Note that $\mathcal{A}_i$ depends on $\mathcal{A}_{i+1}$ and $\mathcal{A}_{i+2}$. Conditional on $\mathcal{A}_0$ and $\mathcal{A}_1$, we can solve for all of the rest of the functions.

We treat $\mathcal{A}_0$ and $\mathcal{A}_1$ as unknown—they are part of the solution. With a_{ni} replacing $\mathcal{A}_i$, our method generates a system of ODEs that is formally identical to the first $n+1$ equations in (4.21) except that $a_{ni} = 0$ for $i > n$—in particular, $a_{n,n+1} = a_{n,n+2} = 0$. The result is a system of $n+1$ linear first-order ODEs with

[27]For comparison with expressions in Campbell (1993) note that the first-order approximation to $\log(\Pi(x,\infty))$ is

$$\log(\Pi(x,\infty)) \approx -\log(\mathcal{R}_0) - \frac{\mathcal{R}_1}{\kappa + \mathcal{R}_0}\,(x - \bar{X}),$$

which shows that $\Pi(x,\infty)$—a weighted average of exponentials—is approximately exponential itself around $\bar{X}$.

constant coefficients:[28]

$$\begin{aligned}
a'_{n0}(\tau) &= 1 - \mathcal{R}_0\, a_{n0}(\tau) + s_X^2\, \varsigma(\bar{X})\, a_{n2}(\tau) \\
a'_{ni}(\tau) &= -\mathcal{R}_1\, a_{n,i-1}(\tau) - (i\,\kappa + \mathcal{R}_0)\, a_{ni}(\tau) \\
&\qquad + s_X^2 \big(\alpha\, c_i\, a_{n,i+1}(\tau) + \varsigma(\bar{X})\, c_{i+1}\, a_{n,i+2}(\tau)\big) \quad \text{for } i = 1, 2, \ldots, n-2 \\
a'_{n,n-1}(\tau) &= -\mathcal{R}_1\, a_{n,n-2}(\tau) - ((n-1)\,\kappa + \mathcal{R}_0)\, a_{n,n-1}(\tau) + s_X^2 \big(\alpha\, c_{n-1}\, a_{n,n}(\tau)\big) \\
a'_{nn}(\tau) &= -\mathcal{R}_1\, a_{n,n-1}(\tau) - (n\,\kappa + \mathcal{R}_0)\, a_{nn}(\tau)
\end{aligned} \tag{4.22}$$

System (4.22) (along with the initial conditions $a_{ni}(0) = 0$) has a unique solution. For finite n, however, $a_{ni} \neq \mathcal{A}_i$, although $\lim_{n\to\infty} a_{ni} = \mathcal{A}_i$ for all i. We can use (4.12) to investigate the convergence of the factor loadings in this case. For simplicity, let $\alpha = 0$ so that $\varsigma(x) \equiv 1$. In this case, $B_1(\tau)$ is as given in (4.13), while

$$B_0(\tau) = -\mathcal{R}_0\,\tau + \frac{s_X^2\,\mathcal{R}_1^2\left(2\,\kappa\,\tau - 3 + e^{-\kappa\,\tau} - e^{-2\,\kappa\,\tau}\right)}{4\,\kappa^3}.$$

Given values for $\mathcal{R}_0$, $\mathcal{R}_1$, κ, $\bar{X}$, and s_X, one can compare $a_{ni}(\tau)$ with $\mathcal{A}_i(\tau)$. What one discovers is that as κ approaches zero, holding n fixed, the approximation $a_{n,i}$ to the annuity factor loadings $\mathcal{A}_i$ worsens. This is due to the fact that $\mathcal{A}_{n+1}$ and $\mathcal{A}_{n+2}$ become larger and larger as $\kappa \to 0$, making the approximation $a_{n,n+1} = a_{n,n+2} = 0$ worse and worse. Of course as long as endowment bond prices can be computed directly (as in this example), the series coefficients for the endowment annuity can be computed by integrating the series coefficients for the bond prices.

A nuisance term. Adding a nuisance term to the PDE eliminates the possibility of using bond prices to solve for the value of the annuity, since the endowment interest rate and the price of endowment risk depend on the volatility of the annuity. (See (3.11).) Nevertheless, our method allows us to solve the PDE for the value of the annuity. The presence of the nuisance term introduces nonlinear terms into the system of ODEs without otherwise changing the character of the problem or its solution. The additional terms for the first two ODEs in (4.21) are

$$(\varepsilon/d_1)\, s_X^2\, \frac{1}{2}\left(\frac{\frac{1}{2}\,\varsigma(\bar{X})\,\mathcal{A}_1(\tau)^2}{\mathcal{A}_0(\tau)^2}\right) \quad \text{and}$$

$$(\varepsilon/d_1)\, s_X^2\, \frac{1}{2}\left(\frac{-\varsigma(\bar{X})\,\mathcal{A}_1(\tau)^3}{\mathcal{A}_0(\tau)^3} + \frac{\frac{1}{2}\,\mathcal{A}_1(\tau)^2 + 2\,\varsigma(\bar{X})\,\mathcal{A}_1(\tau)\,\mathcal{A}_2(\tau)}{\mathcal{A}_0(\tau)^2}\right).$$

Our solution method proceeds as before, truncating the system and setting $a_{n,i} = 0$ for $i = n+1, n+2, \ldots$. The system of ODEs for the functions $a_{ni}(\tau)$ has a unique solution (given the initial conditions) that converges on the true solution to the PDE as $n \to \infty$.

