Board of Governors of the Federal Reserve System

International Finance Discussion Papers

Number 814

August 2004

PPP Rules, Macroeconomic (In)stability and Learning

Luis-Felipe Zanna

PPP Rules, Macroeconomic (In)stability and Learning

Luis-Felipe Zanna*

August 2004

Abstract

Governments in emerging economies have pursued real exchange rate targeting through Purchasing Power Parity (PPP) rules that link the nominal depreciation rate to either the deviation of the real exchange rate from its long run level or to the difference between the domestic and the foreign CPI-inflation rates. In this paper we disentangle the conditions under which these rules may lead to endogenous fluctuations due to self-fulfilling expectations in a small open economy that faces nominal rigidities. We find that besides the specification of the rule, structural parameters such as the share of traded goods (that measures the degree of openness of the economy) and the degrees of imperfect competition and price stickiness in the non-traded sector play a crucial role in the determinacy of equilibrium. To evaluate the relevance of the real (in)determinacy results we pursue a learnability (E-stability) analysis for the aforementioned PPP rules. We show that for rules that guarantee a unique equilibrium, the fundamental solution that represents this equilibrium is learnable in the E-stability sense. Similarly we show that for PPP rules that open the possibility of sunspot equilibria, a common factor representation that describes these equilibria is also E-stable. In this sense sunspot equilibria and therefore aggregate instability are more likely to occur due to PPP rules than previously recognized.

Keywords: Small Open Economy, PPP rules, Multiple Equilibria, Sunspot Equilibria, Indeterminacy, Expectational Stability and Learning

JEL Classifications: C62, D83, E32, F41

---

*Email address: luis-felipe.zanna@frb.gov. This paper is based on a chapter of my Ph.D. dissertation at the University of Pennsylvania. I am grateful to Martín Uribe, Stephanie Schmitt-Grohé and Frank Schorfheide for comments on early versions of this paper. I also thank David Bowman, George Evans and Dale Henderson for comments and suggestions. All errors remain mine. The views in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of any other person associated with the Federal Reserve System.

1 Introduction

It has been claimed that the real exchange rate is perhaps the most popular real target in developing economies. The reason is that policy makers in these economies are always concerned about avoiding losses in competitiveness in foreign markets, or similarly, about maintaining purchasing power parity (PPP). In order to achieve the real exchange target policy makers often follow PPP rules. Such rules link the nominal rate of devaluation of the domestic currency to the deviation of the real exchange rate from its long run level or to the difference between the domestic inflation rate and the foreign inflation rate. For instance, Calvo et al. (1995) argued that Brazil, Chile and Colombia followed such rules in the past.

The characterization of the channels through which real exchange rate targeting affects the business cycles in emerging economies is a central issue in the design and implementation of the PPP rules. The theoretical literature about PPP rules has tried to disentangle these channels.[1] One of these important attempts is made by Uribe (2003) who analyzes a PPP rule whereby the government increases the devaluation rate when the real exchange rate is below its steady-state level. He pursues a determinacy of equilibrium analysis and argues that PPP rules may lead to aggregate instability in the economy by inducing endogenous fluctuations due to self-fulfilling expectations.

From the economic policy-design perspective, this result has important implications. It states that the aforementioned rules may open the possibility of sunspot equilibria and lead the economy to equilibria with undesirable properties such as a large degree of volatility. This implication in turn suggests that a determinacy of equilibrium analysis can be used to differentiate among rules favoring those that at least avoid sunspot equilibria by guaranteeing a unique equilibrium with a lower degree of volatility.[2] Although appealing this argument is still far from complete and may suffer from some drawbacks. The reason is that in the typical determinacy of equilibrium analysis, it is implicitly assumed that agents can coordinate their actions and learn the equilibria (unique or multiple) induced by the rule. But relaxing this assumption may have interesting consequences for the design of PPP rules. On one hand, if agents cannot learn the unique equilibrium targeted by the rule then the economy may end up diverging from this equilibrium. But if this is the case then it is clear that there are some rules that although guaranteeing a unique equilibrium, do not insure that the economy will reach it.[3] On the other hand, if agents cannot learn sunspot equilibria then

[1] See Dornbusch (1980,1982), Adams and Gros (1986), Lizondo (1991), Montiel and Ostry (1991) and Calvo et al. (1995), among others.

[2] This idea is not specific to PPP rules. In fact the idea that a rule that leads to indeterminacy of equilibrium may be seen as undesiderable has been emphasized by recent studies about interest rate rules. See Benhabib, Schmitt-Grohé and Uribe (2001), Carlstrom and Fuerst (2001), Clarida, Gali and Gertler (2000), Rotemberg and Woodford (1999) and Woodford (2003) among others.

[3] Bullard and Mitra (2002) have emphasized the importance of this point in the interest rate rule literature.

one may doubt about the relevance of characterizing rules that lead to multiple equilibria as "bad" ones. After all, if agents cannot learn sunspot equilibria then they are less likely to occur.

Therefore, it seems clear that a determinacy of equilibrium analysis of PPP rules should in principle be accompanied by a learnability of equilibrium analysis. Both analyses would help policy makers to distinguish and design PPP rules satisfying two requirements: uniqueness and learnability of the equilibrium. The first requirement would prevent the economy from achieving sunspot equilibria with undesirable properties such as a large degree of volatility. Whereas, the second requirement would guarantee that agents can indeed coordinate their actions on the equilibrium the policy makers are targeting.

The present paper is motivated by the interest of studying if particular representations of the equilibria (unique or multiple) induced by PPP rules are learnable in the Expectational - Stability (E-Stability) sense proposed by Evans and Honkapohja (1999, 2001).[4,5] In fact our purpose in the present paper is three-fold. First we study and disentangle the structural conditions of an open economy under which an Uribe-type PPP rule may generate multiple equilibria (real indeterminacy).[6] We use a small open economy model with traded and non-traded goods. We assume flexible prices for the former and sticky prices for the latter. Under this set-up we show how the aforementioned conditions depend not only on the responsiveness of the rule to the real exchange rate but also on some important structural parameters of the economy. For instance we find that ceteris paribus, given the sensitivity of the rule to the real exchange rate, the lower the degree of openness of the economy (the lower the share of traded goods), the more likely that the rule will induce aggregate instability in the economy by generating multiple equilibria. In addition, keeping the rest constant, the lower (the higher) the degree of price stickiness (the degree of monopolistic competition) in the non-traded sector, the more likely that the rule will lead to real indeterminacy.

The second goal of this paper consists of showing that under real determinacy the fundamental solution that describes the unique equilibrium induced by the PPP rule is learnable in the E-

[4] Henceforth we will use the terms "learnability", "E-stability" and "expectational stability" interchangeably in this paper.

[5] Evans and Honkapoja (1999, 2001) have argued that a unique equilibrium and sunspot equilibria are not "fragile" if they are learnable in the sense of E-stability. Technically what they propose is to assume that agents in the model initially do not have rational expectations but are endowed with a mechanism to form forecasts using recursive learning algorithms and previous data from the economy. Then they develop some E-stability conditions which govern whether or not a given rational expectations equilibrium is aymptotically stable under least squares learning.

[6] From now on we will use the terms "multiple equilibria" and "real indeterminacy" (a "unique equilibrium" and "real determinacy") interchangeably. By real indeterminacy we mean a situation in which the behavior of one or more (real) variables of the model is not pinned down by the model. This situation implies that there are multiple equilibria and opens the possibility of the existence of sunspot equilibria.

stability sense.[7] In addition we use the recent work by Evans and McGough (2003) to prove that under real indeterminacy some common factor representations of stationary sunspot equilibria are also E-stable.[8] This result suggests that under some reasonable assumptions agents can learn and coordinate their actions to achieve sunspot equilibria, making them "more likely" to occur under PPP rules. In this sense these equilibria should not be perceived as mere mathematical and theoretical curiosities.

The natural question that arises from these results is whether under a different timing of the PPP rule, it is possible for policy makers to design a simple rule that avoids sunspot equilibria but still induces a unique equilibrium whose characterization is learnable. In accord with the findings in the interest rate rule literature, we find that a PPP rule that is backward-looking in the sense of being defined in terms of the (deviation of the) past real exchange rate (from its long run level) satisfies these two requirements.

Finally the third goal of this paper is associated with the original work by Dornbusch (1980, 1982) that studies how a PPP rule whereby the nominal exchange rate is linked to the (deviation of the) current domestic price level (from its long-run level), may affect the output price-level stability trade-off by playing a role as an absorber of fundamental shocks.[9] We analyze a rule motivated by Dorbunsch's works assuming that the nominal devaluation rate is positively linked to the difference between the domestic and foreign CPI-inflation rates. In fact this specification tries to capture the previously mentioned stylized facts about PPP rules in Brazil, Colombia and Chile. As before we state the conditions under which this rule leads to real indeterminacy. We also show that the common factor representation of stationary sunspot equilibria as well as the fundamental solution that describes the unique equilibrium induced by the rule are learnable in the E-stability sense.

The remainder of this paper is organized as follows. Section 2 presents the set-up of a sticky-price model with its main assumptions. Section 3 pursues the determinacy of equilibrium analysis for a PPP rule defined in terms of the current real exchange rate. Section 4 deals with the learnability analysis for the aforementioned rule. Section 5 pursues all the previous analyses for a PPP rule defined in terms of the CPI-inflation rate. Finally Section 6 concludes.

[7] This fundamental solution is also well-known as the Minimal State Variable solution. See McCallum (1983).

[8] The common factor representation is an alternative representation of a Rational Expectations Equilibria. See Evans and Honkapoja (1986).

[9] Dornbusch (1980,1982) uses a Mundell and Fleming small open economy model with sticky wages *á la* Taylor and finds that the aforementioned PPP rule affects the output price-level stability trade-off through two different channels. On one hand, it tries to maintain constant the real exchange rate stabilizing net exports and therefore the demand side. On the other hand, it affects the supply side by its effect on the price of imported intermediate goods. Dornbusch shows that in such a model if the economy is hit by supply shocks then the price volatility always increases with tighter PPP rules. If the demand channel dominates the supply channel then the PPP rule reduces the volatility of output. But if the supply channel dominates the demand channel then the volatility of output is increased.

2 A Sticky-Price Model

2.1 The Household-Firm Unit

Consider a small open economy inhabited by a large number of identical household-firm units indexed by $j \in [0,1]$. The household-firms live infinitely and the preferences of the representative agent j can be described by the intertemporal utility function[10]

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[A(c_t^T(j), c_t^N(j), h_t^T(j), h_t^N(j)) - \frac{\phi}{2} \left(\frac{P_t^N(j)}{P_{t-1}^N(j)} - 1 - \bar{\pi}^N \right)^2 \right] \tag{1}$$

$$A(c_t^T(j), c_t^N(j), h_t^T(j), h_t^N(j)) = \alpha \log(c_t^T(j)) + (1-\alpha) \log(c_t^N(j)) + \psi \left[1 - h_t^T(j) - h_t^N(j)\right] \tag{2}$$

where $\alpha,\ \beta \in (0,1)$, ψ, $\phi > 0$; $c_t^T(j)$ and $c_t^N(j)$ denote the consumption of traded and non-traded goods respectively, $h_t^T(j)$ and $h_t^N(j)$ are the labor allocated to the production of the traded good and the non-traded good. E_t denotes the expectational operator.[11] Equations (1) and (2) imply that the representative agent derives utility from consuming traded and non-traded goods, and from not working in either sector.

