Tests for Non-Linear Dynamics in Systems of Non-Stationary Economic Time Series:

The case of short-term US interest rates

by

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Abstract: Using Hall and Heyde's (1980) representation theorem, we show that the stationary co-integration relations of an integrated system are generally non-linear stochastic processes. We propose a sequential non-parametric procedure to test stationary co-integration relations for non-linear dynamics, and apply this procedure to short-term US interest rates as an illustration. We demonstrate that the weekly federal funds rate is co-integrated with Treasury bill and commercial paper rates and that the co-integration relations are non-linear.

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Macroeconomists often employ time-invariant linear dynamic structural models to estimate the responses of economic variables to stochastically independent structural shocks. Econometric models of this form implicitly assume that the economic variables in the system are linear stochastic processes.[1] The response of a linear stochastic process to a shock is completely characterized by the coefficients in its moving average representation, called the impulse response function. Analysis of the impulse response functions has been widely used to study linear macroeconomic models of the business cycle, money demand, and the monetary transmission mechanism.[2]

If the economy is non-linear, its response to a shock is much more complex, and an econometric model that erroneously assumes linearity will not capture important characteristics of the response to a shock. In particular linear systems exhibit the principle of superposition, which means that, if the input to a linear system is a summation of sine waves of given frequencies, the output will be a summation of sine waves with the same frequencies. The principle of superposition does not hold for non-linear systems. Therefore, non-linear economic systems have highly complex responses to structural shocks, which makes it very difficult to isolate the sources of business cycle variation. For example, in a linear economic model, a policy shock can induce variation in real output only at the same frequency as the shock. This assumption is implicit in many studies of the business cycle.[3] In a non-linear economic system, business cycle variation can be induced by a policy shock of any frequency, because the policy shock can interact non-linearly with other structural disturbances.[4]

In this paper, we adapt a well-known univariate test for non-linearity to test a system of non-stationary macroeconomic variables for non-linear dynamics. We focus on the case of integrated and possibly co-integrated systems. We use Hall and Heyde's (1980) representation theorem to show these stationary co-integration relations will, generally, be non-linear stochastic processes, and develop a sequential estimation procedure to test the stationary co-integration relations for non-linearity.

In Section I, we provide basic results on testing stationary time series for non-linear dynamics using polyspectral methods. Hinich (1982) derives a test for non-linearity based on the property that stationary linear stochastic processes have flat bispectra.[5] The Hinich bispectrum test explicitly requires stationarity of the process being tested.[6]

In order to apply the bispectrum test, non-stationary data must be transformed to produce stationary data. One type of non-stationarity that has been extensively investigated in economics is integration. A time series that is integrated of order d has a stationary d^{th} difference. Integrated time series can be reduced to stationarity either by differencing the level of the series, or through co-integration.[7] Co-integration has become one of the most important empirical characterizations of macroeconomic time series, and empirical analyses are often conditioned on the integration and co-integration properties of the data.[8]

In Section II, we use a representation theorem due to Hall and Heyde (1980) to show that the co-integration relations of an integrated system are generally stationary non-linear processes.[9] We propose a sequential non-parametric testing procedure to test stationary co-integration relations for non-linear dynamics. Bierens (1997, 1998) developed a non-parametric test for co-integration, based on Hall and Heyde's theorem. This test does not require linearity, which implies that the estimated co-integration relations are valid even if the system is non-linear. We can test the resulting co-integration relations, which are stationary, for non-linear dynamics using variants of Hinich's (1982) non-parametric test for non-linearity. Granger (1991) proposes several non-linear generalizations of co-integration including non-linear error-correction. If our test rejects linearity of the stationary co-integration relations, non-linear error-correction is one possibility.

In Section III, we apply our test to a system of short-term U.S. interest rates as an illustration. We show that the Treasury bill rate, the commercial paper rate, and the federal funds rate are co-integrated over the period 1971-1997. Short-term interest rates are well suited for non-linear testing, because the power of tests for non-linear dynamics increase substantially with sample size, and short-term interest rates have large sample sizes relative to other business cycle variables, such as real output and inflation. In addition, discrete sampling of a continuous time series causes aliasing, which Hinich and Patterson (1989) found empirically biases tests for non-linearity toward finding linearity.[10] We constructed high frequency weekly interest rate data using an anti-aliasing filter, following Hinich and Patterson (1985b, 1989), that minimizes the effect of aliasing.[11] Using variants of Hinich's (1982) test, we find that the resulting stationary linear combinations are non-linear. These results are robust to sample period and suggest that the untested linearity assumption implicit in many macroeconomic models may be incorrect. In particular, our analysis suggests that interest rate spreads, which are important to the monetary transmission mechanism, exhibit non-linear dynamics.

Appendix 1 describes our bispectrum estimator and the null distributions of the test statistics. Appendix 2 discusses aliasing and anti-aliasing filter design.

I. Non-Linearity and Polyspectral Analysis

Let X_t be a real, mean zero, third-order stationary stochastic process.[12] Define the first three cumulants as $c_X(t) = E[X_t]$, $c_{XX}(t_1, t_2) = E[X_{t_1} X_{t_2}]$, and $c_{XXX}(t_1, t_2, t_3) = E[X_{t_1} X_{t_2} X_{t_3}]$.[13] Third-order stationarity implies $c_X(t) = 0$ for all *t*, $c_{XX}(t_1, t_2)$ is a function only of $\tau = (t_1 - t_2)$, and $c_{XXX}(t_1, t_2, t_3)$ is a function only of $\tau_1 = (t_1 - t_2)$ and $\tau_2 = (t_2 - t_3)$. We therefore denote the second and third-order cumulant functions by $c_{XX}(\tau)$ and $c_{XXX}(\tau_1, \tau_2)$ respectively. These functions are assumed to be absolutely summable. Under mild regularity conditions, X_t has a representation of the form:

$$X_t = \sum_{u=-\infty}^{\infty} g_u \varepsilon_{t-u} , \tag{1}$$

where g_u is a sequence of coefficients, and ε_t is a serially uncorrelated white noise input sequence.[14] In this representation, X_t is the output of a time-invariant linear filter applied to white noise input, but it is not necessarily a linear process. X_t is a *linear sequence* if ε_t is stochastically independent.[15] In general, whiteness is not sufficient for stochastic independence unless the white noise sequence is Gaussian.

The response of a linear sequence to a shock is completely characterized by the *transfer function* of the filter:

$$G(f) = \sum_{u=-\infty}^{\infty} g_u e^{-i(2\pi f)u} . \tag{2}$$

If the input to a linear sequence is a sine wave of frequency *f*, the output will also be a sine wave with frequency *f*. The amplitude will be scaled by $|G(f)|$, and the phase will be shifted by $\tan^{-1}(\mathrm{Im}\,G(f)/\mathrm{Re}\,G(f))$.[16]

A general model for a *non-linear* sequence is

$$X_t = h(\ldots, \varepsilon_{t-2}, \varepsilon_{t-1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}, \ldots) \tag{3}$$

where ε_t is stochastically independent. If X_t is causal, it does not depend on the future values of ε_t. This is a common assumption that would not substantively affect our discussion. If *h* is a well-behaved function it can be represented as a Volterra series:[17]

$$X_t = \sum_{u=-\infty}^{\infty} g_u \varepsilon_{t-u} + \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} g_{u,v}\varepsilon_{t-u}\varepsilon_{t-v} + \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty}\sum_{w=-\infty}^{\infty} g_{u,v,w}\varepsilon_{t-u}\varepsilon_{t-v}\varepsilon_{t-w} + \ldots \tag{4}$$

The response of the non-linear sequence to a shock will depend on *generalized transfer functions* of the form:

$$G(f) = \sum_{u=-\infty}^{\infty} g_u e^{-i(2\pi f)u}\ ,\ G(f,g) = \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} g_{u,v} e^{-i2\pi(fu+gv)}\ ,\ \ldots \tag{5}$$

If the input to a non-linear sequence contains components with frequencies *f* and *g*, then the output will contain components with frequencies *f*, *g*, $(f+g)$, 2*f*, 2*g*, $2(f+g)$, 3*f*, 3*g*, $3(f+g), \ldots$, and the amplitudes and phases of these components will depend on the generalized transfer functions.