As noted above, as the rate of mean reversion decreases, the accuracy of the Taylor coefficients decreases (holding n fixed). When the nuisance term is present, we cannot compute endowment bond prices directly. Nevertheless, we can improve

[28] Some equations may need modification for $n < 3$.

the accuracy as follows. We can solve a linear PDE (that is related to the quasi-linear PDE in a purely formal way) for *pseudo* endowment bond prices. The Taylor expansion for these pseudo bond prices can be integrated to provide better approximations for $a_{n,n+1}$ and $a_{n,n+2}$. In particular, we solve (4.10) where $P^e(x,\tau)$ is replaced by $\widehat{P}^e(x,\tau)$, the value of a pseudo endowment bond (since $\varepsilon \neq 0$). If the data are such that the Taylor expansion for $\log(\widehat{P}^e(x,\tau))$ is finite order, we can compute the Taylor expansion for $\widehat{P}^e(x,\tau)$ and integrate it to obtain values for $a_{n,n+1}$ and $a_{n,n+2}$ to be used in place of zero in the system of ODEs. This procedure delivers the exact Taylor coefficients if the nuisance term is absent. Moreover our numerical investigation indicates that when the nuisance term is present, this procedure delivers significant improvement.

An example where $\eta = 1$. We now turn to solving (4.4). For this example, we assume $y = c$, so that

$$\mathcal{F}(x,\tau,\partial\Xi) := \theta\,(\Xi(x,\tau) - \log(\theta)) - \bar{\mu}_\Xi(x,\tau) - q(\tau)\,\widetilde{\mu}_C(x) \\ - (1-\gamma)\,\frac{1}{2}\,\|q(\tau)\,\sigma_C(x) + \bar{\sigma}_\Xi(x,\tau)\|^2. \quad (4.23)$$

For this example, let $\mu_X(x) = \kappa\,(\bar{X} - x)$, $\widetilde{\mu}_C(x) = x$,

$$\sigma_X(x) = \sqrt{\varsigma(x)}\begin{pmatrix} s_x \\ 0 \end{pmatrix}, \quad \text{and} \quad \sigma_C(x) = \sqrt{\varsigma(x)}\begin{pmatrix} s_{c1} \\ s_{c2} \end{pmatrix},$$

where (as before) $\varsigma(x) = (1-\alpha) + \alpha\,x$.[29] In this example the data are first-order polynomials in x, so the solution is also a first-order polynomial

$$\Xi(x,\tau) = \mathcal{A}_0(\tau) + \mathcal{A}_1(\tau)\,(x - \bar{X}).$$

The system of ODEs is

$$\mathcal{A}_0'(\tau) = \theta\,(\log(\theta) - \mathcal{A}_0(\tau)) + q(\tau)\,\bar{X} + \varsigma(\bar{X})\,D$$
$$\mathcal{A}_1'(\tau) = q(\tau) - (\kappa+\theta)\,\mathcal{A}_1(\tau) + \alpha\,D,$$

subject to $\mathcal{A}_0(0) = \log(\theta)$ and $\mathcal{A}_1(0) = 0$, where

$$D := (1-\gamma)\,\frac{1}{2}\,\left((q(\tau)\,s_{c1})^2 + (q(\tau)\,s_{c2} + s_x\,\mathcal{A}_1(\tau))^2\right).$$

When $\alpha \neq 0$ and $\gamma \neq 1$, explicit analytical expressions for $\mathcal{A}_0(\tau)$ and $\mathcal{A}_1(\tau)$ are difficult to obtain. (Numerical solutions are easy in any case.) When $\alpha = 0$, we have

$$\mathcal{A}_1(\tau) = \frac{\xi(\tau)}{\kappa+\theta}, \quad \text{where } \xi(\tau) = 1 - e^{-\theta\,\tau}\left(1 + q(\tau)\,\frac{\theta}{\kappa}\right). \quad (4.24)$$

[29] With $\alpha = 1$, this example is essentially the term structure example in Duffie and Epstein (1992a), while with $\alpha = 0$ it is essentially the term structure example of Duffie, Schroder, and Skiadas (1997) without the signal extraction problem. (The signal extraction problem puts a trend in s_x.)

Note that $\xi(0) = 0$ and $\xi(\infty) = 1$. Given that $\sigma_\psi = \bar{\sigma}_\Xi = \mathcal{A}_1 \sigma_X$, equation (4.24) implies

$$\sigma_\psi(t) = \begin{pmatrix} \dfrac{s_x}{\kappa + \theta}\, \xi(T-t) \\ 0 \end{pmatrix}. \tag{4.25}$$

The term structure and the equity premium. Continuing with this example where $\alpha = 0$, we investigate the term structure of interest rates and the premium on the capital account. Inserting (4.25) into (2.39), we have

$$r(t) = \theta + x(t) + \left(\frac{1}{2} - Q(T-t)\right)(s_{c1}^2 + s_{c2}^2) + (1-\gamma)\,\frac{s_{c1}\, s_x\, \xi(T-t)}{\kappa+\theta}$$

$$\lambda(t) = \begin{pmatrix} Q(T-t)\, s_{c1} + (\gamma - 1)\,\dfrac{s_x\, \xi(T-t)}{\kappa+\theta} \\ Q(T-t)\, s_{c2} \end{pmatrix}.$$

When $\eta = 1$, $\pi(t) = q(T-t)/\theta$ is deterministic, so that $\sigma_k(t) = \sigma_c(t)$. Because we also know from (2.25) that $\sigma_\phi(t) = \sigma_k(t)$, we see that the premium on the capital account is

$$\lambda(t)^\top \sigma_\phi(t) = Q(T-t)\, \|\sigma_c(t)\|^2 + (\gamma-1)\, \sigma_c(t)^\top \sigma_\psi(t).$$