We assume that the non-traded good is a composite good. We introduce monopolistic competition in the model by assuming that the household-firm unit j can choose the price of the non-traded good it supplies, $P_t^N(j)$, subject to a particular demand constraint described by

$$y_t^N(j) \geq c_t^N \left(\frac{P_t^N(j)}{P_t^N} \right)^{-\mu} \tag{3}$$

where $c_t^N = \left[\int_0^1 c_t^N(j) dj \right]$ represents the aggregate demand for the non-traded good and $\mu > 1$.

We also assume that there are sticky prices in the production of the non-traded good. This assumption is useful to understand the last term of the intertemporal utility function (1). Following Rotemberg (1982) we suppose that the household-unit dislikes having its price of non-traded goods grow at a rate different from $\bar{\pi}^N$, the steady-state level of the non-traded goods inflation rate.[12] We introduce sluggish adjustment in prices not only because this will enrich our analysis, but also

[10] The set-up of this model is very similar to Uribe (2003) and Zanna (2003a). However we endogenize labor in both sectors and introduce technology shocks. Moreover we use specific functional forms to be able to convey the main message of this paper. In particular we assume separability in terms of both types of consumption. A CES utility function will not affect the qualitative results of this paper but will make the derivation of our analytical results cumbersome.

[11] For the first part of the paper we will assume that agents have rational expectations. However for the E-stability analysis we will relax this assumption.

[12] Benhabib et al. (2001a,b) and Dupor (2001) also follow this approach to model price stickiness. An alternative approach follows Calvo (1983). Our results are invariant to this approach.

because, as Uribe (2003) points out, one of the main reasons that explains and motivates the real exchange targeting through PPP rules, is that policy makers believe in eliminating the real rigidities imposed by a fixed exchange rate system in an environment with nominal rigidities.

The production of traded and non-traded goods only requires labor and uses the following technologies

$$y_t^T(j) = z_t^T \left(h_t^T(j)\right)^{\theta_T} \qquad and \qquad y_t^N(j) = z_t^N \left(h_t^N(j)\right)^{\theta_N} \tag{4}$$

where $\theta_T, \theta_N \in (0,1)$, z_t^T and z_t^N are random productivity parameters that satisfy

$$\hat{z}_t^T = \varrho^T \hat{z}_{t-1}^T + \zeta_t^T \qquad and \qquad \hat{z}_t^N = \varrho^N \hat{z}_{t-1}^N + \zeta_t^N \tag{5}$$

where $\hat{z}_t^k = \log(z_t^k)$, $\zeta_t^k \backsim N(0, \sigma_\zeta^2)$ and $\varrho^k \in (0,1)$ for $k = T, N$. For simplicity in the analysis we assume no correlation between the productivity shocks.

The law of one price holds for the traded good and to simplify the analysis we normalize the foreign price of the traded good to one. Therefore, the domestic currency price of traded goods (P_t^T) is equal to the nominal exchange rate ($\mathcal{E}_t$). This simplification in tandem with (1) and (2) can be used to derive the consumer price index (CPI)

$$p_t = \frac{(\mathcal{E}_t)^\alpha \left(P_t^N\right)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \tag{6}$$

Using equation (6) and defining the nominal devaluation rate as

$$\epsilon_t = \mathcal{E}_t / \mathcal{E}_{t-1} - 1 \tag{7}$$

it is straightforward to derive the CPI-inflation rate, π_t, as a weighted average of the nominal depreciation rate, ϵ_t, and the inflation of the non-traded goods, $\pi_t^N = P_t^N / P_{t-1}^N - 1$; that is

$$1 + \pi_t = (1 + \epsilon_t)^\alpha \left(1 + \pi_t^N\right)^{(1-\alpha)} \tag{8}$$

We define the real exchange rate (e_t) as the ratio between the price of traded goods and the aggregate price of non-traded goods

$$e_t = \mathcal{E}_t / P_t^N \tag{9}$$

From this definition of the real exchange rate we deduce that

$$e_t = e_{t-1} \left(\frac{1 + \epsilon_t}{1 + \pi_t^N} \right) \tag{10}$$

We assume that in each period $t \geq 0$ the representative agent can purchase two types of financial assets: fiat money $M_t^d(j)$ and nominal state contingent claims, $D_{t+1}(j)$, that pay one unit of currency in a specified state of period $t+1$ and that are traded internationally. There exists a set of these state contingent claims that completely spans the fundamental (intrinsic) uncertainty associated with productivity shocks. Moreover following Kimbrough (1986), we suppose that money reduces the transaction costs in goods markets. These costs measured in terms of the traded good can be described by

$$g_t(j) = A\left(c_t^T(j) + \frac{c_t^N(j)}{e_t}\right)^{1+\gamma}\left(\frac{M_t^d(j)}{\mathcal{E}_t}\right)^{-\gamma} \tag{11}$$

where $A, \gamma > 0$.

Using the previous assumptions the representative agent's flow constraint each period can be written as[13]

$$M_t^d(j)+E_tQ_{t,t+1}D_{t+1}(j) \leq W_t(j)+\mathcal{E}_t y_t^T(j)+P_t^N(j)y_t^N(j)-\mathcal{E}_t\tau_t-\mathcal{E}_t c_t^T(j)-P_t^N c_t^N(j)-\mathcal{E}_t g_t(j) \tag{12}$$

where $Q_{t,t+1}$ refers to the period-t price of a claim to one unit of currency delivered in a particular state of period $t+1$ divided by the probability of occurrence of that state and conditional of information available in period t. Hence $E_tQ_{t,t+1}D_{t+1}(j)$ denotes the cost of all contingent claims bought at the beginning of period t. Constraint (12) says that the total end-of-period nominal value of the financial assets can be worth no more than the value of the financial wealth brought into the period, W_t, plus non-financial income during the period net of the value of taxes, $\mathcal{E}_t\tau_t$, the value of consumption spending and the value of the liquidity transaction costs.

To derive the period-by-period budget constraint of the representative agent, it is important to notice that total beginning-of-period wealth in the following period is given by

$$W_{t+1}(j) = M_t^d(j) + D_{t+1}(j) \tag{13}$$

and that $E_tQ_{t,t+1}$ corresponds to the price at period t of a claim that pays one unit of currency in every state in period $t+1$ and represents the inverse of the risk-free gross nominal interest rate, $1+i_t$; that is

$$E_tQ_{t,t+1} = \frac{1}{1+i_t} \tag{14}$$

Then we can use equations (12), (13) and (14) to derive the budget constraint of the representative agent as

$$\begin{aligned} E_tQ_{t,t+1}W_{t+1}(j) \quad \leq \quad & W_t(j) + \mathcal{E}_t y_t^T(j) + P_t^N(j)y_t^N(j) - \mathcal{E}_t\tau_t - \frac{i_t}{1+i_t}M_t^d(j) \\ & -\mathcal{E}_t c_t^T(j) - P_t^N c_t^N(j) - \mathcal{E}_t g_t(j) \end{aligned} \tag{15}$$

[13] We follow Woodford (2003) to construct the budget constraint of the representative agent.

The agent is also subject to a Non-Ponzi game condition described by

$$\lim_{s\to\infty} E_t q_{t+s} W_{t+s}(j) \geq 0 \tag{16}$$

at all dates and under all contingencies, where q_t represents the period-zero price of one unit of currency to be delivered in a particular state of period t divided by the probability of occurrence of that state, given information available at time 0. It is given by

$$q_t = Q_1 Q_2 .....Q_t \tag{17}$$

with $q_0 \equiv 1$.

Under this sticky-price set-up the problem of the representative agent is reduced to choose the sequences $\{c_t^T(j), c_t^N(j), h_t^T(j), h_t^N(j), M_t^d(j), W_{t+1}(j), P_t^N(j)\}_{t=0}^{\infty}$ in order to maximize (1) subject to (2), (3), (4), (11), (15) and (16), and given $W_0(j)$, and $\bar{\pi}^N$ and the time paths for i_t, $\mathcal{E}_t$, P_t^N, c_t^N, Q_{t+1}, τ_t and z_t^T and z_t^N. Note that the utility function specified in (1) and (2) implies that the preferences of the agent display non-sasiation. This means that constraints (15) and (16) both hold with equality.

The first order conditions correspond to (15) and (16) both with equality and

$$\frac{\alpha}{c_t^T(j)} = \lambda_t(j)\left[1 + \Gamma\left(\frac{i_t}{1+i_t}\right)^{\frac{\gamma}{1+\gamma}}\right] \tag{18}$$

$$\frac{\alpha c_t^N(j)}{(1-\alpha)c_t^T(j)} = e_t \tag{19}$$

$$\left(\frac{P_t^N(j)}{P_t^N} - \frac{\delta_t(j) e_t}{\lambda_t(j)}\right)\frac{z_t^N \theta_N \left[h_t^N(j)\right]^{(\theta_N - 1)}}{e_t} = z_t^T \theta_T \left[h_t^T(j)\right]^{\theta_T - 1} \tag{20}$$

$$\frac{M_t^d(j)}{\mathcal{E}_t} = \left[(A\gamma)\left(1 + \frac{1}{i_t}\right)\right]^{\frac{1}{1+\gamma}}\left(c_t^T(j) + \frac{c_t^N(j)}{e_t}\right) \tag{21}$$

$$\frac{\lambda_t(j)}{\mathcal{E}_t} Q_{t,t+1} = \frac{\lambda_{t+1}(j)}{\mathcal{E}_{t+1}}\beta \tag{22}$$

$$\beta\phi E_t\left\{\left[\frac{P_{t+1}^N(j)}{P_t^N(j)} - 1 - \bar{\pi}^N\right]\frac{P_{t+1}^N(j)}{\left[P_t^N(j)\right]^2}\right\} = \phi\left[\frac{P_t^N(j)}{P_{t-1}^N(j)} - 1 - \bar{\pi}^N\right]\frac{1}{P_{t-1}^N(j)} + \frac{\lambda_t(j) z_t^N \left[h_t^N(j)\right]^{\theta_N}}{P_t^N e_t} - \frac{\delta_t(j)\mu c_t^N}{P_t^N}\left[\frac{P_t^N(j)}{P_t^N}\right]^{-(1+\mu)} \tag{23}$$

where $\lambda_t(j)/\mathcal{E}_t$ corresponds to the multiplier of the budget constraint, $\delta_t(j)$ is the multiplier associated with the demand constraint (3) and $\Gamma = A(1+\gamma)(A\gamma)^{-\frac{\gamma}{1+\gamma}}$. The interpretation of the first order conditions is straightforward. In particular, equation (18) is the usual intertemporal envelope condition that makes the marginal utility of consumption of traded goods equal to the marginal utility of wealth ($\lambda_t(j)$) multiplied by the intertemporal price of consuming traded goods.[14] Condition (19) implies that the marginal rate of substitution between traded and non-traded goods must be equal to the real exchange rate. In addition, condition (20) equalizes the marginal revenue products of labor among sectors. Equation (21) represents the demand for real balances of money as an increasing function of consumption expenditure and a decreasing function of the risk-free nominal interest rate. And finally condition (22) implies a standard pricing equation for one-step-ahead nominal contingent claims. Note that in each period t there is one condition of this type for each possible state in period $t+1$.