Tests for linearity and Gaussianity can be based on the higher-order polyspectra of stationary sequences.[18] In general, the kth-order polyspectrum is the Fourier transform of the kth-order cumulant function. The second-order cumulant polyspectrum (*power spectrum*) is defined as the Fourier transform of $c_{XX}(\tau)$:

$$P_X(f) = \sum_{\tau=-\infty}^{\infty} c_{XX}(\tau) e^{-i2\pi f\tau}\ ,\ |f| < \frac{1}{2}. \tag{6}$$ [19]

Similarly, the third-order cumulant polyspectrum (*bispectrum*) is defined as the second-order Fourier transform of $c_{XXX}(\tau_1, \tau_2)$:

$$B_X(f,g) = \sum_{\tau_1=-\infty}^{\infty}\sum_{\tau_2=-\infty}^{\infty} c_{XXX}(\tau_1, \tau_2) e^{-i2\pi(f\tau_1 + g\tau_2)}\ , \tag{7}$$

$(f,g) \in D = \{(f,g):\ 0 < f < (1/2),\ g < f,\ 2f + g < 1\}$.[20] If the second and third-order cumulant functions are absolutely summable, then the power spectrum and the bispectrum exist and are well defined.

The power spectrum and bispectrum can be interpreted using the Cramér spectral representation of X_t:

$$X_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi ft} dZ_X(f)\,, \tag{8}$$

where,

$$E[dZ_X(f)] = 0\,,\ E[dZ_X(f)dZ_X^*(g)] = \begin{cases} 0 & f \neq g \\ P(f)df & f = g \end{cases} \tag{9}$$

and

$$E[dZ_X(f)dZ_X(g)dZ_X^*(h)] = \begin{cases} 0 & f+g \neq h \\ B(f,g)dfdg & f+g = h \end{cases}. \tag{10}$$

[21]

The power spectrum describes the contribution to the expectation of the product of two Fourier components whose frequencies are the same, whereas the bispectrum describes the contribution to the expectation of the product of three Fourier components where one frequency is equal to the sum of the other two. The integral of the power spectrum is equal to the variance of the sequence, $c_{XX}(0)$, and the power spectrum can be interpreted as a decomposition of the variance by frequency. Similarly, the bispectrum decomposes the skewness of the sequence, $c_{XXX}(0,0)$, by pairs of frequencies.

Define the *skewness function*, $\Gamma_X(f,g)$, as the normalized square modulus of the bispectrum:

$$\Gamma_X(f,g) = \frac{|B_X(f,g)|^2}{P_X(f)P_X(g)P_X(f+g)}\,. \tag{11}$$

Let ε_t be a stochastically independent sequence, then $P_\varepsilon(f) = c_{\varepsilon\varepsilon}(0)$ and $B_\varepsilon(f,g) = c_{\varepsilon\varepsilon\varepsilon}(0,0)$ for all $(f,g) \in D$. This implies that a linear process has a constant skewness function equal to $\Gamma_X(f,g) = c_{\varepsilon\varepsilon\varepsilon}(0,0)^2 c_{\varepsilon\varepsilon}(0)^{-3}$, because $P_X(f) = |G(f)|^2 P_\varepsilon(f)$ and $B_X(f,g) = G(f)G(g)G^*(f+g)B_\varepsilon(f,g)$.[22] If the stochastically independent input sequence is also Gaussian then $c_{\varepsilon\varepsilon\varepsilon}(0,0) = 0$ and $\Gamma_X(f,g)$ will be identically zero. These properties form the basis of Hinich's (1982) and Rao and Gabr's (1980) tests of Gaussianity and linearity. The tests used in this paper are based on Hinich (1982).[23] The null hypothesis of the Hinich test that $\Gamma_X(f,g)$ is constant for all frequency pairs is a necessary but not sufficient condition for linearity.[24] These tests and their distributions under the null are described in Appendix 1.

II. Integration and Co-integration

Macroeconomic time series often appear to be subject to permanent shocks, and it has become a standard practice to model these time series as non-stationary integrated processes.[25] The polyspectral tests for non-linearity cannot be applied directly to integrated time series, but they can be applied to either the first differences or the co-integration relations. In this section, we develop a model, which proves that stationary co-integration relations are generally non-linear processes.

Initially, we establish some notational conventions. Let (Ω, F, μ) denote a probability space. Let $T:\Omega \to \Omega$ denote a one to one ergodic measure-preserving shift transformation. If $X_0(\omega)$ is a random variable, then $X_t(\omega) = X_0(T^t\omega)$ defines a strictly stationary ergodic sequence.[26] Hall and Heyde (1980, pp. 136) prove that $X_t(\omega)$ has a representation of the form:

$$X_t(\omega) = Y_t(\omega) + Z_t(\omega) - Z_{t+1}(\omega)\ , \tag{12}$$

where $Y_t(\omega) = Y_0(T^t\omega)$ is a stationary martingale difference sequence, and $Z_t(\omega) = Z_0(T^t\omega)$ such that $Z_0(\omega)$ is in L^1.[27] Explicit formulas for the representation are given by:

$$Y_0 = \sum_{k=-\infty}^{\infty} (E[X_k \mid F_0] - E[X_k \mid F_{-1}])\ ; \tag{13}$$

and,

$$Z_0 = \sum_{k=0}^{\infty} (E[X_k \mid F_{-1}]) - \sum_{k=-\infty}^{-1} (X_k - E[X_k \mid F_{-1}])\ , \tag{14}$$

where $\{F_s\}$ is the filtration generated by the shift transform.

A stochastic sequence is said to be *integrated of order one*, $I(1)$, if the first difference of the sequence is stationary.[28] Let S_t be a q-dimensional $I(1)$ vector sequence, $S_t = (S_{1t},\ldots,S_{qt})^T$. The first-difference sequence is stationary and has the following representation:

$$S_t - S_{t-1} = \Delta S_t = X_t = Y_t + Z_t - Z_{t+1}\ , \tag{15}$$

where Y_t is a stationary vector martingale difference sequence, and Z_t is a stationary vector sequence.[29] The level of the integrated sequence

$$S_t = \sum_{s=0}^{t} Y_s - Z_{t+1} + Z_1 + S_0 \tag{16}$$

is non-stationary and is dominated by the accumulated martingale difference which gives rise to the permanent shocks.

A system of integrated time series is *co-integrated* if some linear combinations of the time series are stationary. Co-integration can be defined as a reduced rank condition involving the covariance matrix of the vector martingale difference. Let the covariance matrix have the form:

$$E[Y_t\, Y_s^T] = \begin{cases} CC^T & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} . \tag{17}$$[30]

If C has reduced rank, $(q-r)$, then there will exist r non-trivial vectors $\beta_1, ..., \beta_r$, called *co-integration vectors*, such that $\beta_j^T C = 0^T$, for all $j = 1, \ldots, r$. Let β be the q by r matrix $\left[\beta_1\ \beta_2 \ldots \beta_r \right]$. The linear combinations, $\beta_j^T S_t$, called *co-integration relations* are stationary, for all $j = 1, \ldots, r$.[31] By contrast, the $(q-r)$-dimensional sequence, $\beta_\perp^T S_t$, called the *common stochastic trends*, is integrated but not co-integrated, where $\beta_\perp$ is the q by $(q-r)$ orthogonal compliment matrix of β.[32]

The q-dimensional sequences ΔS_t and $\left[\beta^T \quad \beta_\perp^T \Delta \right] S_t$ are both stationary.[33] The components of ΔS_t have the form:

$$\Delta S_{jt} = Y_{jt} + Z_{jt} - Z_{j,t+1} . \tag{18}$$

Both Y_{jt} and Z_{jt} generally exhibit non-linear dependence, although Y_{jt} is a martingale difference and is non-forecastable in the mean square metric, see Hinich and Patterson (1987). The co-integration relations have the form,

$$\beta_j^T S_t = \beta_j^T (Z_1 - Z_{t+1}) + \beta_j^T S_0 , \tag{19}$$

and have been purged of the effects of the permanent shocks generated by the martingale difference. These relations are generally non-linear, because they are linear combinations of the potentially non-linear components of $(Z_1 - Z_{t+1})$.[34]

Our proposed method for testing for whether co-integration relations are non-linear is to first determine the number of co-integrating vectors using Bierens' (1997, 1998) non-parametric test. We then test the estimated co-integrating relations for Gaussianity and linearity using Hinich's (1982) tests. This sequential method allows us to test the stationary components of the system for non-linear dynamics.

Our method contrasts with the standard approach to co-integration. Stationary linear combinations of integrated variables are usually specified to follow a linear ARMA process or are included in linear structural models. The standard linear vector error correction model (VECM) has the form:

$$\Delta S_t = \alpha\beta^T S_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta S_{t-j} + \varepsilon_t \ . \tag{20}$$

If the model is co-integrated then the *q* by *r* parameter matrices, α and β, have rank *r*. The co-integration relations enter the model linearly, through the coefficient vector α. The error-correction model is usually estimated under the assumption that ε_t is stochastically independent, which implies that the co-integration relations are linear stochastic processes. Our discussion shows that co-integration does not generally imply linearity, therefore, there is no reason to expect ε_t to be either Gaussian or independent.