The average slope of the term structure at the origin depends on the sign of the average of $\lambda(t)^\top \sigma_r(t) = \lambda(t)^\top \sigma_X(t)$. (A negative sign produces a positively sloped term structure.) In this case we have

$$\lambda(t)^\top \sigma_\phi(t) = Q(T-t)\,(s_{c1}^2 + s_{c2}^2) + (\gamma-1)\,\frac{s_{c1}\, s_x\, \xi(T-t)}{\kappa+\theta}$$

and

$$\lambda(t)^\top \sigma_r(t) = \lambda(t)^\top \sigma_x(t) = Q(T-t)\, s_{c1}\, s_x + (\gamma-1)\,\frac{s_x^2\, \xi(T-t)}{\kappa+\theta}.$$

Let us consider two cases. First, suppose $s_{c1} = 0$, so that the expected growth rate of consumption is uncorrelated with the actual growth rate ($\sigma_x(t)^\top \sigma_c(t) = 0$). In this case, the premium on the capital account is unambiguously positive and increases with γ, while the slope of the average yield curve depends on γ (the slope is positive if $\gamma < 1$, negative otherwise). An increase in γ reduces both the interest rate and the average slope of the yield curve at the origin. Second, suppose $\gamma = 0$, so that the agent is risk neutral and has a preference for late resolution of uncertainty. Suppose also that the horizon is infinite; in this case, the sign of the premium on the capital account is determined by the sign of the covariance between expected and actual growth rates of consumption, while the yield curve unambiguously slopes upward on average at the origin.

Existence and convergence: An example. Although the Cauchy–Kowaleskaya theorem guarantees a unique solution exists in the neighborhood of $\tau = 0$, it does not guarantee the existence of a solution for all finite τ, not does it guarantee, even when a solution exists for all finite τ, that the solution converges as $\tau \to \infty$. To begin to address these issues, consider the case where the nuisance term is absent. In this case, $r_e(t) = R_e(X(t))$ and $\lambda_e(t) = \Lambda_e(X(t))$, where

$$R_e(x) = d_0 + d_1\,\widetilde{\mu}_Y(x) - d_1^2\,\|\sigma_Y(x)\|^2 \quad \text{and} \quad \Lambda_e(x) = d_1\,\sigma_Y(x),$$

where we have used $d_2 = -d_1$. If

$$R_e(x), \quad \mu_X(x) - \sigma_X(x)^\top \Lambda_e(x), \quad \text{and} \quad \sigma_X(x)^\top \sigma_X(x)$$

are all affine in x, then we have an exponential-affine model of endowment bond prices. For example, suppose there are three Brownians and two independent state variables, each of which has dynamics given by

$$dX_i(t) = \kappa_i\,(\bar{X}_i - X_i(t))\,dt + s_i\,\sqrt{\varsigma_i(X_i)}\,dW_i(t),$$

where[30] $\varsigma_i(x) = (1-\alpha)\,\bar{X}_i + \alpha\,x$, so that

$$\mu_X(x) = \begin{pmatrix} \kappa_1\,(\bar{X}_1 - x_1) \\ \kappa_2\,(\bar{X}_2 - x_2) \end{pmatrix} \quad \text{and} \quad \sigma_X(x) = \begin{pmatrix} s_1\,\sqrt{\varsigma_1(x_1)} & 0 \\ 0 & s_2\,\sqrt{\varsigma_2(x_2)} \\ 0 & 0 \end{pmatrix}.$$

Now further suppose that we are solving the planning problem. We choose a one-factor model of the term structure and allow the second factor to affect the equity market. In particular, $r(t) = R(X(t))$ and $\lambda(t) = \Lambda(X(t))$, where

$$R(x) = x_1 \quad \text{and} \quad \Lambda(x) = \begin{pmatrix} \frac{q_1\,\sqrt{\varsigma_1(x_1)}}{s_1} & 0 & \frac{q_2\,\sqrt{\varsigma_2(x_2)}}{s_2} \end{pmatrix}^\top.$$

Note that

$$\sigma_X(x)^\top \Lambda(x) = \begin{pmatrix} q_1\,\varsigma_1(x_1) & 0 \end{pmatrix}^\top,$$

ensuring that X_2 does not affect real bond prices. Turning to the PDE for $\eta \neq 1$, note that $\widetilde{\mu}_Y(x) = R(x) + \frac{1}{2}\,\|\Lambda(x)\|^2$ and $\sigma_Y(x) = \Lambda(x)$.

First consider the case where $\alpha = 0$. In this case

$$R_e(x) = (1-\eta)\,x_1 + \eta\,\theta + \eta\,(1-\eta)\frac{1}{2}\left(\bar{X}_1\,q_1^2/s_1^2 + \bar{X}_2\,q_2^2/s_2^2\right).$$

Although the parameters of the dynamics of the second state variable appear, the second state variable itself does not. In fact, the model devolves to a single-state-variable model in this case. We observe that if $R_e(\bar{X}) \leq 0$ then the solution will not converge, since the value of a perpetuity is not defined in this case. There is however a more telling condition. We can solve for the asymptotic rate. If it is not

[30]The definition of $\varsigma(\,\cdot\,)$ here is slightly different from the definition in the previous section.

positive, the solution will not converge. The asymptotic forward rate can be shown to be

$$R_e(\bar{X}) - \frac{(1-\eta)^2\,\bar{X}_1\,\left(s_1^2 + 2\,\kappa_1\,q_1\right)}{2\,\kappa_1^2}.$$

Consider the parameters be given in Table 2 for example.[31] With these parameters the solution will not converge for $\eta > 1.6218$.

i	$\bar{X}_i$	κ_i	s_i^2	q_i
1	3/100	1/15	1/50	−11/75
2	1/100	3/500	1/2500	−1/50

TABLE 2. Parameter values.