Finally we postpone the explanation of condition (23). The reason is that it will be used to derive the augmented Phillips curve for non-traded goods, that is actually one of the relevant equations for the determinacy and learnability of equilibrium analyses.

2.2 The Government

The government issues two nominal liabilities: money, M_t^s, and state contingent debt D_{t+1}^s. It also levies taxes, τ_t, pays interest on its debt, and receives revenues from seigniorage. Thus we can write the government budget constraint as

$$E_t(Q_{t,t+1}W_{t+1}^s) = W_t^s - \frac{i_t M_t^s}{1+i_t} - \mathcal{E}_t \tau_t \tag{24}$$

where $W_{t+1}^s = M_t^s + D_{t+1}^s$. The government follows a Ricardian fiscal policy. That is, it picks the path of taxes, τ_t, satisfying the intertemporal version of (24) in conjunction with the transversality condition

$$\lim_{k\to\infty} E_t(q_{t+k}W_{t+k}^s) = 0 \tag{25}$$

Finally we define the monetary policy as in Uribe (2003). The government follows a PPP rule whereby the government sets the nominal devaluation rate as a function of the deviation of the current real exchange rate (e_t) with respect to its long-run level ($\bar{e}$). That is

$$\epsilon_t = \rho(e_t - \bar{e}) \qquad with \qquad \rho_e = \frac{d\rho}{de_t} < 0 \tag{26}$$

where $\rho(.)$ is a continuous function that in steady state satisfies $\bar{\epsilon} = \rho(0)$.

[14] This price is equal to its output cost (=1) plus a term that is a function of the opportunity cost of holding wealth in monetary form.

2.3 The Equilibrium

We will focus on *a symmetric equilibrium* in which all the household-firm units choose the same price for the good they produce. Therefore in equilibrium all agents are identical and we can drop the index j. In equilibrium the money market and the non-traded goods market clear. Thus

$$M_t^d = M_t^s \tag{27}$$

and $y_t^N = \left(h_t^N\right)^{\theta_N} = c_t^N$. As usual we ignore the wealth effects due to inflation by assuming that the transaction liquidity costs, g_t, are rebated to the representative agent in a lump-sum fashion.

We also assume free capital mobility. This implies that the following non-arbitrage condition must hold

$$Q_{t,t+1}^* = Q_{t,t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{28}$$

where $Q_{t,t+1}^*$ refers to the period-t foreign currency price of a claim to one unit of foreign currency delivered in a particular state of period $t+1$ divided by the probability of occurrence of that state and conditional of information available in period t. An equivalent condition to (22) holds for the foreign economy (rest of the world). That is,

$$\frac{\lambda_t^*}{P_t^{T*}} Q_{t,t+1}^* = \frac{\lambda_{t+1}^*}{P_{t+1}^{T*}} \beta^* \tag{29}$$

where λ_t^*, P_t^{T*} and β^* represent the marginal utility of wealth, the price of traded goods and the subjective discount rate in the foreign economy respectively. Using (22), (28), (29), the law of one price for traded goods and the assumption that $\beta^* = \beta$ we can derive that $\frac{\lambda_{t+1}}{\lambda_t} = \frac{\lambda_{t+1}^*}{\lambda_t^*}$, that holds at all dates and under all contingencies.[15] This equation implies that the domestic marginal utility of wealth is proportional to its foreign counterpart. Then $\lambda_t = \xi \lambda_t^*$ where ξ refers to a constant parameter that determines the wealth difference across countries. Since we are dealing with a small open economy, λ_t^* can be taken as an exogenous variable. To simplify the analysis we assume that λ_t^* is constant and equal to λ^*. This assumption implies that λ_t becomes a constant. Consequently

$$\lambda_t = \lambda = \xi \lambda^* \tag{30}$$

But this result of a constant marginal utility and conditions (14) and (22) imply that

$$1 + i_t = \beta^{-1} \left[E_t \left(\frac{1}{1 + \epsilon_{t+1}} \right) \right]^{-1} \tag{31}$$

[15] Note that as a consequence of the aforementioned contingent claims that completely span the uncertainty about productivity shocks the model abstracts from wealth effects due to current account imbalances. In this respect the model is similar to the ones in Clarida, Gali and Gertler (2001) and Gali and Monacelli (2004).

where E denotes the expectation operator. Note that condition (31) is very similar to the uncovered interest parity condition.

Utilizing (20), (23), $\pi_t^N = P_t^N / P_{t-1}^N - 1$, the symmetry in equilibrium, the equilibrium condition in the non-traded sector ($y_t^N = \left(h_t^N\right)^{\theta_N} = c_t^N$) and (30), we obtain

$$E_t\left[(\pi_{t+1}^N - \bar{\pi}^N)(1+\pi_{t+1}^N)\right] = \frac{1}{\beta}(\pi_t^N - \bar{\pi}^N)(1+\pi_t^N) + \frac{(\mu-1)\lambda c_t^N}{\phi\beta e_t} - \frac{\mu\psi}{\phi\beta\theta_N}\left(\frac{c_t^N}{z_t^N}\right)^{\frac{1}{\theta_N}} \tag{32}$$

that corresponds to the augmented Phillips curve for the non-traded goods inflation.[16]

Furthermore applying the symmetry in equilibrium and recalling (30), we can rewrite (18), (19) and (21) as

$$\frac{\alpha}{c_t^T} = \lambda\left[1+\Gamma\left(\frac{i_t}{1+i_t}\right)^{\frac{\gamma}{1+\gamma}}\right] \tag{33}$$

$$\frac{\alpha c_t^N}{(1-\alpha)c_t^T} = e_t \tag{34}$$

$$\frac{M_t^d}{\mathcal{E}_t} = \left[(A\gamma)\left(1+\frac{1}{i_t}\right)\right]^{\frac{1}{1+\gamma}}\left(c_t^T + \frac{c_t^N}{e_t}\right) \tag{35}$$

We proceed giving the definition of a symmetric equilibrium for a government that pursues a Ricardian fiscal policy and follows a PPP rule that responds to the current real exchange rate as described by (26).

Definition 1 *Given, W_0, $\bar{\epsilon}$, $\bar{\pi}^N$ and e_0 and the exogenous stochastic processes $\left\{z_t^N, z_t^T\right\}_{t=0}^{\infty}$, a Symmetric Equilibrium under a Ricardian fiscal policy is defined as a set of stochastic processes $\{c_t^T,\ c_t^N,\ M_t,\ \tau_t,\ e_t,\ Q_{t+1},\ q_t,\ \mathcal{E}_t,\ \epsilon_t,\ \pi_t^N,\ i_t\}_{t=0}^{\infty}$ satisfying conditions (32), (33), (34), (35), the intertemporal version of (24) together with (25), the PPP rule defined by (26), the money market clearing condition (27), definitions (7), (17) and equations (10), (14), and (31).*

3 The Determinacy of Equilibrium Analysis

To pursue the determinacy of equilibrium analysis we reduce the model further. To do so we can use conditions (33) and (34) to obtain

$$\frac{1-\alpha}{c_t^N} = \frac{\lambda}{e_t}\left[1+\Gamma\left(\frac{i_t}{1+i_t}\right)^{\frac{\gamma}{1+\gamma}}\right] \tag{36}$$

that together with the PPP rule (26) and equations (5), (10), (31) and (32), are the only equations necessary to pursue the determinacy of equilibrium analysis in our model. They help us to find

[16] We would have derived a similar augmented Phillips curve if we had follow Calvo's (1983) approach.

the stochastic processes $\{c_t^N,\ e_t,\ \epsilon_t,\ \pi_t^N,\ i_t\}_{t=0}^{\infty}$. These set of equations are also useful to define the non-stochastic steady state. It corresponds to

$$\bar{\pi}^N = \bar{\epsilon} \qquad (1+\bar{\imath})\,\beta = (1+\bar{\epsilon}) \qquad \bar{e} = \frac{\lambda \bar{c}^N}{(1-\alpha)}\left[1+\Gamma\left(\frac{\bar{\imath}}{1+\bar{\imath}}\right)^{\frac{\gamma}{1+\gamma}}\right] > 0$$

where $\left(\bar{c}^N\right)^{\frac{1}{\theta_N}} = \frac{(1-\alpha)(\mu-1)\theta_N}{\mu\psi\left[1+\Gamma\left(\frac{\bar{\imath}}{1+\bar{\imath}}\right)^{\frac{\gamma}{1+\gamma}}\right]}$ and $\bar{\epsilon}$ denotes the long-run nominal devaluation rate that is determined by the government.

We point out that we do not need to consider in the determinacy analysis equations (24) and (25). The reason is that under a Ricardian fiscal policy, the intertemporal version of the government's budget constraint in conjunction with its transversality condition will be always satisfied. Moreover the stochastic processes $\{c_t^T,\ M_t,\ Q_{t+1},\ q_t,\ \mathcal{E}_t\}_{t=0}^{\infty}$ can be recovered using (7), (14), (17), (27), (34) and (35).[17]

We can go further reducing and log-linearizing the model. Using equations (5), (10), (26), (31), (32), and (36) yields[18]

$$E_t\hat{\pi}_{t+1}^N = \beta^{-1}\hat{\pi}_t^N - K\hat{e}_t + HE_t\hat{\epsilon}_{t+1} + K\hat{z}_t^N \tag{37}$$

$$\hat{e}_t = \hat{e}_{t-1} + \frac{\bar{\epsilon}}{1+\bar{\epsilon}}\left(\hat{\epsilon}_t - \hat{\pi}_t^N\right) \tag{38}$$

$$\hat{z}_t^N = \varrho^N \hat{z}_{t-1}^N + \zeta_t^N \tag{39}$$

$$\hat{\epsilon}_t = \frac{\rho_e \bar{e}}{\bar{\epsilon}}\hat{e}_t \tag{40}$$

where $\hat{x}_t = \log(x_t) - \log(\bar{x})$, $\bar{x}$ represents the steady-state level of x_t,

$$K = \frac{\mu\psi\left(\bar{c}^N\right)^{\frac{1}{\theta_N}}}{\beta\theta_N^2\phi\bar{e}(1+\bar{\epsilon})} > 0, \qquad H = (1-\theta_N)\Omega K > 0 \tag{41}$$

and $\Omega = \left[\frac{\bar{\epsilon}\Gamma\gamma\mu\psi}{\bar{\imath}(1+\bar{\epsilon})(1+\gamma)(1-\alpha)(\mu-1)\theta_N}\right]\left(\frac{\bar{\imath}}{1+\bar{\imath}}\right)^{\frac{\gamma}{1+\gamma}}\left(\bar{c}^N\right)^{\frac{1}{\theta_N}}$. For our future analyses it is important to observe that $K > 0$ and $H > 0$. To see this recall our assumptions about the values that are feasible to assign to the structural parameters of the model and the definition of the steady-state.