Granger (1991) proposes three non-linear generalizations of co-integration. The first generalization is that non-linear functions of the time series may be co-integrated in the sense that $g_1(x_{1t})$ and $g_2(x_{2t})$ have a dominant property that the linear combination of non-linearly transformed variables $z_t = g_1(x_{1t}) - A g_2(x_{2t})$ does not exhibit. A second generalization is to allow time-varying co-integration vectors. A third generalization is non-linear error correction, in which the co-integration relations would enter the error-correction model through a non-linear function *f*, i.e.

$$\Delta S_t = f(\beta^T S_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta S_{t-j} + \varepsilon_t \ . \tag{21}$$

A natural non-linear error correction specification is to allow mean reversion only for large deviations of the co-integration relations from their means, so that *f* has the form:

$$f(z) = \begin{cases} \text{z if z} > k \\ 0 \text{ if z} \le k \end{cases} . \tag{22}$$

In this case, $z_t = \beta^T S_t$ behaves like a unit root in a neighborhood of its mean, but exhibits mean reversion when it is outside the neighborhood. This model is a straightforward generalization of the standard error-correction model that exhibits non-linear dynamics, but the linear combination, $z_t = \beta^T S_t$, is not generally stationary.[35] This last point is the main difference between our approach and non-linear error correction. We show that co-integration relations can be non-linear stationary stochastic processes. If our test rejects linearity, then the class of stationary non-linear error correction models could be investigated as a possible model for the non-linearity. An alternative approach would be to model the co-integration relation using any of the standard univariate non-linear models such as bi-linearity, threshold auto-regression, non-linear moving average, etc.[36]

III. Empirical Results

In this section, we test a co-integrated system of short-term U.S. interest rates for non-linear dynamics. We focus on interest rates because we can obtain a long, high frequency sample, which is essential in testing for non-linearity. Ashley, Patterson, and Hinich (1986, pp. 171-173) show that the power of the bispectrum test for non-linearity improves considerably as sample sizes increase, and has very high power at sample sizes over one thousand against certain non-linear models.

Short-term interest rates on federal funds and on unsecured corporate and government debts are frequently included in studies of the business cycle, money demand, and the monetary transmission mechanism. Improper sampling of high frequency time series leads to a type of bias called aliasing. Hinich and Patterson (1989) found that correcting for aliasing improved the performance of the bispectrum test for non-linearity. We obtained business daily data for the federal funds rate, the secondary market rate on one-month treasury bills, and the interest rate on one-month commercial paper from 4/08/1971 to 8/29/1997.[37] These interest rates are converted to one-month holding period yields on a bond interest basis, and are passed through an anti-aliasing filter. The anti-aliasing filter is designed to remove the high-frequency power in the daily rate series to minimize the bias caused by converting the daily time series to weekly time series.[38] The daily rates are converted to weekly rates by sampling the filtered daily rates every seven days. These weekly rates are denoted as CP1M, TB1M, and FF. We show that these rates are integrated and co-integrated and exhibit non-linear serial dependence.

The method we use to solve the aliasing problem follows Hinich and Patterson (1985b, 1989). The problems of aliasing and anti-aliasing filter design are discussed in detail in Appendix 2.

III.1 Integration and Tests for Non-Linearity

The assumption that short-term U.S. interest rates are integrated is quite common.[39] We ran a battery of univariate tests of the unit root and stationarity hypotheses on ln(CP1M), ln(TB1M), ln(FF), and their first differences. ADF1 and ADF2 are augmented Dickey-Fuller tests of the unit root and unit root with drift hypotheses against stationarity and trend stationarity respectively.[40] PP1 and PP2 are the Phillips-Perron tests of the same hypotheses.[41] KPSS1 and KPSS2 are tests of the stationarity and trend stationarity hypotheses against the alternatives of unit root and unit root with drift respectively.[42] Finally, the Bierens (1997) non-parametric test for the existence of co-integration is run as a univariate test of the unit root with drift hypothesis against trend stationarity on each variable. These tests, which are reported in Table 1, show that ln(CP1M), ln(TB1M), and ln(FF) are $I(1)$. Figures 1 and 2 display the levels and first differences of the interest rates.

Let S_t denote the vector of logged interest rates. We can test each component of the stationary first difference vector, ΔS_t, for non-linearity by testing for flatness of the skewness functions, using the tests described in Appendix 1. We pre-whiten each of the components using an AR(6) filter to eliminate bias in the spectral estimation prior to testing, to decrease the likelihood of falsely rejecting the null of linearity.

The results of the non-linearity tests are reported in Table 2. The Z_ρ test statistics are normally distributed under the null of linearity, and we treat these tests as two tailed tests.[43] Using the full sample, the tests provide strong evidence of non-linearity. Broadly speaking the values of the 10%, 20%, 40%, and 60% fractiles of the skewness function are too negative and the 80% and 90% fractiles are too positive to be consistent with the null hypothesis of linearity. Linearity is rejected for $\Delta\ln$(TB1M) by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, $Z_{.6}$, $Z_{.8}$ using the 99% critical values and by $Z_{.9}$ using the 95% critical values. Linearity is rejected for $\Delta\ln$(CP1M) by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, $Z_{.6}$, and $Z_{.9}$ using the 99% critical values. Linearity is rejected for $\Delta\ln$(FF) by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, using the 99% critical values and by $Z_{.8}$ and $Z_{.9}$ using the 95% critical values. These tests provide overwhelming evidence of non-linear dynamics for the first differences of these short-term interest rates.

III.2 Co-Integration

Friedman and Kuttner (1992, 1993) claimed that the spread between the commercial paper rate and the Treasury bill rate was stationary. We conducted a co-integration analysis of the system $S_t = (\ln(\text{CP1M}), \ln(\text{TB1M}), \ln(\text{FF}))^T$.

The co-integration analysis is conducted in two steps: rank identification and estimation. The rank identification, which determines the number of co-integration relations, is based on the non-parametric test procedure developed by Bierens (1997, 1998). The number of co-integration relations is determined by a set of hypothesis tests, called λ-min tests that are essentially non-parametric versions of the well-known Johansen (1988) parametric λ-max tests. The test is non-parametric because the matrices involved are constructed from the data independently of the data-generating process; see Bierens (1997).

The number of co-integration relations can also be estimated using a function of the eigenvalues, $\hat{g}_m(r)$. The value of r that minimizes $\hat{g}_m(r)$ is a consistent estimate of the true number of co-integration relations, see Bierens (1997, Section 4.4).

The rank of the system is determined using both the estimation and hypothesis test methods. The estimated number of co-integration relations is 2. The λ-min tests are reported in Table 3. The tests are run in sequence, starting with the null hypothesis that the number of co-integrating vectors is zero, followed by a test of the null hypothesis that there is one co-integrating vector, and so on until the null cannot be rejected. In Table 3, we find that $r = 0$ (no co-integration) is decisively rejected, as is the hypothesis that $r = 1$ (one co-integrating vector), but we cannot reject the hypothesis that $r = 2$.

We also applied the maximum likelihood λ-max and trace tests developed by Johansen (1988, 1991, 1992) and Johansen and Juselius (1990). The $I(1)$ maximum likelihood method estimates a finite-order VECM, as in (20), where the coefficient matrices $\Pi, \Gamma_1, ..., \Gamma_{p-1}$ are 3 by 3. If the system is co-integrated then the matrix Π has reduced rank $r < 3$, and can be decomposed into $\Pi = \alpha\beta^T$. The matrices α and β are full rank 3 by r matrices, and the columns of β are the co-integration vectors. Pantula (1989) and Johansen (1992) have suggested a procedure to jointly identify the deterministic components and the rank of Π. The idea is to test the models sequentially, beginning with the most restrictive model considered. Each hypothesis can be tested using either the trace or λ-max test statistics. We conducted these tests for a set of lag lengths $p = 4, 5, ..., 20$. These tests uniformly find that there are two co-integration vectors and that the correct deterministic component is a constant that is restricted to the co-integration space. This specification is therefore extremely robust to the lag length and agrees with the rank determination of the non-parametric test. Table 4 reports these tests for a lag length of $p = 6$.[44]

The co-integration vectors can be estimated either parametrically or non-parametrically. The parametric estimates are more useful for purposes of forecasting and Bierens (1998) has argued that hypothesis tests in the parametric model have higher power than comparable tests in the non-parametric model. This, however, is not necessarily true for our analysis, because the hypothesis tests are predicated on linearity. The parametric estimate of the co-integration vectors is $\beta = [\beta_1, \beta_2]$, where $\beta_1 = (1, -1.031, 0)^T$ and $\beta_2 = (0, 1, -0.913)^T$.[45] The first basis vector β_1 reflects the near stationarity of the spread between the logarithms of the commercial paper rate and the Treasury bill rate. The second basis vector β_2 reflects the near stationarity of the spread between the Treasury bill rate and the federal funds rate. Chi-squared tests in the VECM(6) accept the hypothesis that $\beta_1 = (1, -1, 0)^T$ but reject the hypothesis that $\beta_1 = (0, 1, -1)^T$.[46] Nevertheless, these results are broadly consistent with stationarity of short-term interest rate spreads. The common stochastic trend for the system is given by $\beta_\perp^T S_t$, where $\beta_\perp = (1, (1/1.031), (1/1.031)(1/0.913))^T$. The non-parametric estimate of the co-integration vectors is $\beta_{NP} = [\beta_{1,NP}, \beta_{2,NP}]$, where $\beta_{1,NP} = (1, -1.075, 0)^T$ and $\beta_{2,NP} = (0, 1, -0.863)^T$, which are consistent with the parametric estimates. Figures 3 and 4 show that the parametric and non-parametric estimates of the co-integration relations are almost identical.