Now consider the case where $\alpha = 1$. In this case, we have

$$R_e(x) = \eta\,\theta + \frac{(1-\eta)\left(\eta\,q_1^2 + 2\,s_1^2\right)x_1}{2\,s_1^2} + \frac{(1-\eta)\left(\eta\,q_2^2\right)x_2}{2\,s_2^2}.$$

Here we have the following problem. If one or both of the coefficients on the state variables in $R_e(x)$ is negative, there may not be a solution for all finite τ. In the present case, we run into problems as soon as $\eta > 1$.

When a nuisance term is present we cannot determine in advance the regions of existence and convergence because the endowment interest rate and price of risk are not known absent the solution for the value of the endowment annuity. Nevertheless, we believe the considerations are essentially the same. In closing, we note that our preliminary numerical investigations indicate that when $\alpha = 1$ solutions do not exist for all finite horizons when $\eta\,\gamma \gtrapprox 1$.[32]

5. Conclusion

Summary. In a nutshell, we we have made two complementary but independent contributions to asset pricing theory under recursive preferences—the first theoretical and the second numerical. On the theoretical front, we present a representation of continuation utility that reduces the general-equilibrium problem to a bond pricing problem. On the numerical front, we extend the class of Markovian models for which we can find the term structure of interest rates in terms of the state variables.

For any parameter values and any process for the forcing variable, we reduce the solution (as long as a solution exists) to that of finding bond prices under a derived process for the interest rate and price of risk, via a standard PDE. But this

[31] These parameter values are for illustrative purposes only; we have not attempted to calibrate the model to actual data. Nevertheless, the parameters for X_1 are representative of the estimates of Brown and Schaefer (1996) for the real term structure.

[32] As noted in the Introduction, we intend to include a numerical investigation in a future version of this paper. In the meantime, we have included a complete *Mathematica* package that implements of method.

transformation of the original problem into that of a term-structure problem would only carry us so far, if we did not know how to solve the resulting PDE. Duffie and Kan (1996) show how to solve the bond pricing PDE for the class of exponential-affine term-structure model by breaking it into a finite set of ordinary differential equations (one for each of the state variables, plus one for the constant term) that can be numerically solved in a fraction of a second on any modern computer. We show that as long as the dynamics of the forcing variables are sufficiently smooth (but not necessarily affine) the equilibrium-based annuity PDE also breaks down into a set of ODEs, though the set may be infinite. Solving all of these equations provides an exact solution, while solving for a finite number provides an approximate solution.

Further research. Notwithstanding some existing claims, we are not yet convinced that the framework of recursive preferences cannot rationalize the major asset-price puzzles. In our view, a complete and thorough examination has been hampered by the lack of tractable tools that would allow an unrestricted exploration. Most of the model-building work in this area has been conducted in a discrete-time setting where the number and, perhaps more importantly, the nature of shocks has been limited. Moreover, there has been the tendency to equate the horizon for the short-term risk-free interest rate with the sampling frequency of the consumption growth data.

Therefore, in our view, the empirical results that bear on the ability of a model such as we have focused on in this paper to resolve asset pricing puzzles is mixed. On the one hand, Cochrane and Hansen (1992) provide evidence that recursive preferences are consistent with the unconditional moments of the data. On the other hand, the attempts to build models consistent with the conditional moments have failed thus far. For example, Weil (1989) reaches the negative conclusion that the equity premium puzzle remains for reasonable values of the preference parameters. In his setting for the exchange problem, however, the shocks are homoskedastic. Yet heteroskedasticity, for example, might prove important for resolving the equity-premium puzzle, which stems from the inability of the model to generate sufficient volatility of the wealth-consumption ratio. If the term premium in (3.14) is stochastic rather than deterministic, then the volatility of a perpetuity depends on the volatility of the term premium as given in footnote 21. Depending on the covariance between endowment interest rate and the term premium, $\widehat{\sigma}_{r_e}(t,u)^\top \sigma_{\beta_e}(t,u)$, the relative variance of the wealth-consumption ratio, $\|\sigma_\pi(t)\|^2$, may be bigger in an economy with state-dependent volatilities than in an economy in which the expectations hypothesis holds for the endowment term structure.

Our continuous-time formulation of the model provides significant advantages over discrete-time formulations in that we have available both the tools of stochastic calculus that allow us to manipulate expressions and the solution techniques for differential-equations. Any serious calibration attempt, however, will require care in matching moments with the data. Although matching moments for asset prices may be fairly straightforward, matching moments for consumption growth will require some effort. In particular, we have no closed-form solutions for consumption growth-rates over discrete time periods in general. These must be computed from simulations given the parameters of the instantaneous dynamics. Yet this very difficulty provides some additional hope that the model may fit tolerably well, since the link between the instantaneous dynamics and the finite-horizon dynamics may contain the needed flexibility.

We cannot claim that the framework of recursive preferences will turn out to be compatible with all of the major known asset-pricing puzzles. But we can claim to have provided some tools that will be prove useful for a thorough study of the question. In future research, we intend to pursue this study.