[17] Note that the spirit of the PPP rule and the assumption that P_t^N is sticky imply that e_t is a predetermined variable. As a consequence assuming that P_0^N and e_0 are given corresponds to assume that $\mathcal{E}_0$ is given which in turn avoids the possibility of nominal indeterminacy.

[18] Observe that we have not included equation $\hat{z}_t^T = \varrho^T\hat{z}_{t-1}^T + \zeta_t^T$. The reason is that $\hat{z}_t^T$ does not affect the other equations. It may affect the current account. But as in Clarida et al. (2001) and Gali and Monacelli (2004) we are abtracting from wealth effects due to current account imbalances.

As we mentioned before in this analysis we only study the possibilities of *real indeterminacy* or *real determinacy* of the equilibrium. By *real indeterminacy* we mean a situation in which the behavior of one or more (real) variables of the model are not pinned down by the model. This situation implies that there are multiple equilibria and opens the possibility of the existence of sunspot equilibria.

Before we analyze the conditions under which PPP rules may lead to real indeterminacy, it is worth constructing some intuition using the model of why these rules may induce equilibria in which expectations are self-fulfilled. In order to accomplish this task we can assume perfect foresight (no uncertainty). Then we rewrite equations (37) and (38) as

$$\hat{\pi}_{t+1}^{N} = \beta^{-1}\hat{\pi}_{t}^{N} - K\hat{e}_t + H\hat{\epsilon}_{t+1} \tag{42}$$

$$\hat{e}_t = \hat{e}_{t-1} + \frac{\bar{\epsilon}}{1+\bar{\epsilon}}\left(\hat{\epsilon}_t - \hat{\pi}_t^N\right)$$

In addition we can iterate forward both equations and derive[19]

$$\hat{\pi}_t^N = \sum_{k=0}^{\infty}\beta^k\left[K\hat{e}_{t+k} - H\hat{\epsilon}_{t+1+k}\right] \tag{43}$$

$$\hat{e}_{t+1} - \hat{e}_t = \frac{\bar{\epsilon}}{1+\bar{\epsilon}}\left(\hat{\epsilon}_{t+1} - \hat{\pi}_{t+1}^N\right) \tag{44}$$

Equation (43) implies that current inflation of non-traded goods is determined by the discounted sum of the expected future real exchange rates and nominal depreciation rates. The first term inside of the parenthesis is associated with future real exchange rates. It captures the fact that higher expected future real exchange rates make non-traded goods become relatively cheaper than traded goods. This leads to a higher expected future excesses of demand for non-traded goods to which the firm-unit responds raising the current price of non-traded goods up and therefore increasing the current non-traded goods inflation rate. On the other hand, the second term in (43) that is associated with future nominal depreciation rates captures the effect of the intertemporal price of consumption on the determination of the current non-traded goods inflation. In essence, expectations of nominal appreciation (negative nominal depreciation rates) decrease the nominal interest rate provided that the uncovered interest parity condition holds under perfect foresight. But a decrease in the nominal interest rate pushes the liquidity transaction costs down, which in turn expands consumption of non-traded (and traded) goods. This increase in consumption lead to a positive excess of demand for non-traded goods and therefore to a higher current inflation.

[19] Here we assume that $\hat{\pi}_t^N$ is a bounded sequence and that $\underset{T\to\infty}{Lim}\,\beta^T\hat{\pi}_{t+T}^N = 0$.

Equation (44) simply describes the depreciation (or appreciation) of the real exchange rate as a difference between the nominal depreciation rate and the non-traded goods inflation rate.

With these two last equations, equation (42) and the PPP rule, $\hat{\epsilon}_t = \frac{\rho_e \bar{e}}{\bar{\epsilon}} \hat{e}_t$, we can show that multiple equilibria are possible by constructing a self-fulfilling equilibrium. Assume that at time $t-1$, when the economy is in its steady state, private agents expect a real appreciation after time t. More explicitly they expect the following path for the (deviation of the) real exchange rate.[20] At time $t-1$, the real exchange rate is at the steady state level ($\hat{e}_{t-1} = 0$). At times t and $t+1$ the real exchange rate is above its steady state level ($\hat{e}_t > 0$ and $\hat{e}_{t+1} > 0$) and satisfies $\hat{e}_t > \hat{e}_{t+1}$, showing a real appreciation and convergence to the steady state level over time. Given this path of the real exchange rate and given the PPP rule, the government will induce a nominal appreciation ($\hat{\epsilon}_{t+k} = \frac{\rho_e \bar{e}}{\bar{\epsilon}} \hat{e}_{t+k}$ with $\rho_e < 0$ and $k \geq 0$) over time after time t. Then using our interpretation of (43) we can infer that the expected path for the real exchange rate and the expected nominal appreciation will motivate the household-firm unit to raise the price of non traded goods in period t. In other words inflation at time t will go up ($\hat{\pi}_t^N > 0$). By equation (42) this effect will increase inflation of non traded goods in period $t+1$ ($\hat{\pi}_{t+1}^N > 0$), if it is strong enough to overcome the opposite effects that the assumed path for the real exchange rate ($\hat{e}_t > 0$) and the rule-induced path for the nominal depreciation rate ($\hat{\epsilon}_{t+1} < 0$) have over inflation of non-traded goods at period $t+1$ ($\hat{\pi}_{t+1}^N$). Observe that this possibility is determined by the values of the structural parameters of the model that affect H and K, and by the value of the nominal depreciation response coefficient to the real exchange rate (ρ_e). But if inflation of non-traded goods goes up ($\hat{\pi}_{t+1}^N > 0$) and people expect a nominal appreciation ($\hat{\epsilon}_{t+1} < 0$) in period $t+1$ accordingly with equation (44), we conclude that the real exchange rate will appreciate over time (($\hat{e}_{t+1} - \hat{e}_t) < 0$). Since all the variables of the system, including the real exchange rate, converge to their steady state level over time then the original expectations of a future real appreciation will be self-fulfilled.

Although this intuitive argument points out the possibility of self-fulfilling equilibria induced by a PPP rule, it is important to disentangle the conditions under which these equilibria are possible. The following proposition achieves this goal characterizing locally the equilibrium for the model described by equations (37)-(40).

Proposition 1 *Suppose the government follows a PPP rule that is described by* $\epsilon_t = \rho(e_t - \bar{e})$ *with* $\rho_e = \frac{d\rho}{de_t} < 0$. *Let* K *and* H *be defined as in (41) and define*

$$\tilde{\rho}_e = \frac{\left[2(1+\bar{\epsilon})\left(1+\frac{1}{\beta}\right) + K\bar{\epsilon}\right]}{\left[\bar{e}\left(1+\frac{1}{\beta} - H\right)\right]}$$

[20]It is important to remember that in the log-linearized set-up all the variables are expressed as deviations from their steady state level.

a) *If* $1+\frac{1}{\beta}<H$ *and* $\rho_e<\tilde{\rho}_e$ *then there is real indeterminacy.*

b) *If* $1+\frac{1}{\beta}<H$ *and* $\tilde{\rho}_e<\rho_e$ *then there is real determinacy.*

c) *If* $H<1+\frac{1}{\beta}$ *then there is real determinacy for any* $\rho_e<0$.

Proof. See Appendix. ■

From Proposition 1 it is clear that conditions under which PPP rules lead to multiple equilibria do not simply depend on the response coefficient ρ_e. On the contrary some of the structural parameters of the model play a fundamental role in the determinacy of equilibrium. In essence all the parameters that affect H and K are relevant for the analysis. In order for the PPP rule to induce multiple equilibria two conditions must be satisfied. The first one constrains the possible values that the structural parameters may take ($1+\frac{1}{\beta}<H$); the second one points out the importance of the PPP rule on inducing real indeterminacy. It sets a threshold for the nominal depreciation response coefficient that depends on the structural parameters of the model ($\rho_e<\tilde{\rho}_e$). Even more interesting is the result of part **c)** in the proposition. It says that for some values that the structural parameters may take and regardless of the value of the nominal devaluation response coefficient (ρ_e), the model displays real determinacy. This result contrasts with the results of Uribe (2003) that claims that if the elasticity of the PPP rule is sufficiently large then a model with sticky-prices always displays real indeterminacy.

To understand the important role that some of the structural parameters of the model may play in the determinacy of equilibrium analysis, we study how the aforementioned threshold ($\tilde{\rho}_e$) varies with respect to some of these structural parameters. Specifically we consider the share of traded goods (α), the degree of monopolistic competition in the non-traded sector (μ) and the degree of price stickiness in the non-traded sector (ϕ).[21] Note that the share of traded goods can be considered a measure of the degree of openness of the economy with $\alpha\rightarrow 0$ describing a very closed economy. The following corollary summarizes the main results.

Corollary 1 *Suppose the government follows a PPP rule given by* $\epsilon_t=\rho(e_t-\bar{e})$ *with* $\rho_e<0$. *Let* $\tilde{\rho}_e$ *be defined as in Proposition 1 and assume* $1+\frac{1}{\beta}<H$, *then* **a)** $\frac{\partial(\tilde{\rho}_e\bar{e})}{\partial\alpha}<0$; **b)** $\frac{\partial(\tilde{\rho}_e\bar{e})}{\partial\mu}>0$ *and* **c)** $\frac{\partial(\tilde{\rho}_e\bar{e})}{\partial\phi}<0$.

Proof. See Appendix. ■

Using Proposition 1 and Corollary 1 we can understand the effects of varying some of the structural parameters and the semi-elasticity of the rule ($\rho_e\bar{e}$), on the determinacy of equilibrium.

[21] It is possible to do the same exercise with respect to other structural parameters such as the share of labor in the production function (θ_N). These results are available upon request. We focus our analyzis on the share of traded goods (α) because this is a particular feature of open economies. We also concentrate the analysis on the degrees of monopolistic competition (μ) and price stickiness (ϕ) in the non-traded sector because these parameters capture an important asymmetry between the traded and non-traded sectors.