The short-run dynamics can be removed from the co-integration relations by computing $\beta^T R_{pt}$, where R_{pt} are the residuals from regressing S_{t-p} on $\Delta S_{t-1},...,\Delta S_{t-p+1}$. The relations $\beta^T R_{pt}$ are the ones actually tested for stationarity in the maximum likelihood procedure, see Hansen and Juselius (1995).

III.3 Robustness

Miyao (1996) has argued that maximum likelihood co-integration tests should be carefully checked for robustness. He argues that the maximum likelihood test should be used in conjunction with other tests of the same null, and should be carefully tested for robustness to lag length and sample period. Therefore, we test for co-integration using both non-parametric and parametric tests, and test the results for robustness to sample period. The results of those tests are consistent, as are the estimated co-integration relations. We also tested these results for robustness to sample period. We examined the integration and co-integration properties of the data over two sub-periods: 1974:09:13 through 1979:09:19, and 1984:3:1 through 1996:12:31. These are periods over which a target for the federal funds rate can be constructed, see Rudebusch (1995).

The results of the non-parametric and parametric co-integration tests for the two sub-samples are reported in Tables 3a/3b, and 4a/4b respectively. The results show that the rank identifications are consistent with those from the full sample. Further, the estimated co-integration vectors are consistent with the estimated vectors from the full sample, because we cannot reject the joint hypothesis, H_0 : $\beta_1 = (1,-1.031,0)^T$ and $\beta_2 = (0,1,-0.913)^T$, for either sub-sample.[47]

III.4 Tests for Non-Linearity of the Co-integration Relations

The stationary components of the system are the two co-integration relations and the first difference of the common stochastic trend. We test the estimated co-integration relations for non-linear serial dependence using the bispectrum tests, described in Appendix 1. We note that the co-integration vectors, β_1 and β_2 are basis vectors for the co-integration space, so that any linear combination of β_1 and β_2 are also stationary. Thus, evidence of non-linearity in one of the basis co-integration relations is actually evidence that the stationary components of the system are non-linear.

Each of the co-integration relations is pre-whitened by an AR(6) filter prior to testing to eliminate bias in the spectral estimation. As a robustness test, we tested these relations for stationarity using the frequency domain test derived by Hinich and Wild (1999). The Hinich and Wild (HW) test checks for residual non-stationarity due to the existence of a waveform with random phase and amplitude. This test has a very different alternative than the co-integration test, and should detect non-stationarity at seasonal frequencies. The test is chi-square under the null of stationarity. The HW - stationarity tests, which are reported in Table 5, confirm that the co-integration relations are stationary.

If the time series are Gaussian then the real and imaginary components of the bispectrum are zero. The test statistics for these two hypotheses, called Gauss1 and Gauss2 respectively, are also reported in Table 5. If either the real or imaginary components of the bispectrum are non-zero then Gaussianity is rejected. If the imaginary component is non-zero then the sequence is not time-reversible as discussed in Appendix 1. These tests indicate that the stationary components of the system are highly non-Gaussian and, in particular, are not time-reversible.

Table 6 gives the results of the tests for non-linearity for the full sample. The tests are computed for three estimates of the stationary co-integration relations: the parametric estimates $\beta_1^T S_t$ and $\beta_2^T S_t$; the parametric estimates purged of short-run dynamics $\beta_1^T R_{pt}$ and $\beta_2^T R_{pt}$; and the non-parametric estimates as $\beta_{1,NP}^T S_t$ and $\beta_{2,NP}^T S_t$. The test statistics are distributed as standard normal variates under the null of linearity, and rejections indicate the existence of non-linear dynamics. Broadly speaking the values of the 10%, 20%, 40%, and 60% fractiles of the skewness function are too negative and the 80% and 90% fractiles are too positive to be consistent with the null hypothesis of linearity, which is consistent with our findings in Section III.1 for the first differences.

The strongest evidence for non-linearity is for the first co-integration relation. Linearity is rejected for $\beta_1^T S_t$ by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, $Z_{.6}$, and by $Z_{.9}$ using the 99% critical values and by $Z_{.8}$ using the 95% critical values. Purging these relations of their short-run dynamics does not change the results. Linearity is rejected for $\beta_1^T R_{pt}$ by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, $Z_{.6}$ using the 99% critical values, by $Z_{.8}$ using the 95% critical values and by $Z_{.9}$ using the 90% critical values. The results for the non-parametric estimate, $\beta_{1,NP}^T S_t$, are consistent with those from the parametric estimates. Linearity is rejected for $\beta_{1,NP}^T S_t$ by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, $Z_{.6}$, and $Z_{.8}$ using the 99% critical values and by $Z_{.9}$ using the 95% critical values.

There is also evidence that the second co-integration relation is non-linear, although this evidence is somewhat weaker. Linearity is rejected for $\beta_2^T S_t$ by $Z_{.1}$ using the 95% critical values and by $Z_{.2}$ using the 99% critical values. Linearity is rejected for $\beta_2^T R_{pt}$ by $Z_{.1}$, $Z_{.2}$, $Z_{.4}$, and $Z_{.8}$ using the 99% critical values, by $Z_{.9}$ using the 95% critical values, and by $Z_{.6}$ using the 90% critical values. The evidence is similar for the non-parametric estimates. Linearity is rejected for $\beta_{2,NP}^T S_t$ by $Z_{.1}$ using the 95% critical values, by $Z_{.2}$ using the 99% critical values, and by $Z_{.9}$ using the 95% critical values.

The common stochastic trend is a linear combination of the first differences of the variables, which have already been shown to be non-linear. It is therefore not surprising that the tests reject linearity for $\beta_\perp^T \Delta S_t$ and $\beta_\perp^T \Delta R_{pt}$.

It is possible that structural shifts over the long period being analyzed could be mistaken for non-linear dynamics.[48] We attempt to address this issue by deleting the period 1979-1983, during which the Federal Reserve targeted non-borrowed reserves and many interest rates were deregulated. We consider the two sub-samples 9/13/1974 through 9/19/1979, and 3/1/1984 through 12/31/1996, which have been studied by Rudebusch (1995). In Section III, we found that the co-integration rank and the estimates of the co-integration relations are not statistically different in these sub-samples to the results over the full sample. In Tables 6a and 6b, we provide the results of the non-linearity tests over these two sub-samples. If the co-integration relations are linear over the two sub-samples then the structural shift hypothesis could be true. In addition, we ran the linearity tests on $\Delta \ln(\text{CP1M})$, $\Delta \ln(\text{TB1M})$, and $\Delta \ln(\text{FF})$ for each sub-sample; these tests are reported in Table 2.

The evidence is mixed. The tests generally fail to reject linearity for the first sub-sample, 1974-1979, but they provide very strong evidence of non-linearity over the second sub-sample 1984-1996. The number of data points for the shorter sub-sample is 258 versus 669 for the longer sub-sample and 1363 for the full sample. The evidence reported in Ashley, Patterson, and Hinich (1986) would indicate that the power of these tests is substantially higher over the longer sub-sample and over the full sample. We conclude that there is strong evidence of non-linearity in the stationary components of the system, but there is some lack of robustness to the sample of data being tested.

IV. Conclusion

We have argued that the co-integration relations in I(1) systems are generally non-linear. Because the co-integration relations derived from I(1) systems are stationary; they can be tested for non-linear serial dependence using standard polyspectral techniques.

Tests for the existence of non-linear dynamics require large sample sizes and may be adversely affected by aliasing and other problems associated with time aggregation. Interest rates are measured with high frequency and aliasing can be controlled by adequate attention to filter design. For these reasons, the conditions are more favorable to testing interest rate data for non-linear dynamics than for most other variables that are important to the business cycle, money demand, and the monetary transmission mechanism. We have shown that an integrated system of short-term US interest rates is co-integrated and there is strong evidence that the stationary components of the system are non-linear. The test we employ is one of the most conservative tests for non-linearity available, strengthening the impact of our findings.