Appendix A. The absence of arbitrage

The state-price deflator. We adopt the stochastic framework studied in Duffie (1996), to which we refer the reader for all omitted details. We restrict attention to a Brownian environment, by which we mean that we are given a l-dimensional vector of orthonormal Brownian motions, $W(t)$, defined on a fixed probability space, and the filtration is that generated by $W(t)$. In other words, the information that agents have at time t is that contained in the path of $W(s)$ for $s < t$.

We assume the existence of a *state-price deflator*, which follows a strictly positive Itô process $m(t)$ that we write as:

$$\frac{dm(t)}{m(t)} = -r(t)\,dt - \lambda(t)^\top dW(t)\,, \tag{A.1}$$

where "$^\top$" denotes the transpose, $r(t)$ is the instantaneous rate of interest and $\lambda(t)$ is the price of risk. Observe that we are free to model $r(t)$ and $\lambda(t)$ independently, as long as a solution to (A.1) exists.[33]

A state-price deflator $m(t)$ guarantees that asset prices are free of arbitrage possibilities. The price of any asset (expressed in a given unit of account) is determined by the formula that its *deflated gain* is a martingale. To see what this means, consider an asset with *cumulative dividend* $D(t)$ and value $V(t)$, both Itô processes. For simplicity of exposition, assume that $V(t)$ is strictly positive and that $D(t)$ is locally riskless, so their processes can be written as:

$$\frac{dV(t)}{V(t)} = \mu_V(t)\,dt + \sigma_V(t)^\top dW(t), \quad \text{and} \quad dD(t) = Z(t)\,dt\,,$$

where $Z(t)$ is the flow of dividends. The *gain* is the sum of the asset's value and its cumulative dividend, $G(t) := V(t) + D(t)$, while the deflated gain is $G(t)m(t)$. To say that $G(t)m(t)$ is a martingale is equivalent to saying that the price process $V(t)$ obeys

$$V(t) = E_t\left[\left(\frac{m(T)}{m(t)}\right)V(T) + \int_{s=t}^{T}\left(\frac{m(s)}{m(t)}\right)Z(s)\,ds\right], \tag{A.2}$$

[33] In the example given in Cox, Ingersoll, Jr., and Ross (1985b), the solution to (A.1) does not exist.

for any $T > t$, where E_t stands for the expectation conditional on time-t information. A direct implication of the pricing equation (A.2) is the no-arbitrage condition:

$$\mu_V(t) + \zeta(t) = r(t) + \lambda(t)^\top \sigma_V(t), \tag{A.3}$$

where $\zeta(t) := Z(t)/V(t)$ is called the *dividend rate.*

Changing the numeraire. Two kinds of assets play important roles in the sequel: zero-coupon bonds and assets that pay a continuous flow of dividends forever. Starting with zero-coupon bonds, let $p(t, T)$ denote the price at time t of the bond paying one unit of account at time T. According to the pricing formula (A.2), the terminal condition $p(T, T) = 1$ implies

$$p(t,\, T) = E_t \left[\frac{m(T)}{m(t)} \right],$$

so that the term structure theory reduces to the problem of producing conditional forecasts of the state-price deflator. Turning to the case of an asset that pays a continuous dividend flow $Z(t)$ forever, the pricing formula (A.2) implies that its price process obeys

$$V(t) = E_t \left[\int_{s=t}^{\infty} \frac{m(s)}{m(t)}\, Z(s) ds \right], \tag{A.4}$$

assuming $\lim_{T\to\infty} \; E_t[m(T)\, V(T)] = 0$. We now show that a change of numeraire transforms $V(t)$ into the price of a consol. Changing numeraire is thus often convenient, not only to study both real and nominal (or foreign and domestic) yield curves, but also to turn many other asset pricing problems into term structure problems.

To illustrate, let $b(t, T)$ denote the value of a claim to the single strictly positive payment of $S(T)$ at time T, so that

$$b(t, T) = E_t \left[\frac{m(T)}{m(t)}\, S(T)] \right].$$

We can think of this asset as a zero-coupon bond that makes its payment in a different "currency." Define $b_S(t, T) := b(t, T)/S(t)$ to be the value of the bond in the new currency units, and let $m_S(t) := m(t)\, S(t)$; then, the pricing equation above becomes

$$b_S(t, T) = E_t \left[\frac{m_S(T)}{m_S(t)} \right], \tag{A.5}$$

which we recognize as the value of a zero-coupon bond when $m_S(t)$ replaces $m(t)$ as the state-price deflator.

In general, given the choice of any strictly positive Itô process $S(t)$ as the new numeraire, $m_S(t) := m(t)\, S(t)$ defines a new state-price deflator. Gains that are measured in the new units, $G(t)/S(t)$, and deflated by the new deflator are martingales. Since $m_S(t)$ is a state-price deflator, its drift and diffusion are (minus) the short rate, r_S, and price of risk, λ_S, in the new units, and we are free to model r_S and λ_S independently (as long as the Itô process they define exists). Moreover, we are free to model independently any two of the processes $m(t)$, $m_S(t)$, and $S(t)$, leaving the third process to inherit the dynamic properties of the other two from the definition $m_S = m\,S$.

If the original units are those of a consumption good and $S(t)$ is the real value of a unit of currency (so that $1/S(t)$ is the price level), then $b_S(t,T)$ given in equation (A.5) is the nominal value of a zero-coupon currency-denominated bond, and $b(t,T)$ is its real value.