In fact when $1 + \frac{1}{\beta} < H$ we can conclude that given the semi-elasticity of the PPP rule, the less open the economy is (the lower α is), the more likely that the PPP rule will induce aggregate instability in the economy by generating multiple equilibria. In addition, given the semi-elasticity of the PPP rule and keeping the rest constant, the higher the degree of monopolistic competition in the non-traded sector (the higher μ), the more likely that the rule will lead to real indeterminacy. Finally under ceteris paribus and given the semi-elasticity of the rule we find that the lower the degree of price stickiness in the non-traded sector (the lower ϕ), the more feasible that the rule will induce multiple equilibria.

Notwithstanding the relevance of these analytical results, it is crucial to investigate their quantitative importance. To accomplish this we rely on a specific parametrization of the model. Since this exercise is merely indicative we borrow some values of the parameters from previous studies about emerging and small open economies.[22] Following Schmitt-Grohé and Uribe's (2001) study about Mexico we assign the following values to some of the relevant structural parameters of the model: $\beta = 0.98$ per quarter, $\bar{\epsilon} = 0.0157$ per quarter, $\gamma = 5.25$, $A = 0.55$, $\mu = 10$, and $\theta_N = 0.36$. We set $\alpha = 0.44$, that corresponds roughly to the imports to GDP share in Mexico during the 90's. Finally we set $\phi = 2.80$, that corresponds to Dib's (2001) estimate of ϕ for Canada in a model with only nominal rigidities.[23] With these values, we will perform four exercises characterizing locally the equilibrium. We will vary the semi-elasticity of the rule, $\rho_e \bar{\epsilon}$ and one and only one of the following structural and policy parameters: α, μ, ϕ and $\bar{\epsilon}$. We summarize the parametrization in the following table.

Table 1

θ_N	ϕ	μ	γ	A	β	α	$\bar{\epsilon}$
0.36	2.8	10	5.25	0.55	0.98	0.44	0.0157

The results of our exercises are presented in Figure 1, where "I" stands for real indeterminacy and "D" stands for real determinacy. As can be observed, this figure confirms the results in Proposition 1 and Corollary 1 showing how significant these results are in quantitative terms. Consider the top left panel. From this panel we can infer the following. Suppose that the government in response to a 1 per cent appreciation of the real exchange rate, devalues the nominal exchange rate

[22] Note that for this exercise we do not need to assign values to all the parameters. We only present the parametrization of the relevant parameters.

[23] There is no clear consensus about the value that this parameter must take in emerging economies. One of the reasons is the lack of studies that have tried to estimate Phillips curves for these economies and that may give information about possible values for this parameter. Even for an industrialized economy such as Canada, this parameter varies between 2.80 and 44.07, depending on the model specification (type of nominal and real rigidities). See Dib (2001).

by 2 percent. In other words, assume that the semi-elasticity of the PPP rule is -2. Whereas this PPP rule may induce multiple equilibria in an economy whose degree of openness is 0.2, the same rule leads to a unique equilibrium in an economy whose degree of openness is 0.6.

Similar inferences can be pursued from the top right and bottom left panels of Figure 1. That is although a rule with semi-elasticity of -2 guarantees a unique equilibrium in an economy with a degree of monopolistic competition of 5 (a degree of price stickiness of 5), the same rule induces multiple equilibria when the aforementioned degree corresponds to 15 (2).

Although it is not possible to derive an analytical result to see how varying the implied nominal depreciation target ($\bar{\epsilon}$) and the semi-elasticity of the rule ($\tilde{\rho}_e \bar{e}$) affects the determinacy of equilibrium, it is possible to evaluate this quantitatively as presented in the bottom right panel of Figure 1. This panel illustrates that given the semi-elasticity of the rule the lower the implied nominal depreciation target the more likely is that the PPP rule will induce real indeterminacy. In addition, it is important to observe that in all four panels of Figure 1 there are regions for which the model always displays real determinacy regardless of the semi-elasticity of the rule. This agrees with part c) of Proposition 1.

To finalize this section we want to point out that similar qualitative results to the ones presented in this section can be obtained if the PPP rule is defined in terms of the real depreciation rate. That is $\epsilon_t = \rho(\Delta e_t)$ where $\Delta e_t = \frac{e_t}{e_{t-1}}$.[24]

4 The Learnability Analysis

The importance of the result from the previous section, that a PPP rule may induce aggregate instability by generating multiple equilibria in the economy, stems from the fact that such rule opens the possibility of expectations driven fluctuations in economic activity. In particular, the model may admit self-fulfilling rational expectations equilibria driven by extraneous processes known as sunspots.[25]

However the previous results, as the ones in Uribe (2003), do not discuss the attainability of these PPP rule induced sunspot equilibria. They do not even mention how attainable the unique equilibrium is. Strictly speaking, and regardless of real determinacy or real indeterminacy, it is not clear whether and how agents may coordinate their actions in order to achieve a particular equilibrium in the model. The purpose of this section is to address this issue. We want to study the potential of agents to learn the unique equilibrium characterized by the fundamental solution and sunspot equilibria described by a common factor solution.

As a criterion of "learnability" of an equilibrium we will use the concept of "E-stability" proposed

[24] See Zanna (2003b).

[25] The idea of expectation driven fluctuations dates back to Keynes (1936).

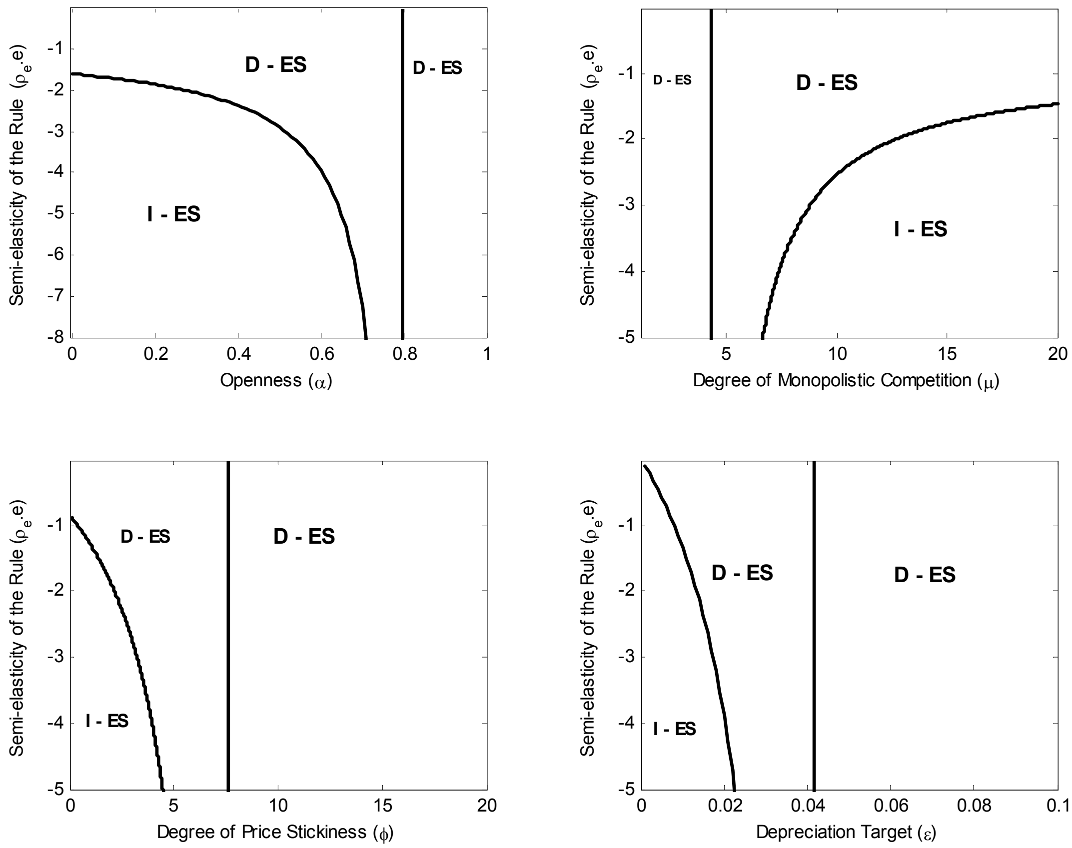

Figure 1: This figure shows how the local determinacy of equilibrium varies with respect to the semi-elasticity of the rule ($\rho_e \bar{e}$), the share of traded goods (α), the degree of monopolistic competition in the non-traded sector (μ), the degree of price stickiness in the non-traded sector (ϕ) and the implied nominal depreciation rate target ($\bar{\epsilon}$). "I" stands for real indeterminacy (multiple equilibria) and "D" stands for real determinacy (a unique equilibrium). "ES" corresponds to E-Stability.

by Evans and Honkapohja (1999, 2001). That is, an equilibrium is "learnable" if it is "E-Stable".[26] Consequently we start by assuming that agents in our model no longer are endowed with rational expectations. Instead they have adaptive rules whereby agents form expectations using recursive least squares updating and data from the system. Then we derive the conditions for expectational stability (E-stability).

In our analysis we will focus on the expectational stability concept for the following reasons. First, in models that display a unique equilibrium (real determinacy models), Marcet and Sargent (1989) and Evans and Honkapohja (1999, 2001) have shown that under some general conditions, the notional time concept of expectational stability of a rational expectation equilibrium governs the local convergence of real time adaptive learning algorithms. Specifically they have shown that under E-stability, recursive least-squares learning is locally convergent to the rational expectations equilibrium. Second, Evans and McGough (2003) have numerically argued that under some assumptions about the parameters of a linear stochastic univariate model, with a predetermined variable, the same argument applies when this model displays sunspot equilibria. Formally they have stated that under a strict subset of the structural parameter space, there exist stationary sunspot equilibria that are locally stable under least square learning provided that agents use a common factor representation for their estimated law of motion.

We adapt the methodology of Evans and Honkapohja (1999, 2001) and Evans and McGough (2003) to pursue the learnability (E-stability) analysis. Accordingly we need to define the concept of E-stability. In order to define it we give an idea of the methodology we apply for the case of real determinacy.

To grasp the methodology, it becomes useful to reduce our model to the following linear stochastic difference equations system. Use (37), (38), (39) and (40) to rewrite the model as

$$\hat{e}_t = \hat{\alpha} + \hat{\beta} E_t \hat{e}_{t+1} + \hat{\delta} \hat{e}_{t-1} + \hat{\kappa} \hat{z}_t^N \qquad and \qquad \hat{z}_t^N = \varrho^N \hat{z}_{t-1}^N + \zeta_t^N \tag{45}$$

where $\hat{\alpha} = 0$,

$$\hat{\beta} = \frac{\beta \left(1 + \bar{\epsilon} - \rho_e \bar{e} + H \rho_e \bar{e}\right)}{\hat{\sigma}} \qquad \hat{\delta} = \frac{1 + \bar{\epsilon}}{\hat{\sigma}} \qquad \hat{\kappa} = \frac{\beta K \bar{\epsilon}}{\hat{\sigma}} \tag{46}$$

$\hat{\sigma} = (\beta + 1)(1 + \bar{\epsilon}) - \rho_e \bar{e} + \beta K \bar{\epsilon}$ and E_t denotes in general (non-rational) expectations. Next, assume that the agents follow a perceived law of motion (PLM) that in this case of real determinacy corresponds to the fundamental solution[27]

[26] It is important to observe that for models with multiple stationary equilibria this statement lacks of technical formality. As pointed out by Evans and McGough (2003) for a model with multiple equilibria, a rational expectations equilibrium may have different representations. Therefore one should not strictly speak of learnable rational expectations equilibrium, but whether a rational expectations equilibrium representation is learnable (E-stable).