These results suggest that the untested assumption of linearity, which is implicit in many macroeconomic studies, may be incorrect. The failure to find robust evidence of non-linearity in lower frequency macroeconomic time series may be due to the low sample sizes that can be obtained for those time series and to problems associated with sampling and time aggregation. Our particular example shows that the spread between both the commercial paper rate and the Treasury bill rate, and the Federal Funds rate exhibits third-order non-linearity. Our results are consistent with work that suggests there are asymmetric effects of monetary policy on interest rates, such as Choi (1999). Our results suggest that better forecasts of these spreads might be obtained with non-linear models, such as bi-linear models.

Appendix 1: Bispectrum Estimation and the Hinich Tests for Gaussianity and Linearity

Let $\{X_0,...,X_{N-1}\}$ be a finite data record of N observations. If necessary, standardize the entire sample by subtracting the sample mean and then dividing by the sample variance. Segment the record into K non-overlapping frames of length L, called the frame-length, $\{X_0^k,...,X_{L-1}^k\} = \{X_{(k-1)L},...,X_{kL-1}\}$, $k=1,\ldots,K$.[49] Let $\bar{X}_i = \sum_{k=1}^{K} X_i^k / K$ be the mean of the ith elements of each frame. The elements of each frame are standardized so that $\{\hat{X}_0^k,...,\hat{X}_{L-1}^k\} = \{(X_0^k - \bar{X}_0),...,(X_{L-1}^k - \bar{X}_{L-1})\}$. The motivation for this standardization is to remove any non-stationarity due to the existence of a purely deterministic waveform in the data. Hinich and Wild (1999) have provided a test for non-stationarity due to the existence of a waveform with stochastic phase and amplitude, but this test should be used only after correcting for unit roots and trends in practice. The Hinich and Wild test is chi-square under the null of stationarity.

The finite Fourier transform of the kth standardized frame is defined as

$$d_{X^k}(f_n) = \sum_{s=0}^{L-1} \hat{X}_s^k e^{-i2\pi f_n [s+(k-1)L]}, \quad k=1,\ldots,K, \tag{23}$$

where $f_n = n/L$, for n = 0,…,L/2. The power spectrum estimator is defined as

$$\hat{P}_X(f_n) = \frac{1}{K}\sum_{k=1}^{K} I_k(f_n), \tag{24}$$

where $I_k(f_n) = (1/L)|d_{X^k}(f_n)|^2$ is the second-order periodogram for the k^{th} frame.[50] The bispectrum estimator is defined as

$$\hat{B}_X(f_n, f_m) = \frac{1}{K}\sum_{k=1}^{K} G_k(f_n, f_m) \tag{25}$$

where $G_k(f_n, f_m) = (1/L)d_{X^k}(f_n)d_{X^k}(f_m)d^*_{X^k}(f_n + f_m)$ is the third-order periodogram for the k^{th} frame.[51] These estimators are consistent if $\ln(L) < .5\ln(N)$ and have resolution bandwidth $\delta = (1/L)$.[52] Increasing the frame length, L, increases resolution, but at the cost of higher variance.[53] If the frame length is larger than .5ln(N) the estimators are not consistent. Brillinger and Rosenblatt (1967,b pp. 203-5) show that the data should be pre-filtered prior to estimation to eliminate bias, which might lead to spurious rejection of the null of linearity or Gaussianity.

These estimators are used to construct the test statistic. Let $\hat{X}_{n,m}$ be defined as

$$\hat{X}_{n,m} = \delta\sqrt{N}\frac{\hat{B}_X(f_n, f_m)}{\sqrt{\hat{P}_X(f_n)\hat{P}_X(f_m)\hat{P}_X(f_n + f_m)}} = \sqrt{\frac{K}{L}}\frac{\hat{B}_X(f_n, f_m)}{\sqrt{\hat{P}_X(f_n)\hat{P}_X(f_m)\hat{P}_X(f_n + f_m)}}. \tag{26}$$

All of the tests used in this paper derive from the asymptotic distribution of $\hat{X}_{n,m}$. Hinich (1982) proved that the estimators $2|\hat{X}_{n,m}|^2$ are asymptotically distributed as independent non-central chi-squared random variables, $2|\hat{X}_{n,m}|^2 \sim \chi^2(2,\lambda_{n,m})$, with non-centrality parameter $\lambda_{n,m} = 2(\delta^2 N)\Gamma_X(f_n, f_m)$.[54]

Gaussianity and Time Reversibility

The null hypothesis for the Gaussianity test is $H_0:\ \Gamma_X(f,g) = 0,\quad \forall f,g \in D$. Under the null, $\lambda_{n,m} = 2(\delta^2 N)\Gamma_X(f_n, f_m) = 0$ and $2\sum_n\sum_m|\hat{X}_{n,m}|^2$ is asymptotically distributed $\chi^2(2P,0)$ (the summations are over n and m such that $(f_n, f_m) \in D$, and P is the number of such pairs).[55] The null hypothesis is rejected if $2\sum_n\sum_m|\hat{X}_{n,m}|^2$ is large relative to $\chi^2(2P,0)$.

The bispectrum can be written using the Euler relation as follows:

$$B_X(f,g) = \sum_{\tau_1=-\infty}^{\infty}\sum_{\tau_2=-\infty}^{\infty} c_{XXX}(\tau_1,\tau_2)\cos(2\pi f\tau_1 + 2\pi g\tau_2) + i\sum_{\tau_1=-\infty}^{\infty}\sum_{\tau_2=-\infty}^{\infty} c_{XXX}(\tau_1,\tau_2)\sin(2\pi f\tau_1 + 2\pi g\tau_2). \tag{27}$$

If the time series is time reversible, such that $c_{XXX}(\tau_1,\tau_2) = c_{XXX}(-\tau_1,-\tau_2)$, then the second summation will be identically zero. Therefore, time reversibility implies that the bispectrum is real valued.[56] Hinich and Rothman (1998) proved that $2\sum_n\sum_m|\operatorname{Im}\hat{X}_{n,m}|^2$ is asymptotically distributed $\chi^2(P,0)$ under the null $H_0:\ \operatorname{Im}B_X(f,g) = 0,\quad \forall f,g \in D$. The null hypothesis of time reversibility is rejected if $2\sum_n\sum_m|\operatorname{Im}\hat{X}_{n,m}|^2$ is large relative to $\chi^2(P,0)$. Similarly, $2\sum_n\sum_m|\operatorname{Re}\hat{X}_{n,m}|^2$ is asymptotically distributed $\chi^2(P,0)$ under the null hypothesis $H_0:\ \operatorname{Re}B_X(f,g) = 0,\quad \forall f,g \in D$.

Linearity

The null hypothesis for the linearity test is $H_0: \Gamma_X(f,g) = k\,\forall f,g \in D$, where k is a constant. Under the null, the estimators $2|\hat{X}_{n,m}|^2$ are asymptotically distributed as P independent draws from a non-central chi-square distribution $\chi^2(2,\lambda_0)$, and the non-centrality parameter, λ_0, can be consistently estimated by $\hat{\lambda}_0 = 2\sum_n \sum_m (|\hat{X}_{n,m}|^2 / P) - 2$. If the sequence is non-linear then the $\lambda_{n,m}$ are not constant for all (*n*, *m*) pairs, and the sample dispersion of $2|\hat{X}_{n,m}|^2$ should exceed that expected under the hypothesized distribution. The dispersion can be measured using the quantiles of the empirical distribution. Let $\hat{\xi}_\rho$ be the $(100\rho)\%$ quantile of the empirical distribution of $2|\hat{X}_{n,m}|^2$, where $0 < \rho < 1$. It can be shown that $\hat{\xi}_\rho$ is asymptotically distributed $N(\xi_\rho, \sigma_\rho^2)$, where $\sigma_\rho^2 = \frac{\rho(1-\rho)}{Pf^2(\xi_\rho)}$.[57] The statistic $Z_\rho \equiv (\hat{\xi}_\rho - \xi_\rho)/\sigma_\rho$ is $N(0,1)$ under the null hypothesis. We consider tests based on ρ = .1, .2, .4, .6, .8, and .9. Linearity is rejected if $|Z_\rho|$ is large relative to N(0,1).