We apply this change-of-numeraire technique to infinitely-lived assets in two ways. In the first case, we set $S(t) = V(t)$, the strictly positive price process for an asset with dividend flow $Z(t)$. The new state-price deflator is $m_V(t) := m(t)\,V(t)$, whose dynamics, by Itô's lemma and the absence-of-arbitrage condition (A.3), are

$$\frac{d\,m_V(t)}{m_V(t)} = -\zeta(t)\,dt - (\lambda(t) - \sigma_V(t))^\top\,dW(t), \tag{A.6}$$

so that the short rate and price of risk are $r_V(t) = \zeta(t)$ and $\lambda_V(t) = \lambda(t) - \sigma_V(t)$. In the second case, we set $S(t) = Z(t)$, so that the new state-price deflator is $m_Z(t) := m(t)Z(t)$ (here we require that the dividend $Z(t)$ be strictly positive). The value of the asset in the new units is $V_Z(t) := V(t)/Z(t) = 1/\zeta(t)$. Let $b_Z(t,T)$ denote the value (in Z units) of a zero-coupon bond paying $Z(T)$ (*i.e.*, one Z unit) at time T. With the foregoing definitions equation (A.4) produces

$$\frac{1}{\zeta(t)} = V_Z(t) = E_t\left[\int_{s=t}^{\infty} \frac{m_Z(s)}{m_Z(t)}\,ds\right] = \int_{s=t}^{\infty} b_Z(t,s)\,ds. \tag{A.7}$$

Thus, the inverse of the dividend rate, $1/\zeta(t)$, is the value of a consol, which has a unit dividend flow and a yield of $\zeta(t)$ (measured in the new units).