[27] The Minimal State Variable (MSV) solution according to McCallum (1983).

$$\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{z}_t^N$$

Iterating forward this law of motion and using it to eliminate all the forecasts in the model we can derive the implied actual law of motion (ALM)

$$\hat{e}_t = \hat{a}_A + \hat{b}_A\hat{e}_{t-1} + \hat{c}_A\hat{z}_t^N$$

Then we obtain the T-mapping $T(\hat{a}, \hat{b}, \hat{c}) = (\hat{a}_A, \hat{b}_A, \hat{c}_A)$, whose fixed points correspond to the rational expectations equilibria. An equilibrium is said to be E-stable if this mapping is stable at the equilibrium in question. More formally a fixed point of the T-mapping is E-stable provided that the differential equation

$$\frac{d(\hat{a}, \hat{b}, \hat{c})}{d\tau} = T(\hat{a}, \hat{b}, \hat{c}) - (\hat{a}, \hat{b}, \hat{c})$$

is locally asymptotically stable at that particular fixed point, where τ is defined as the "notional" time.[28]

For the case of sunspot equilibria we apply the same methodology but in that case the PLM is augmented by the sunspot and its particular structure. In particular we will focus on the common factor representation proposed by Evans and McGough (2003). Due to space constraint we refer the readers to the aforementioned references for a detailed explanation.[29]

It is important to observe that a fundamental part in the learnability analysis consists of making explicit what agents know when they form their forecasts. In the E-stability analysis literature it is common to assume that when agents form their expectations $E_t\hat{e}_t$, they do not know $\hat{e}_t$. In this paper this assumption may be inconsistent with the assumptions that we use to derive the equations of the model. In particular notice that for the derivation of the first order conditions of the representative agent we assume that $E_tP_t^N(j) = P_t^N(j)$ (or in a symmetric equilibrium $E_tP_t^N = P_t^N$) and $E_te_t = e_t$. Therefore assuming in the learnability analysis that the agents do not know e_t when forming expectations would have some implications for the specification of the model. Specifically it would require to replace $\hat{\pi}_t^N$ and $\hat{e}_t$ in equations (37), (38) and (40) with the expectations of $\hat{\pi}_t^N$ and $\hat{e}_t$, given current information ($\hat{\pi}_{t-1}^N$, $\hat{e}_{t-1}$ and exogenous shocks). Henceforth for the learnability analysis of the model (45) we will assume that when forming expectations agents know $\hat{e}_t$.

We proceed to present the results of the learnability analysis for the fundamental solution of the model (45) in the following Proposition.

[28] Observe that this definition suggests that to prove E-stability of a fixed point corresponds to prove that all the eigenvalues of the matrix of derivatives $DT(\hat{a}, \hat{b}, \hat{c})$ are less than 1.

[29] The proof of learnability of common factor representations of sunspot equilibria in this paper also goes over this methodology.

Proposition 2 *Suppose the government follows a PPP rule that is described by* $\epsilon_t = \rho(e_t - \bar{e})$ *and* $\rho_e = \frac{d\rho}{de_t} < 0$*. Let* K*,* H*,* $\hat{\beta}$*,* $\hat{\delta}$*, and* $\hat{\kappa}$ *be defined as in (41) and (46) respectively, and* $\hat{\alpha} = 0$*. Consider the following AR(1) representation*

$$\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{z}_t^N \tag{47}$$

where $\hat{a} = 0$*,* $\hat{b}$ *is defined as a stable root of the quadratic equation* $\hat{\beta}\hat{b}^2 - \hat{b} + \hat{\delta} = 0$*, and* $\hat{c}$ *is defined by* $\hat{c} = \frac{\hat{\kappa}}{1-\hat{\beta}(\hat{b}+\varrho^N)}$*. Under the real determinacy conditions specified in Proposition 1, there is a unique equilibrium of the model (45) characterized by the fundamental solution (47), with* $\hat{b} \in (-1, 1)$*, and this solution is learnable in the E-stability sense.*

Proof. See Appendix. ■

Proposition 2 points out that when the model displays a unique equilibrium (real determinacy) then the fundamental solution is E-stable. This is the reason of denoting as "D-ES" the regions of the four panels of Figure 1 for which the model displays not only real determinacy but also E-stability. The importance of this result stems from the fact that policy makers will face less difficulties in implementing PPP rules that lead to a unique equilibrium since they know that agents will coordinate on that equilibrium and the macroeconomic system will not diverge away from the targeted equilibrium.

It is also possible to show that under real indeterminacy the fundamental solution or MSV solution can be E-stable. However in this case policy makers will face other difficulties. In particular under multiple equilibria there might be self-fulfilling rational expectations equilibria driven by extraneous processes known as sunspot. These equilibria may be characterized by undesirable features such as larger volatility of macroeconomic variables suggesting that policy makers should avoid rules that in principle may induce multiple equilibria.

Although the previous argument may sound appealing, it may suffer from some drawbacks. For instance, it is not clear whether agents are able to coordinate their actions on a particular sunspot equilibria. To clarify this issue the next proposition illustrates that some particular representations of stationary sunspot equilibria can be E-stable. To simplify the analysis and to be able to derive analytical results we assume that $\varrho^N = 0$.

Proposition 3 *Suppose the government follows a PPP rule that is described by* $\epsilon_t = \rho(e_t - \bar{e})$ *and* $\rho_e = \frac{d\rho}{de_t} < 0$*. Let* K*,* H*,* $\hat{\beta}$*,* $\hat{\delta}$*, and* $\hat{\kappa}$ *be defined as in (41) and (46) respectively, and* $\hat{\alpha} = 0$*, and assume that* $\varrho^N = 0$*. Consider the following common factor representation*

$$\hat{e}_t = \hat{a} + \hat{b}_i\hat{e}_{t-1} + \hat{d}\hat{\xi}_t + \left(\hat{\beta}\hat{b}_j\right)^{-1}\hat{\zeta}_t^N \tag{48}$$

$$\hat{\xi}_t = \hat{b}_j\hat{\xi}_{t-1} - \left(\hat{\beta}\hat{b}_j\right)^{-1}\hat{\zeta}_t^N + \hat{\eta}_{e_t} \tag{49}$$

where $\hat{a} = 0$, $\hat{b}_i, \hat{b}_j \in (-1, 1)$ *are unique and correspond to the real roots of the quadratic equation* $\hat{\beta}\hat{b}^2 - \hat{b} + \hat{\delta} = 0$, $\hat{d}$ *is arbitrary,* $\hat{\zeta}_t^N = \hat{\kappa}\zeta_t^N$ *and* $\hat{\eta}_{e_{t+1}} = \hat{e}_{t+1} - E_t(\hat{e}_{t+1})$ *is a martingale difference sequence.*[30]

> **a)** *Under the real indeterminacy conditions specified in Proposition 1, there are stationary sunspot equilibria of (45) characterized by the common factor representation (48) and (49), where* $\hat{b}_i = \hat{b}_1 \in (0, 1)$ and $\hat{b}_j = \hat{b}_2 \in (-1, 0)$, *and this representation is learnable in the E-stability sense.*

> **b)** *Under the real indeterminacy conditions specified in Proposition 1, there are stationary sunspot equilibria of (45) characterized by the common factor representation (48) and (49), where* $\hat{b}_i = \hat{b}_2 \in (-1, 0)$ and $\hat{b}_j = \hat{b}_1 \in (0, 1)$, *and this representation is NOT learnable in the E-stability sense.*[31]

Proof. See Appendix. ■

Proposition 3 demonstrates that some common factor representations of sunspot equilibria induced by PPP rules are learnable in the sense of E-stability. We would like to emphasize the important role that the common factor representations proposed by Evans and McGough (2003) play in the learnability analysis. To see this, observe that the typical stationary sunspot equilibrium representation, $\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{e}_{t-2} + \hat{d}\hat{s}_t + \hat{k}\hat{\zeta}_t^N$, where $\hat{s}_t$ denotes the sunspot, is never E-stable. The reason is that such perceived law of motion leads to an actual law of motion $\hat{e}_t = \hat{a}_A + \hat{b}_A\hat{e}_{t-1} + \hat{k}_A\hat{\zeta}_t^N$ that implies that $\hat{d} = 0$. But this suggests that the typical sunspot representation is not learnable. This argument in tandem with Proposition 3 reveal that common factor representations make stationary sunspot equilibria more likely to arise under private learning than previously recognized.

Our results from the real determinacy and learnability of equilibrium analyses pose the question of whether changing the timing of the PPP rule avoids sunspot equilibria and still induces a unique equilibrium that is E-stable. Similarly to the findings in the interest rate rule literature, we find that a PPP rule that is backward-looking in the sense of being defined in terms of the past real

[30] Note that strictly speaking since we are assuming that the economy starts at $t = 0$ and we have an initial condition for $\hat{e}_t$, then in the common factor representation of a rational expectations equilibrium $\hat{e}_t$, equation (48) should be written as

$$\hat{e}_t = \hat{a} + \hat{b}_i\hat{e}_{t-1} + \hat{d}\hat{\xi}_t + \left(\hat{\beta}\hat{b}_j\right)^{-1}\hat{\zeta}_t^N + N\hat{b}_j^{t+1}$$

for $t = 0, 1, ...$and N arbitrary. However since $\hat{b}_j \in (-1, 1)$, then as $t \to \infty$ the solution $\hat{e}_t$ converges to a process that satisfies (48).

[31] Note that NOT introducing the constant $\hat{a}$ in the perceived law of motion, such that $\hat{e}_t = \hat{b}_i\hat{e}_{t-1} + \hat{d}\hat{\xi}_t + \left(\hat{\beta}\hat{b}_j\right)^{-1}\hat{\zeta}_t^N$, is not innocuous for statement b) of the proposition. In this case the common factor representation with $\hat{b}_i = \hat{b}_2 \in (-1, 0)$ and $\hat{b}_j = \hat{b}_1 \in (0, 1)$ becomes E-stable.

exchange rate satisfies these two requirements.[32] A backward-looking PPP rule can be described as $\epsilon_t = \rho(e_{t-1} - \bar{e})$ and using this specification and equations (37), (38) and (39), we can reduce the model to

$$\hat{e}_t = \hat{\alpha}_p + \hat{\beta}_p E_t \hat{e}_{t+1} + \hat{\delta}_p \hat{e}_{t-1} + \hat{\kappa}_p \hat{z}_t^N \qquad \textit{and} \qquad \hat{z}_t^N = \varrho^N \hat{z}_{t-1}^N + \zeta_t^N \tag{50}$$

where $\hat{\alpha}_p = 0$,

$$\hat{\beta}_p = \left[1 + \frac{1}{\beta} + \frac{K\bar{\epsilon}}{1+\bar{\epsilon}} - \frac{(H-1)\rho_e \bar{e}}{1+\bar{\epsilon}}\right]^{-1} \qquad \hat{\delta}_p = \frac{\hat{\beta}_p}{\beta}\left(1 + \frac{\rho_e \bar{e}}{1+\bar{\epsilon}}\right) \qquad \hat{\kappa}_p = \hat{\beta}_p \left(\frac{K\bar{\epsilon}}{1+\bar{\epsilon}}\right) \tag{51}$$

and E_t denotes in general (non-rational) expectations.