Ashley, Hinich, and Patterson (1986) show that the Hinich non-linearity test has substantial power against several commonly estimated non-linear models, such as bi-linear models, non-linear moving average models, and threshold autoregression models.[58] Nevertheless, the conservatism of the Hinich test has been reflected in empirical studies. Barnett, Gallant, Hinich, Jungeilges, Kaplan, and Jensen (1994) find that the Hinich test was much less likely to reject its null than other competing tests, such as the Brock, Dechert, Scheinkman, and LeBaron (1996) test, and the Kaplan (1993) test.

Appendix 2: Aliasing and the Construction of Anti-Aliasing Filters

Let X_t be a continuous-time series that is sampled at regular intervals of time, $0, \Delta T, 2\Delta T, \ldots, (N-1)\Delta T$. ΔT is called the sampling interval, and $1/\Delta T$ is the sampling rate. The sampled sequence is denoted $X_{k\Delta T}$, $k = 0, \ldots, N-1$.

The power spectrum of the continuous-time series is $g(f) = \int_{-\infty}^{\infty} c_{XX}(\tau) e^{-i(2\pi f)\tau}$. The power spectrum of the discrete-time sampled sequence, $g_{\Delta T}(f)$, is given by the following:

$$g_{\Delta T}(f) = \sum_{j=-\infty}^{\infty} g(f + \frac{j}{\Delta T}), \qquad (28)$$

for $|f| \le (1/2\Delta T)$.[59] The frequency $f_N = (1/2\Delta T)$ is called the Nyquist folding frequency. If $g(f) = 0$ for all $f \ge |f_N|$ then the power spectrum of the continuous-time series and the discrete-time sampled sequence are equal. If the continuous-time series does not have this property then the power spectrum at frequency, *f*, of the sampled sequence is equal to the sum of the values of the power spectrum of the continuous-time series at all frequencies of the form $f+(j/\Delta T)$ for $j=0,\pm 1,\pm 2,\ldots$. Thus, the low frequency harmonics are made indistinguishable from the combined power of higher frequency harmonics because of sampling. This phenomenon is called aliasing.

It is very important to eliminate any power in a time series at frequencies that exceed the Nyquist folding frequency prior to sampling, because failure to do so will lead to biased estimation due to aliasing. Aliasing has traditionally been described in the frequency domain, but Hinich (1998) has recently shown that aliasing corrupts the impulse response functions in the time domain and therefore leads to serious identification problems.

The same problem results if a discrete-time sequence is sampled at a lower frequency, such as sampling a daily interest rate at weekly intervals. In this case, the sampling interval is $\Delta T = 7$ and the Nyquist folding frequency is $(1/2\Delta T) = (1/14)$. If the daily interest rates have power at frequencies exceeding (1/14) then aliasing will occur. The solution to this problem is to filter the daily interest rates in such a way that the power spectrum of the filtered rates will be zero at frequencies exceeding the Nyquist. If $\{g_j\}$ are the filter weights then the power spectrum of the filtered sequence equals the power spectrum of the underlying sequence multiplied by the gain of the filter $|G(f)|^2$ where $G(f)=\sum_{j=-\infty}^{\infty} g_j e^{-i(2\pi f)j}$. The solution to the aliasing problem would be to design a filter with gain:

$$|G(f)|^2 = \begin{cases} 1 & |f| \le f_N \\ 0 & |f| > f_N \end{cases}. \tag{29}$$

This gain function corresponds to the ideal symmetric low-pass filter with weights

$$g_j = \begin{cases} \sin(2\pi f_N)/\pi k & k=\pm 1,\pm 2,\ldots \\ 2f_N = 1/\Delta T & k=0 \end{cases}, \tag{30}$$

which cannot be realized with a finite data sample. In fact, the rate of decrease of the filter weights is too slow to simply truncate the filter at some finite number of leads and lags. The usual solution is to taper the weights of the ideal filter. We taper the ideal weights using a Hanning cosine taper. The gain function of the Hanning tapered filter, with 7 leads and lags, is compared with the ideal gain function in Figure 5 for a cutoff frequency of $(1/14)$. This filter is referred to as an anti-aliasing filter in the text. A common approach in economics is to report un-weighted weekly averages of daily interest rates. The un-weighted averaging filter is similar to the filter used in this paper, but it has larger side lobes. See Figure 6.

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Table 1 – Univariate Stationarity Tests

Variable	ADF1	ADF2	PP1	PP2	KPSS1	KPSS2	Bierens
Ln(CP1M)	-1.9132	-2.271	-8.6587	-10.1641	.9272	.5386	1.14414
Ln(TB1M)	-1.8289	-2.2407	-9.4498	-11.2726	1.0186	.5626	.77406
Ln(FF)	-1.8174	-2.2031	-8.8747	-10.7101	1.0027	.5511	1.02026
ΔLn(CP1M)	-7.4386	-7.4820	-740.9803	-738.3171	.1120	.0524	0.00000
ΔLn(TB1M)	-8.0123	-8.0573	-877.6077	-874.0001	.1210	.0567	0.00000
ΔLn(FF)	-7.3217	-7.3561	-2118.753	-2109.664	.1249	.0584	0.00000
H0:	UR	URD	UR	URD	S	TS	URD
H1:	S	TS	S	TS	UR	URD	TS
5% c.v.	< -3.86	< -3.41	< -14	< -21.5	$> .436$	$> .146$	$< .025$
10% c.v.	< -2.57	< -3.13	< -11.2	< -18.5	$> .347$	$> .119$	$< .006$

Table 2 – Linearity Tests

Variable	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
Full Sample: 1971-1997						
Δln(CP1M)	-2.70	-3.91	-3.80	-3.63	0.28	2.78
Δln(TB1M)	-2.72	-3.95	-4.38	-2.86	2.64	2.51
Δln(FF)	-2.65	-3.65	-3.54	-1.32	2.41	2.48
Sub-Sample #1: 1974-1979						
Δln(CP1M)	-0.92	-1.42	-1.88	-0.15	0.51	1.01
Δln(TB1M)	-1.15	-1.71	-1.15	-0.96	0.44	0.68
Δln(FF)	-0.60	-1.13	-1.00	-0.46	-0.80	0.30
Sub-Sample #2: 1984-1996						
Δln(CP1M)	-1.58	-1.12	0.00	0.16	1.11	0.72
Δln(TB1M)	-1.57	-2.39	-2.14	-2.10	0.16	1.85
Δln(FF)	-2.00	-2.99	-4.44	-0.09	2.74	2.00

Z_ρ is standard normal under the null H0: $B(f,g)$ is constant $\forall(f,g)\in D$

c. v. 1.65 (90%) 1.96 (95%) 2.58 (99%)

Table 3 – Non-Parametric Co-Integration Tests

Full Sample: 1971-1997

Hypothesis	Test Stat	Critical Region	M	Conclusion
H0: r = 0	0.00000	20% (0,.006)	3	Reject
H1: r = 1	0.00000	10% (0,.017)	4	Reject
	0.00000	5% (0,.008)	4	Reject
H0: r = 1	0.00054	20% (0,.077)	3	Reject
H1: r = 2	0.00054	10% (0,.034)	3	Reject
	0.00054	5% (0,.017)	3	Reject
H0: r = 2	0.76618	20% (0,.341)	3	Accept
H1: r = 3	0.76618	10% (0,.187)	3	Accept
	0.76618	5% (0,.111)	3	Accept

Table 3a – Non-Parametric Co-Integration Tests

Sub-Sample #1: 1974-1979

Hypothesis	Test Stat	Critical Region	M	Conclusion
H0: r = 0	0.00000	20% (0,.006)	3	Reject
H1: r = 1	0.00000	10% (0,.017)	4	Reject
	0.00000	5% (0,.008)	4	Reject
H0: r = 1	0.00523	20% (0,.077)	3	Reject
H1: r = 2	0.00523	10% (0,.034)	3	Reject
	0.00523	5% (0,.017)	3	Reject
H0: r = 2	1.33438	20% (0,.341)	3	Accept
H1: r = 3	1.33438	10% (0,.187)	3	Accept
	1.33438	5% (0,.111)	3	Accept

Table 3b – Non-Parametric Co-Integration Tests

Sub-Sample #2: 1984-1997

Hypothesis	Test Stat	Critical Region	M	Conclusion
H0: r = 0	0.00000	20% (0,.006)	3	Reject
H1: r = 1	0.00000	10% (0,.017)	4	Reject
	0.00000	5% (0,.008)	4	Reject
H0: r = 1	0.00008	20% (0,.077)	3	Reject
H1: r = 2	0.00008	10% (0,.034)	3	Reject
	0.00008	5% (0,.017)	3	Reject
H0: r = 2	2.33982	20% (0,.341)	3	Accept
H1: r = 3	2.33982	10% (0,.187)	3	Accept
	2.33982	5% (0,.111)	3	Accept