Appendix B. *Mathematica* code

Here is the *Mathematica* package used to compute the PDE solutions in the paper.

```
(* :Title: AnnuitySolve.m *)

(* :Context: AnnuitySolve` *)

(* :Author: Mark Fisher *)

(* :Summary:
In an economy where the representative agent has recursive preferences
(Kreps-Porteus stochastic differential utility), the optimal
wealth-consumption ratio is the value of an annuity that satisfies a
quasi-linear 2nd-order PDE. This package provides tools to compute the
series solution to the PDE.
*)

(* :Package Version: 1.0 (August 1998) *)

(* :Mathematica Version: 3.0 *)

(* :Sources:
```

```
Fisher, M. and C. Gilles (1998) "Consumption and asset prices with
recursive preferences." Photocopied, Federal Reserve Board.
*)

(* :Discussion:
NAnnuitySolve is designed to compute the series solution to a
quasi-linear 2nd-order PDE in terms of an unknown function A[x,t],
where the data are real analytic and where the boundary condition is
of the form A[x,0] == C, for C constant. For an annuity, C == 0, while
for a zero-coupon bond, C == 1. The Cauchy-Kowaleskaya theorem
guarantees the existence of a unique real analytic solution in the
neighborhood of t == 0. The solution provided by NAnnuitySolve is of
the form Sum[a[i][t](x-x0)^i, {i, 0, order}] where the a[i][t] are
functions of t. The boundary condition is specified indirectly via the
option InitialCondition -> C, which imposes the condition a[0][0] == C
(along with a[i][0] == 0 for i >= 1). For exponential-polynomial bond
prices, the option FunctionalForm -> Exp allows one to solve for
Log[A[x,t]] subject to a[0][0] == 0.
*)

(* :Requirements: Utilities`FilterOptions` *)

(* :Examples:
For examples, see the notebook AnnuitySolveExamples.nb, which is
available from the author upon request.
*)

BeginPackage["AnnuitySolve`", {"Utilities`FilterOptions`"}]

AnnuitySolve::usage = "AnnuitySolve[pde, A, t, {x, x0}, order]
calculates a symbolic series solution for the pde to the order
specified, where A is an unknown function of x and t. It expands a
polynomial of factor loadings in t around the point x0. Multiple state
variables can be specified as in AnnuitySolve[pde, A, t, {x, x0}, {y,
y0}, order]. As specified, AnnuitySolve returns a pure function; if A
is replaced by A[outargs], AnnuitySolve returns the function evaluated
at outargs. AnnuitySolve has the options FunctionalForm and
InitialCondition. In addition, options can be passed to DSolve."

NAnnuitySolve::usage = "NAnnuitySolve[pde, A, {t, min, max}, {x, x0},
order] calculates a numerical series solution for the pde to the order
specified, where A is an unknown function of x and t. It expands a
polynomial of factor loadings in t around the point x0. Multiple state
variables can be specified as in AnnuitySolve[pde, A, {t, min, max},
{x, x0}, {y, y0}, order]. As specified, NAnnuitySolve returns a pure
function; if A is replaced by A[outargs], NAnnuitySolve returns the
function evaluated at outargs. NAnnuitySolve has the options
FunctionalForm, InitialCondition, FactorLoadingSymbol,
PolynomialOrderDifferential, and DifferentialLoadings. In addition,
```

```
options can be passed to NDSolve."

BondToAnnuitySeries::usage = "BondToAnnuitySeries[bond, t, {x, x0},
order] takes an expression for zero-coupon bond prices as a function
of x and t and returns series coefficients (as functions of t that
have been symbolically integrated) up to the order specified for the
associated annuity. Multiple state variables can be specified as in
BondToAnnuitySeries[bond, t, {x, x0}, {y, y0}, order]. Options can be
passed to Integrate."

NBondToAnnuitySeries::usage = "NBondToAnnuitySeries[bond, {t, min,
max}, {x, x0}, order] takes an expression for zero-coupon bond prices
as a function of x and t and returns series coefficients (as functions
of t that have been numerically integrated) up to the order specified
for the associated annuity. Multiple state variables can be specified
as in NBondToAnnuitySeries[bond, {t, min, max}, {x, x0}, {y, y0},
order]. The output is designed to be used with the option
DifferentialLoadings (in conjunction with PolynomialOrderDifferential)
for NAnnuitySolve. Options can be passed to NDSolve."

AbsValuePDE::usage = "AbsValuePDE[pde, A, soln] returns the absolute
value of the deviations of the PDE from zero (as a pure function),
where the function A has been replaced by the trial solution. The
function is useful for determining how well the PDE is satisfied.
AbsValuePDE[pde, A[args], soln] returns the function evaluated at
args. "

FunctionalForm::usage = "FunctionalForm is an option for AnnuitySolve
and NAnnuitySolve. The default setting is FunctionalForm -> Identity.
The setting FunctionalForm -> Exp can be used to solve for
exponential-polynomial bond prices."

InitialCondition::usage = "InitialCondition is an option for
AnnuitySolve and NAnnuitySolve. It specifies the value of the
zero-order factor loading function at t = 0. The default setting for
NAnnuitySolve is InitialCondition -> 10^-100, which avoids division by
zero when the PDE is quasi-linear."

PolynomialOrderDifferential::usage = "PolynomialOrderDifferential is
an option for NAnnuitySolve. The default setting is
PolynomialOrderDifferential -> 0. This option is used in conjunction
with the option DifferentialLoadings. Typically for this purpose the
setting would be PolynomialOrderDifferential -> 2."

DifferentialLoadings::usage = "DifferentialLoadings is an option for
NAnnuitySolve. The default setting is DifferentialLoadings -> {}. This
option is used in conjunction with the option
PolynomialOrderDifferential. Typical use would involve computing
exponential-polynomial bond prices (using NAnnuitySolve with
```

```
FunctionalForm -> Exp), and then calling NBondToAnnuitySeries, the
output of which would be used as the DifferentialLoadings."

FactorLoadingSymbol::usage = "FactorLoadingSymbol is an option for
NAnnuitySolve. It is used to coordinate passing the output of
NBondToAnnuitySeries to NAnnuitySolve via the option
DifferentialLoadings. The default setting is FactorLoadingSymbol ->
$a, which need not be changed unless there is a symbol conflict."

$a::usage = "$a is the symbol for the factor loadings."

MakeCoefficients::usage = "MakeCoefficients[order, n] is an auxiliary
function, called by other functions."

MakePolynomial::usage = "MakePolynomial[xvars, t, coeffs] is an
auxiliary function, called by other functions."

Begin["‘Private‘"]