The following proposition summarizes the aforementioned result.

Proposition 4 *Suppose the government follows a backward-looking PPP rule that is described by $\epsilon_t = \rho(e_{t-1} - \bar{e})$ and $\rho_e = \frac{d\rho(e_{t-1})}{de_{t-1}} < 0$, and consider the model described in (50). Let K, H, $\hat{\beta}_p$, $\hat{\delta}_p$, and $\hat{\kappa}_p$ be defined as in (41), and (51) respectively and $\hat{\alpha}_p = 0$. Consider the AR(1) representation*

$$\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{z}_t^N \tag{52}$$

where $\hat{a} = 0$, $\hat{b}$ is uniquely defined as a stable root of the quadratic equation $\hat{\beta}_p \hat{b}^2 - \hat{b} + \hat{\delta}_p = 0$, and $\hat{c}$ is also uniquely defined by $\hat{c} = \frac{\hat{\kappa}_p}{1 - \hat{\beta}_p(\hat{b} + \varrho^N)}$.

> **a)** *If either $H > 1 + \frac{1}{\beta}$ and $\rho_e < 0$, or $H < 1 + \frac{1}{\beta}$ and $\breve{\rho}_e < \rho_e < 0$, with $\breve{\rho}_e = \frac{2\left(1+\frac{1}{\beta}\right)(1+\bar{\epsilon}) + K\bar{\epsilon}}{\bar{e}\left(H - 1 - \frac{1}{\beta}\right)}$, then the model (50) displays a unique equilibrium (real determinacy) that can be represented by the fundamental solution (52) with $\hat{b} \in (-1, 1)$. Moreover this solution is E-stable.*
>
> **b)** *If $H < 1 + \frac{1}{\beta}$ and $\rho_e < \breve{\rho}_e < 0$ then there exists no equilibrium.*

Proof. See Appendix. ■

5 PPP Rules Defined in Terms of The CPI-Inflation

In this section we analyze a different type of PPP rule. We study rules whereby the government in response to an increase in the CPI-inflation, increases the nominal depreciation rate. The motivation to consider this type of rule is twofold. First, from an empirical point of view, Calvo et al. (1995) mention that starting in 1968, Brazil's government implemented a rule by which the

[32] Forward-looking PPP rules defined in terms of the expected future real exchange rate still open the possibility of sunspot equilibria as shown in Zanna (2003b).

exchange rate was adjusted as a function of the difference between domestic and U.S. inflation. In addition, between 1985 and 1992, Chile used an exchange rate band whose trend was determined by the difference between the domestic inflation rate and a measure of the average inflation in the rest of the world. Second, from a theoretical point of view, Dornbusch (1980, 1982) conceives PPP rules as a means to introduce the necessary real flexibility to cope with intrinsic (fundamental) uncertainty in a world that faces nominal rigidities. He defines a PPP rule as a function whereby the nominal exchange rate is positively linked to the domestic price index.

We try to capture the aforementioned stylized facts and some of the flavor of Dornbusch's work by defining a rule whereby the nominal depreciation rate is positively linked to the difference between the domestic CPI inflation (π_t) and the foreign CPI-inflation.[33] However note that since in our analysis the foreign variables are considered exogenous and constant, then the specification of the PPP rule reduces to

$$\epsilon_t = \rho(\pi_t) \qquad with \qquad \rho_\pi = \frac{d\rho(\pi_t)}{d\pi_t} > 0 \tag{53}$$

where $\rho(.)$ is a continuous function.

As before we proceed in the following way. First we will prove that such rule may induce aggregate instability in the economy by generating multiple equilibria and opening the possibility of sunspot equilibria. Specifically we will study and disentangle the conditions under which this rule leads to real indeterminacy or to real determinacy. Second, we will study the "learnability" properties not only of the fundamental solution but also of the common factor representation of stationary sunspot equilibria.

The following proposition summarizes the conditions under which the aforementioned PPP rule induces either real determinacy or real indeterminacy in the model.

Proposition 5 *Suppose the government follows a PPP rule given by* $\epsilon_t = \rho(\pi_t)$ *with* $\rho_\pi = \frac{d\rho}{d\pi_t} > 0$. *Let* K *and* H *be defined as in (41) and define* $\tilde{\rho}_\pi = \frac{2\left(1+\frac{1}{\beta}\right)+\frac{K\bar{\epsilon}}{1+\bar{\epsilon}}}{2\alpha\left(1+\frac{1}{\beta}\right)+2(1-\alpha)H+\frac{K\bar{\epsilon}}{1+\bar{\epsilon}}} > 0$.

a) *If* $\tilde{\rho}_\pi < \rho_\pi < 1$ *then there is real indeterminacy.*

b) *If either* $\rho_\pi > max\{1, \tilde{\rho}_\pi\}$ *or* $\rho_\pi < \min\{1, \tilde{\rho}_\pi\}$ *then there is real determinacy.*

c) *If* $1 < \rho_\pi < \tilde{\rho}_\pi$ *then there exists no equilibrium.*

Proof. See Appendix. ■

Proposition 5 suggests that multiple equilibria are also possible for PPP rules that depend on the current CPI-inflation. In particular it points out that a *necessary* condition for these rules

[33] This is also the specification in Montiel and Ostry (1991).

to cause real indeterminacy is that the response coefficient to the CPI- inflation be less than one. That means that in response to a one percent increase in the CPI-inflation rate, the government raises the nominal devaluation rate in less than one percent. Interestingly such response seems to be feasible in the practice of economic policy. However in order to generate real indeterminacy the nominal depreciation response coefficient, ρ_π, must be above a threshold, $\tilde{\rho}_\pi$, which in turn depends on the structural parameters that affect K and H. Therefore, as before, we proceed studying analytically and numerically how varying some structural parameters of the model affects the previously mentioned threshold. In particular we focus our analysis on the degree of openness of the economy, α, and the degrees of monopolistic competition, μ, and price stickiness, ϕ, in the non-traded sector. The results are presented in Corollary 2.

Corollary 2 *Suppose the government follows a PPP rule given by* $\epsilon_t = \rho(\pi_t)$ *with* $\rho_\pi > 0$*. Let* $\tilde{\rho}_\pi$ *be defined as in Proposition 5 and satisfy* $\tilde{\rho}_\pi < 1$ *then* **a)** $\frac{\partial \tilde{\rho}_\pi}{\partial \alpha} > 0$, **b)** $\frac{\partial \tilde{\rho}_\pi}{\partial \mu} < 0$ *and* **c)** $\frac{\partial \tilde{\rho}_\pi}{\partial \phi} > 0$.

Proof. See Appendix. ■

Using Proposition 5 and Corollary 2 we can breakdown the effects of varying some of the structural parameters of the model and the PPP rule response coefficient to CPI-inflation ρ_π, on the determinacy of equilibrium analysis. In fact, when $\rho_\pi < 1$ we can conclude the following. Given the PPP rule response coefficient to CPI-inflation, the less open the economy is (the lower α is), the more likely that the PPP rule will induce aggregate instability in the economy by generating multiple equilibria. In addition, given the rule response coefficient to CPI-inflation and keeping the rest constant, the higher the degree of monopolistic competition in the non-traded sector (the higher μ), the more likely that the rule will lead to real indeterminacy. Finally under ceteris paribus and given the rule response coefficient to CPI-inflation, the lower the degree of price stickiness in the non-traded sector (the lower ϕ), the more feasible that the rule will induce multiple equilibria.

Under the parametrization of Table 1 we construct Figure 2 that corroborates these results quantitatively. To some extent it also validates numerically how likely is that the aforementioned PPP rule may destabilize the economy by generating multiple equilibria. Moreover although it is not possible to derive an analytical result to see how varying the implied nominal depreciation target ($\bar{\epsilon}$) and the response coefficient to the CPI-inflation (ρ_π) affect the determinacy of equilibrium, it is possible to evaluate this quantitatively. In short the bottom right panel of Figure 2 illustrates that given the response coefficient of the rule the lower the nominal depreciation target is, the more likely is that the PPP rule will induce real indeterminacy.

It is also important to observe that similar qualitative results to the ones described in Proposition 5 and Corollary 2 can be obtained if the PPP rule is defined in terms of the non-traded goods inflation rate. That is $\epsilon_t = \rho(\pi_t^N)$ with $\frac{d\rho(\pi_t^N)}{d\pi_t^N} > 0$. Furthermore it is also possible to prove that a backward-looking rule defined in terms of the past CPI-inflation rate will avoid multiple