Table 4 – Parametric Co-Integration Tests

Full Sample: 1971-1997

Hypothesis	λ-max	Trace	λ-max c.v. 90% *95%*	Trace c.v. 90% *95%*	Result
Ho: r = 0 constant restricted to co-integration space	103.48	158.63	14.09	31.88 *34.78*	Reject
Ho: r = 0 Unrestricted constant	103.47	158.60	13.39	26.70 *29.38*	Reject
Ho: r = 0 unrestricted constant trends in co-integration space	114.26	172.02	16.13	39.08 *42.20*	Reject
Ho: r = 1 constant restricted to co-integration space	50.59	55.15	10.29	17.79 *19.99*	Reject
Ho: r = 1 Unrestricted constant	50.59	55.13	10.60	13.31 *15.34*	Reject
Ho: r = 1 unrestricted constant trends in co-integration space	51.41	57.77	12.39	22.95 *25.47*	Reject
Ho: r =2 constant restricted to co-integration space	4.55	4.55	7.50	7.50 *9.13*	Accept
Ho: r = 2 Unrestricted constant	4.54	4.54	2.71	2.71 *3.84*	N/A
Ho: r = 2 unrestricted constant trends in co-integration space	6.36	6.36	10.56	10.56 *12.386*	N/A

Table 4a – Parametric Co-Integration Tests

Sub-Sample #1: 1974-1979

Hypothesis	λ-max	Trace	λ-max c.v. 90% *95%*	Trace c.v. 90% *95%*	Result
Ho: r = 0 constant restricted to co-integration space	25.51	45.50	14.09	31.88 *34.78*	Reject
Ho: r = 0 Unrestricted constant	25.46	44.55	13.39	26.70 *29.38*	Reject
Ho: r = 0 unrestricted constant trends in co-integration space	42.51	71.10	16.13	39.08 *42.20*	Reject
Ho: r = 1 constant restricted to co-integration space	19.24	19.99	10.29	17.79 *19.99*	Reject
Ho: r = 1 Unrestricted constant	18.86	19.08	10.60	13.31 *15.34*	Reject
Ho: r = 1 unrestricted constant trends in co-integration space	21.04	28.59	12.39	22.95 *25.47*	Reject
Ho: r =2 constant restricted to co-integration space	.75	.75	7.50	7.50 *9.13*	Accept
Ho: r = 2 Unrestricted constant	.22	.22	2.71	2.71 *3.84*	N/A
Ho: r = 2 unrestricted constant trends in co-integration space	7.55	7.55	10.56	10.56 *12.386*	N/A

Table 4b – Parametric Co-Integration Tests

Sub-Sample #2: 1984-1996

Hypothesis	λ-max	Trace	λ-max c.v. 90% *95%*	Trace c.v. 90% *95%*	Result
Ho: r = 0 constant restricted to co-integration space	58.93	91.86	14.09	31.88 *34.78*	Reject
Ho: r = 0 Unrestricted constant	58.93	90.97	13.39	26.70 *29.38*	Reject
Ho: r = 0 unrestricted constant trends in co-integration space	69.81	103.99	16.13	39.08 *42.20*	Reject
Ho: r = 1 constant restricted to co-integration space	29.90	32.93	10.29	17.79 *19.99*	Reject
Ho: r = 1 Unrestricted constant	29.83	32.04	10.60	13.31 *15.34*	Reject
Ho: r = 1 unrestricted constant trends in co-integration space	31.96	34.19	12.39	22.95 *25.47*	Reject
Ho: r =2 constant restricted to co-integration space	3.03	3.03	7.50	7.50 *9.13*	Accept
Ho: r = 2 Unrestricted constant	2.21	2.21	2.71	2.71 *3.84*	N/A
Ho: r = 2 unrestricted constant trends in co-integration space	2.23	2.23	10.56	10.56 *12.386*	N/A

Table 5 – Stationarity, Gaussianity, and Time Reversibility Tests

Full Sample 1971-1997

Parametric Estimates	HW	Gauss1	Gauss2
$\beta_1^T S_t$	22.6120 (0.9321)	539.2411 (0.0000)	610.5936 (0.0000)
$\beta_1^T R_{pt}$	20.4589 (0.9676)	334.3938 (0.0000)	442.1985 (0.0000)
$\beta_2^T S_t$	33.9137 (0.4719)	343.3336 (0.0000)	945.0921 (0.0000)
$\beta_2^T R_{pt}$	13.1550 (0.9995)	296.2166 (0.0000)	330.7980 (0.0000)
$\beta_\perp^T \Delta S_t$	29.0369 (0.7095)	495.1035 (0.0000)	244.4961 (0.0000)
$\beta_\perp^T \Delta R_{pt}$	32.7482 (0.5289)	281.7567 (0.0000)	283.7022 (0.0000)
Non-Parametric Estimates			
$\beta_{1,NP}^T S_t$	25.2276 (0.8620)	638.6265 (0.0000)	707.1962 (0.0000)
$\beta_{2,NP}^T S_t$	34.2187 (.4572)	387.4622 (0.0000)	465.5005 (0.0000)

HW is the Hinich and Wild (1999) test. Test statistic $\chi^2(34)$ under the null of stationarity.
Gauss1 is $\chi^2(72)$ under the null H0: $\mathrm{Re}(B(f,g))= 0\ \forall(f,g)\in D$
Gauss2 is $\chi^2(72)$ under the null H0: $\mathrm{Im}(B(f,g))= 0\ \forall(f,g)\in D$

Table 6 – Linearity Tests

Full Sample: 1971-1997

Parametric Estimates	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
$\beta_1^T S_t$	-2.81	-3.95	-5.42	-3.20	2.24	2.72
$\beta_1^T R_{pt}$	-2.73	-3.79	-4.37	-2.65	2.46	1.88
$\beta_2^T S_t$	-2.25	-2.91	-1.59	0.25	1.06	1.47
$\beta_2^T R_{pt}$	-2.23	-3.02	-3.52	-1.90	2.58	2.15
$\beta_\perp^T \Delta S_t$	-2.61	-3.54	-4.92	-3.32	1.06	2.46
$\beta_\perp^T \Delta R_{pt}$	-2.44	-3.13	-4.17	-4.04	1.44	1.44
Non-Parametric Estimates	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
$\beta_{1,NP}^T S_t$	-2.81	-4.09	-5.39	-2.65	2.78	2.19
$\beta_{2,NP}^T S_t$	-2.26	-2.66	-0.93	-0.38	1.29	1.94

Z_ρ is standard normal under the null H0: $B(f,g)$ is constant $\forall (f,g) \in D$. Linearity is rejected if $|Z_\rho|$ exceeds the critical value. These values are 1.65 (90%) 1.96 (95%) 2.58 (99%)

Table 6a – Linearity Tests

Sub-Sample #1: 1974-1979

Parametric Estimates	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
$\beta_1^T S_t$	-1.08	-1.38	-2.01	-0.38	0.75	1.13
$\beta_1^T R_{pt}$	-0.61	-0.66	-1.04	-1.72	0.32	0.58
$\beta_2^T S_t$	-1.15	-1.72	0.68	0.63	0.34	0.73
$\beta_2^T R_{pt}$	-0.25	1.44	1.57	2.29	1.10	1.26
$\beta_\perp^T \Delta S_t$	0.31	-0.17	-1.00	0.19	0.24	0.70
$\beta_\perp^T \Delta R_{pt}$	-0.27	-0.67	-0.37	0.75	0.40	0.34
Non-Parametric Estimates	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
$\beta_{1,NP}^T S_t$	-1.07	-1.39	-1.99	-0.36	0.98	1.12
$\beta_{2,NP}^T S_t$	-1.15	-1.72	0.50	0.79	0.15	0.67

Z_ρ is standard normal under the null H0: B(f,g) is constant $\forall (f,g) \in$ D. Linearity is rejected if $|Z_\rho|$ exceeds the critical value. These values are 1.65 (90%) 1.96 (95%) 2.58 (99%)

Table 6b – Linearity Tests

Sub-Sample #2: 1984-1996

Parametric Estimates	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
$\beta_1^T S_t$	-2.00	-2.96	-4.44	-3.76	1.89	1.96
$\beta_1^T R_{pt}$	-1.97	-2.65	-3.48	.039	1.07	1.96
$\beta_2^T S_t$	-2.00	-2.83	-2.41	0.11	2.43	1.92
$\beta_2^T R_{pt}$	-1.94	-2.89	-3.36	-0.69	1.78	1.99
$\beta_\perp^T \Delta S_t$	-1.93	-2.81	-3.08	-3.23	1.92	1.96
$\beta_\perp^T \Delta R_{pt}$	-1.62	-2.23	-2.69	-0.89	0.86	1.50
Non-Parametric Estimates	$Z_{.1}$	$Z_{.2}$	$Z_{.4}$	$Z_{.6}$	$Z_{.8}$	$Z_{.9}$
$\beta_{1,NP}^T S_t$	-2.00	-2.95	-4.43	-3.88	1.95	1.96
$\beta_{2,NP}^T S_t$	-1.99	-2.82	-2.14	0.05	2.40	1.91

Z_ρ is standard normal under the null H0: $B(f,g)$ is constant $\forall (f,g) \in D$. Linearity is rejected if $|Z_\rho|$ exceeds the critical value. These values are 1.65 (90%) 1.96 (95%) 2.58 (99%)

End Notes

[1] A linear stochastic process is defined as the output of a time invariant linear filter applied to a stochastically independent input process.