AnnuitySolve::badargs = "The arguments to the function in the PDE are
not properly specified."

Options[AnnuitySolve] = {FunctionalForm -> Identity,
    InitialCondition -> 0}

AnnuitySolve[pde_Equal, A_Symbol, t_Symbol, xargs:{_Symbol, _}..,
        order_Integer?Positive, opts___?OptionQ] :=
    Module[{ff, ic0, fls, dopts, x, x0, n, arglist, args, coeffs, a, poly,
        polyeqn, seriesargs, le, initconds, odes},
    {ff, ic0} = {FunctionalForm, InitialCondition} /. {opts} /.
        Options[AnnuitySolve];
    fls = FactorLoadingSymbol /. {opts} /. Options[NAnnuitySolve];
    dopts = FilterOptions[DSolve, opts];
    {x, x0} = Transpose[{xargs}];
    n = Length[x];
    arglist = Union @ Cases[pde, f_[A][x__] | A[x__] :> {x}, Infinity];
    args = First @ arglist;
    If[Length[arglist] != 1 || Union[args] =!= Union[Join[x, {t}]],
        Message[AnnuitySolve::badargs]; Return[$Failed]];
    coeffs = MakeCoefficients[order, n, fls];
    poly = MakePolynomial[x - x0, t, coeffs];
    polyeqn = (Subtract @@ pde) /. A -> Function @@ {args, ff @ poly};
    seriesargs = Sequence @@ Thread[{x, x0, order}];
    le = List @@ LogicalExpand[Series[polyeqn, seriesargs] == 0];
    initconds = Thread[
        Through[coeffs[0]] == Prepend[Table[0, {Length[le] - 1}], ic0]
        ];
    odes = Join[le, initconds];
    Function @@ {args, ff @ poly /.
```

```
        ( First @ DSolve[odes, coeffs, t, Evaluate[dopts]] )}
    ]

AnnuitySolve[pde_Equal, A_Symbol[outargs__], t_Symbol,
        xargs:{_Symbol, _}.., order_Integer?Positive, opts___?OptionQ] :=
    AnnuitySolve[pde, A, t, xargs, order, opts][outargs]

Options[NAnnuitySolve] = {FunctionalForm -> Identity,
    InitialCondition -> 10^-100, PolynomialOrderDifferential -> 0,
    DifferentialLoadings -> {}, FactorLoadingSymbol -> $a}

NAnnuitySolve[pde_Equal, A_Symbol,
        range:{t_Symbol, tmin_?NumericQ, tmax_?NumericQ},
        xargs:{_Symbol, _?NumericQ}.., order_Integer?Positive,
        opts___?OptionQ] :=
    Module[{ff, ic0, diff, diffloads, fls, ndopts, x, x0, n,
        arglist, args, allcoeffs, coeffs, poly, polyeqn, seriesargs,
        le, initconds, odes},
    {ff, ic0, diff, diffloads, fls} = {FunctionalForm,
        InitialCondition, PolynomialOrderDifferential,
        DifferentialLoadings, FactorLoadingSymbol} /. {opts} /.
        Options[NAnnuitySolve];
    ndopts = FilterOptions[NDSolve, opts];
    {x, x0} = Transpose[{xargs}];
    n = Length[x];
    arglist = Union @ Cases[pde, f_[A][x__] | A[x__] :> {x}, Infinity];
    args = First @ arglist;
    If[Length[arglist] != 1 || Union[args] =!= Union[Join[x, {t}]],
        Message[AnnuitySolve::badargs]; Return[$Failed]];
    allcoeffs = MakeCoefficients[order + diff, n, fls];
    coeffs = Select[allcoeffs, ( Max @@ # ) <= order &];
    diffloads = Select[diffloads, ( Max @@ #[[1]] ) > order &];
    poly = MakePolynomial[x - x0, t, allcoeffs] /. diffloads;
    polyeqn = (Subtract @@ pde) /. A -> Function @@ {args, ff @ poly};
    seriesargs = Sequence @@ Thread[{x, x0, order}];
    le = List @@ LogicalExpand[Series[polyeqn, seriesargs] == 0];
    initconds = Thread[
        Through[coeffs[0]] == Prepend[Table[0, {Length[le] - 1}], ic0]
        ];
    odes = Join[le, initconds];
    Function @@ {args, ff @ poly /.
        ( First @ NDSolve[odes, coeffs, range, Evaluate[ndopts],
            StartingStepSize -> 10^-6, MaxSteps -> 10^6] ) }
    ]

NAnnuitySolve[pde_Equal, A_Symbol[outargs__],
        range:{t_Symbol, tmin_?NumericQ, tmax_?NumericQ},
        xargs:{_Symbol, _?NumericQ}.., order_Integer?Positive,
        opts___?OptionQ] :=
```

```
    NAnnuitySolve[pde, A, xargs, order, range, opts][outargs]

BondToAnnuitySeries[bond_, t_Symbol, xargs:{_Symbol, _}..,
        order_Integer?Positive, opts___?OptionQ] :=
    Module[{fls, iopts, x, x0, n, seriesargs, ser, sercoeffs,
        g, loadings},
    fls = FactorLoadingSymbol /. {opts} /. Options[NAnnuitySolve];
    iopts = FilterOptions[Integrate, opts];
    {x, x0} = Transpose[{xargs}];
    n = Length[x];
    seriesargs = Sequence @@ Thread[{x, x0, order}];
    ser = Series[bond, seriesargs];
    sercoeffs = SeriesTable[ser, order, n];
    loadings = Integrate[sercoeffs /. t -> s, {s, 0, t}, Evaluate[iopts]];
    Thread[MakeCoefficients[order, n, fls] -> (Function[t, #]& /@ loadings)]
    ]

NBondToAnnuitySeries::badbond = "The expression for the value of the
bond is not numeric."

NBondToAnnuitySeries[bond_,
        range:{t_Symbol, tmin_?NumericQ, tmax_?NumericQ},
        xargs:{_Symbol, _?NumericQ}..,
        order_Integer?Positive, opts___?OptionQ] :=
    Module[{fls, ndopts, x, x0, n, seriesargs, ser, sercoeffs,
        loadings, g},
    fls = FactorLoadingSymbol /. {opts} /. Options[NAnnuitySolve];
    ndopts = FilterOptions[NDSolve, opts];
    {x, x0} = Transpose[{xargs}];
    If[!NumericQ[bond /. Thread[x -> x0] /. t -> (tmax + tmin)/2],
        Message[NBondToAnnuitySeries::badbond]; Return[$Failed]];
    n = Length[x];
    seriesargs = Sequence @@ Thread[{x, x0, order}];
    ser = Series[bond, seriesargs];
    sercoeffs = SeriesTable[ser, order, n];
    loadings = (g /. First @ NDSolve[{g'[t] == #, g[0] == 0},
            g, range, Evaluate[ndopts], StartingStepSize -> 10^-6,
            MaxSteps -> 10^6])& /@ sercoeffs;
    Thread[MakeCoefficients[order, n, fls] -> loadings]
    ]

AbsValuePDE[pde_Equal, A_Symbol, soln_] :=
    Module[{arglist, args},
    arglist = Union @ Cases[pde, f_[A][x__] | A[x__] :> {x}, Infinity];
    If[Length[arglist] != 1, Message[AnnuitySolve::badargs]; Return[$Failed]];
    args = First @ arglist;
    Function @@ {args,
        Abs[Subtract @@ pde] /. A -> Function @@ {args, soln}}]
```

```
AbsValuePDE[pde_Equal, A_Symbol[outargs__], soln_] :=
    AbsValuePDE[pde, A, soln][outargs]

(* auxiliary functions *)
MakeCoefficients[order_, n_, sym_:$a] :=
    Module[{b},
    Flatten @
        Table[sym @@ Array[b, n],
            Evaluate[ Sequence @@ Thread[{Array[b, n], 0, order}] ]
        ]
    ]

MakePolynomial[xvars_List, t_, coeffs_List] :=
    Apply[Times, xvars^# & /@ (coeffs /. $a -> List), {1}].Through[coeffs[t]]

(* this is a kludge; SeriesCoefficient[series, {n1, n2, ...}] doesn't work *)
SeriesTable[series_, order_, n_] :=
    Flatten @ Fold[
        Map[Table[SeriesCoefficient[#, i], {i, 0, order}]&, #1, {#2}]&,
        series,
        Range[0, n - 1]
        ]

End[]
EndPackage[]
```

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