[2] See, for example, King, Plosser, Stock, and Watson (1991), and Leeper, Sims, and Zha (1996).

[3] For example, King and Watson (1996 pp. 37-39) study the business cycle covariability of various macroeconomic time series after first removing all frequencies outside of the business cycle range.

[4] See Priestly (1988) for further discussion of these points.

[5] The bispectrum, which is the double Fourier transform of the third-order cumulant function, will not exist if the process is non-stationary. The first use of the bispectrum in economics was by M. D. Godfrey (1965).

[6] A set of tests for non-linear dynamics have been used applied to economic data. Barnett et al (1996a, b, c, 1997) apply some of the most widely used tests to real and simulated data.

[7] Integrated time series are dominated by a martingale term, and provide a model for time series that are subject to permanent shocks. If two time series are dominated by the same martingale then a particular linear combination of them is stationary, and they are co-integrated.

[8] See King, Plosser, Stock, and Watson (1987, 1991), Miyao (1996), and Friedman and Kuttner (1992) for examples.

[9] The Wold decomposition theorem proves that the stationary linear combination can be represented as the output of a moving average filter applied to uncorrelated white noise input. The process may nevertheless be non-linear because the uncorrelated input process may not be stochastically independent.

[10] Lowering the sampling rate of a discrete process also cause aliasing.

[11] See also, Ashley and Patterson (1986), Scheinkman and LeBaron (1986), and Brockett, Hinich, and Patterson (1988).

[12] The assumptions of real and mean zero can be relaxed, see Hinich and Messer (1995).

[13] Cumulants and moments are equivalent up to the third-order. This is not true for higher orders.

[14] This is a consequence of the Wold decomposition theorem. See Brockwell and Davis (1991) and Engle and Granger (1987).

[15] See Hinich (1982) and Hinich and Patterson (1989). See also Priestley (1988, pp. 13-16) for related discussion.

[16] The operation | | denotes complex modulus.

[17] For details on Volterra representations, see Schetzen (1980, 1981) and Rugh (1981).

[18] More complete discussions on higher order polyspectra can be found in Nikias and Raghuveer (1987), Mendel (1991), Brillinger and Rosenblatt (1967 a, b), and Brillinger (1965).

[19] Throughout the paper, frequencies are measured in units of inverse time. Multiplying these frequencies by 2π converts them to radian measure.

[20] This region is called the principal domain. See Hinich and Messer (1995).

[21] See Nikias and Raghuveer (1987). The * notation denotes the complex conjugate operation.

[22] This result is due to Brillinger (1965).

[23] See also Hinich and Patterson (1985b, 1989) and Ashley, Hinich, and Patterson (1986).

[24] The Hinich test is based on the fact that independent sequences have a constant bispectrum. In fact, independent sequences have constant polyspectra of all orders. Thus, dependence can be reflected in any of the higher-order polyspectra, and some dependent sequences have constant bispectrum. We could design tests based on higher-order polyspectra, but most economic time series are not long enough for consistent estimation of even the fourth-order polyspectrum.

[25] See Engle and Granger (1987) for properties of integrated processes. Fisher and Seater (1993) discuss the connection between integration and neutrality and super neutrality hypotheses.

[26] See Davidson (1994, pp. 192-193) and Hall and Heyde (1980, pp. 128).

[27] The representation theorem requires mixingale assumptions on the stationary sequence, see Davidson (1994, pp. 247-252). Hall and Heyde also require $E \mid X_0 \mid < \infty$.

[28] Engle and Granger (1987) add the condition that the stationary moving average representation of the first difference, obtained from the Wold decomposition, be invertible.

[29] See Bierens (1997). The operator Δ is the standard difference operator.

[30] This decomposition is valid because vector martingale differences are serially uncorrelated and have positive semi-definite covariance matrices.

[31] This is because $E[\left(\sum_{s=0}^{t}\beta_j^T Y_s\right)\left(\sum_{s=0}^{t}\beta_j^T Y_s\right)^T] = \beta_j^T E[\left(\sum_{s=0}^{t}Y_s\right)\left(\sum_{s=0}^{t}Y_s\right)^T]\beta_j = \beta_j^T C C^T \beta_j = 0^T 0 = 0$.

[32] The orthogonal complement has the property $\beta^T\beta_{\perp} = 0$.

[33] In the absence of co-integration, the two transformations are equivalent. If $r = 0$, then $\beta_{\perp}$ is full rank and can be taken as the identity matrix.

[34] It is possible that a linear combination of non-linear processes could be linear, but this would not hold in general.

[35] For example, the process resulting from the example presented will not be stationary, but will instead behave like a unit root process when near its mean. Granger (1991) gives conditions under which $f(z)$ is stationary.

[36] See Hinich and Patterson (1985a) for a method of estimating the coefficients of a quadratic non-linear process.

[37] The federal funds rate and the commercial paper rate are available from the Federal Reserve Board's website. The commercial paper rate series was discontinued in August 1997. Richard Anderson of the Federal Reserve Bank of St. Louis provided us with the secondary market rate on one-month treasury bills.

[38] The filtering procedure is similar to the five day unweighted averages suggested by the Board of Governors, but, we would argue, is superior for our purposes.

[39] See for example Stock and Watson (1989), Friedman and Kuttner (1993), and King, Plosser, Stock, and Watson (1991).

[40] See Fuller (1996), Said and Dickey (1984), and Said (1991). The lag length, p, is chosen by the formula $p = 5(n)^{25}$.

[41] See Phillips and Perron (1988). The truncation lag for the Newey-West estimator is $p = 5(n)^{25}$.

[42] See Kwaitkowski, Phillips, Schmidt, and Shin (1992). The truncation lag for the Newey-West estimator is $p = 5(n)^{25}$.

[43] Ashley, Patterson, and Hinich (1986) found that one-tailed tests may fail to detect certain types of non-linearity that would be detected by two-tailed versions of the tests.

[44] We computed various information criteria for the VECM. The Schwartz criteria indicated a lag length of 4 and the Akaike criterion indicated a length of 20. Using these as a range of models to consider, we chose to estimate the model with p = 6 on the basis that this model was fairly parsimonious and passed tests for absence of first and fourth order auto-correlation.

[45] The basis for the co-integration space has been transformed into a basis with one zero in each vector. Since any linear combination of co-integration vectors is a co-integration vector this does not change any of our analysis. The form of the basis reported in the text is easier to understand than the untransformed basis. The constant that is subtracted from the co-integration relations is the estimated restricted constant.

[46] These tests are $\chi^2(1)$. The values of the test statistics are 1.31 and 10.87 respectively.

[47] These tests are $\chi^2(2)$. For the 1974-1979 sub-sample the test statistic is 3.26 (p-value of .2) and for the 1984-1996 sub-sample the test statistic is .38 (p-value of .83).

[48] See Barnett, Medio, and Serletis (1999, pp. 61) for discussion of this issue.

[49] If the last frame is incomplete, it is dropped from the calculation of the estimator.

[50] This estimator is described in Welch (1967) and Groves and Hannan (1968). Kay and Marple (1981) discuss various methods of power spectral estimation. We employ a trapezoidal data taper, which leads to minor modifications of these formulas.

[51] This estimator is described in Hinich and Messer (1995).

[52] The resolution bandwidth is the approximate spacing between independent power spectral estimates, see Koopmans (1974, pp. 304).

[53] Koopmans (1974) called this tradeoff the Grenander uncertainty principle.

[54] The proof follows from the fact that the $\hat{X}_{m,n}$ are asymptotically independently normal.

[55] This is true because $|\hat{X}_{m,n}|^2$ are asymptotically independent.

[56] The imaginary part of all polyspectra is zero if the sequence is time reversible, see Brillinger and Rosenblatt (1967a).

[57] See David (1970) Theorem 9.2.

[58] As noted in Nikias and Raghuveer (1987), the bispectrum can be particularly useful in identifying quadratic phase coupling, resulting from interaction between two harmonic components at their sum and/or difference frequencies.
[59] See Koopmans (1974, pp. 66-73).

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