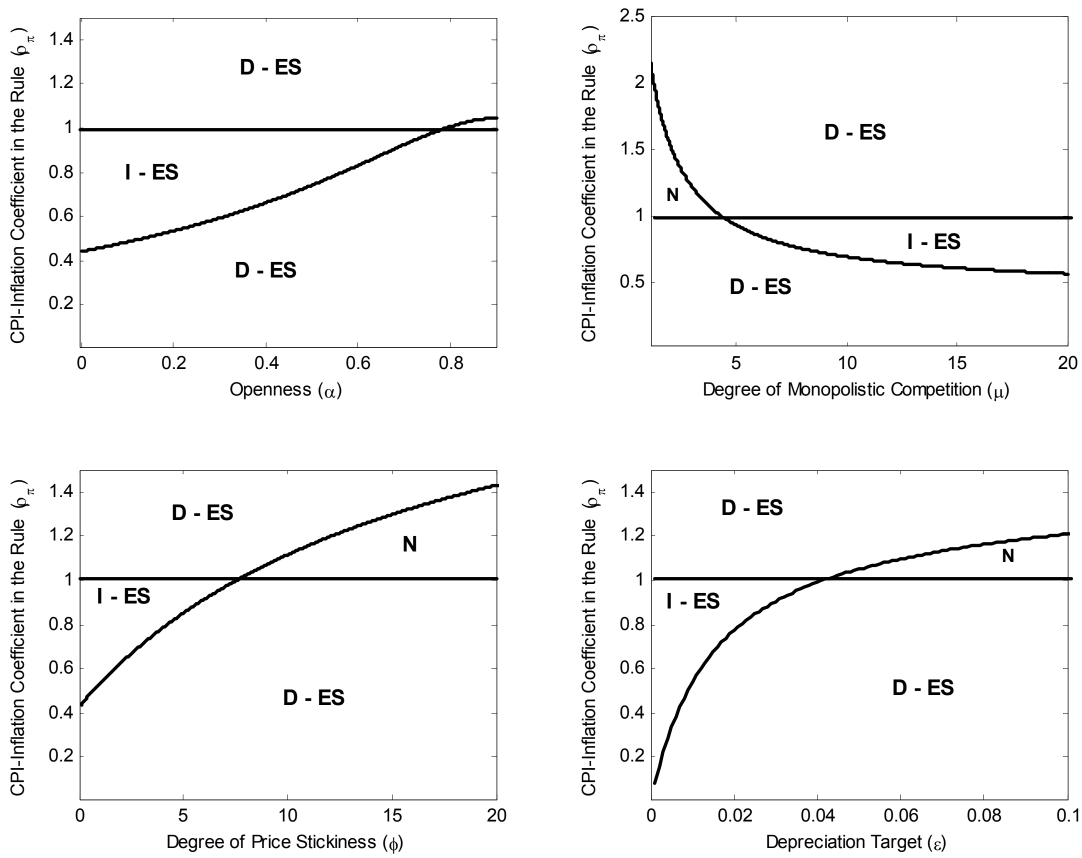

Figure 2: This figure shows how the local determinacy of equilibrium varies with respect to the PPP rule response coefficient to the CPI-inflation (ρ_π) , the share of traded goods (α), the degree of monopolistic competition in the non-traded sector (μ), the degree of price stickiness in the non-traded sector (ϕ) and the implied nominal depreciation rate target ($\bar{\epsilon}$). "I" stands for real indeterminacy (multiple equilibria), "D" stands for real determinacy (a unique equilibrium) and "N" for non-existence of equilibrium. "ES" corresponds to E-Stability.

equilibria.[34]

We proceed by pursuing the learnability analysis. As argued before this analysis is useful to evaluate the attainability of the possible unique equilibrium and multiple equilibria induced by the PPP rule. We use equations (37), (38), (39) and the log-linearized versions of (8) and $\epsilon_t = \rho(\pi_t)$ to reduce the model to

$$\hat{e}_t = \hat{\alpha}_q + \hat{\beta}_q E_t \hat{e}_{t+1} + \hat{\delta}_q \hat{e}_{t-1} + \hat{\kappa}_q \hat{z}_t^N \qquad and \qquad \hat{z}_t^N = \varrho^N \hat{z}_{t-1}^N + \zeta_t^N \tag{54}$$

where $\hat{\alpha}_q = 0$,

$$\hat{\beta}_q = \frac{1 - [\alpha + (1-\alpha)H]\,\rho_\pi}{\hat{\sigma}_q} \qquad \hat{\delta}_q = \frac{1 - \alpha\rho_\pi}{\beta\hat{\sigma}_q} \qquad \hat{\kappa}_q = \frac{K\bar{\epsilon}\,(1-\rho_\pi)}{(1+\bar{\epsilon})\,\hat{\sigma}_q} \tag{55}$$

$\hat{\sigma}_q = 1 - [\alpha + (1-\alpha)H]\,\rho_\pi + \frac{1-\alpha\rho_\pi}{\beta} + \frac{K\bar{\epsilon}(1-\rho_\pi)}{1+\bar{\epsilon}}$ and E_t denotes in general (non-rational) expectations.

As before in order to pursue the learnability analysis we use the methodology proposed by Evans and Honkapohja (1999, 2001). We derive some E-stability conditions and check whether a particular representation of the equilibrium under analysis satisfies or violates them.

The following proposition summarizes the results.

Proposition 6 *Suppose the government follows a PPP rule that is described by $\epsilon_t = \rho(\pi_t)$ and $\rho_\pi = \frac{d\rho(\pi_t)}{d\pi_t} > 0$. Let K, H, $\hat{\beta}_q$, $\hat{\delta}_q$, and $\hat{\kappa}_q$ be defined as in (41) and (55) respectively and $\hat{\alpha}_q = 0$.*

a) *Under the real determinacy conditions specified in Proposition 5 there exists a unique equilibrium characterized by the fundamental solution*

$$\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{z}_t^N \tag{56}$$

where $\hat{a} = 0$, $\hat{b} \in (-1, 1)$ is uniquely defined by the quadratic equation $\hat{\beta}_q \hat{b}^2 - \hat{b} + \hat{\delta}_q = 0$, and $\hat{c}$ is also uniquely defined by $\hat{c} = \frac{\hat{\kappa}_q}{1 - \hat{\beta}_q(\hat{b} + \varrho^N)}$. This solution is learnable in the E-stability sense.[35]

b) *Assume that $\varrho^N = 0$. Under the real indeterminacy conditions specified in Proposition 5 there are stationary sunspot equilibria described by the common factor representation*

$$\hat{e}_t = \hat{a} + \hat{b}_i \hat{e}_{t-1} + \hat{d}\hat{\xi}_t + \left(\hat{\beta}_q \hat{b}_j\right)^{-1} \hat{\zeta}_t^N \tag{57}$$

$$\hat{\xi}_t = \hat{b}_j \hat{\xi}_{t-1} - \left(\hat{\beta}_q \hat{b}_j\right)^{-1} \hat{\zeta}_t^N + \hat{\eta}_{e_t} \tag{58}$$

[34] See Zanna (2003b).

[35] Under real indeterminacy it is also possible to prove that there is an equilibrium characterized by fundamental solution (56) with $\hat{b} \in (0, 1)$ which is E-stable.

where $\hat{a} = 0$, $\hat{b}_i, \hat{b}_j \in (-1, 1)$ are unique and correspond to the roots of the quadratic equation $\hat{\beta}_q \hat{b}^2 - \hat{b} + \hat{\delta}_q = 0$, $\hat{d}$ is arbitrary, $\hat{\zeta}_t^N = \hat{\kappa}_q \zeta_t^N$ and $\hat{\eta}_{e_{t+1}} = \hat{e}_{t+1} - E_t(\hat{e}_{t+1})$ is a martingale difference sequence. In particular, the common factor representation (57) and (58) with $\hat{b}_i = \hat{b}_1 \in (0, 1)$ and $\hat{b}_j = \hat{b}_2 \in (-1, 0)$ is learnable in the E-stability sense.[36]

Proof. See Appendix. ■

Proposition 6 states that when the PPP rule under study induces a unique equilibrium then this equilibrium represented by the fundamental solution, also known as the MSV solution, is learnable in the E-stability sense. This result is important since it means that given that the rule induces a unique equilibrium then agents will be able to coordinate on that particular equilibrium and therefore the economy will converge towards it over time. In addition as was demonstrated in Proposition 2, it is also possible to prove for PPP rules defined in terms of the CPI-inflation, that even under real indeterminacy the fundamental solution is still E-stable. However under real indeterminacy there are other equilibria such as stationary sunspot equilibria whose feasibility is worth evaluating in terms of learnabiliy. Accordingly, the second part of Proposition 6 shows that a common factor representation of stationary sunspot equilibria is learnable in the sense of E-stability. This result is interesting for two reasons. First, as mentioned before, it suggests that sunspot equilibria induced by PPP rules are more likely to occur. Second, it warns policy makers about some of the negative consequences of implementing PPP rules that respond to inflation. Dornbusch (1980, 1982) conceived PPP rules as a means to introduce the necessary real flexibility to cope with intrinsic (fundamental) uncertainty in an economy with nominal rigidities. In contrast our result points out that such PPP rules may open the possibility of learnable representations of sunspot equilibria aggravating the effects of extrinsic (non-fundamental) uncertainty in an economy with nominal rigidities.

6 Conclusions

In this paper we establish and disentangle the conditions under which PPP rules lead to real (in)determinacy in a small open economy that faces nominal rigidities. We find that besides the specification of the rule, structural parameters such as the share of traded goods (that measures the degree of openness of the economy) and the degrees of imperfect competition and price stickiness in the non-traded sector play a crucial role in the determinacy of equilibrium.

More importantly to evaluate the relevance of the determinacy results we also pursue a learnability (E-stability) analysis. We show that for rules that guarantee a unique equilibrium the

[36] Excluding a constant $\hat{a}$ in the perceived law of motion such that $\hat{e}_t = \hat{b}_i \hat{e}_{t-1} + \hat{d}\hat{\xi}_t + \left(\hat{\beta}\hat{b}_j\right)^{-1} \hat{\zeta}_t^N$ is not innocuous for this part of the proposition. In this case the common factor representation with $\hat{b}_i = \hat{b}_2 \in (-1, 0)$ and $\hat{b}_j = \hat{b}_1 \in (0, 1)$ is also E-stable.

fundamental solution that describes this equilibrium is learnable in the E-stability sense. Similarly we show that for PPP rules that open the possibility of sunspot equilibria, some common factor representations of these equilibria are also E-stable. That is, agents can coordinate their actions and learn some representations of stationary sunspot equilibria. In this sense these equilibria are more likely to occur under PPP rules than previously recognized and therefore these rules are more prone to cause aggregate instability in the economy.

Dornbusch (1980, 1982) conceived PPP rules as a means of introducing the real flexibility necessary to cope with intrinsic (fundamental) uncertainty in an economy with nominal rigidities. Our results indicate that PPP rules must be chosen with care in order to avoid the possibility of "learnable" sunspot equilibria and the associated aggravation of the effects of extrinsic (non-fundamental) uncertainty. In other words, PPP rules should satisfy two stability requirements: uniqueness and learnability. On one hand, the rule should avoid sunspot equilibria that are usually associated with undesirable properties such as a large degree of volatility. On the other hand, the rule should guarantee that agents can indeed coordinate their actions on the equilibrium the policy makers are targeting and that the economy will not in fact diverge away from this target.

There are some possible extensions of the analysis presented in this paper. First, one may consider extending the model to have two traded goods: a domestic one and a foreign one. This will enrich the analysis making the model more similar to the ones in Dornbusch (1980, 1982). Under this set-up one can explore how our results may vary when the government responds to different measures of inflation in the PPP rule. Second one may study how our determinacy and learnability of equilibrium results may be affected by following the approach by Preston (2003). That is, instead of imposing the assumption of non-rational expectations on the derived log-linearized model, we may impose this assumption as a primitive one of the model. This assumption implies that agents do not have a complete economic model with which to derive true probability laws since they do not know other agents' tastes and beliefs. In this case agents solve multi-period decision problems whereby their actions depend on forecasts of macroeconomic conditions many periods into the future. We leave these extensions for further research.

7 Appendix

Lemma 1 *In a* 2×2 *linearized system of difference equations whose matrix is denoted by* J *and whose characteristic equation corresponds to* $\mathcal{P}\ (w){=}w^2 + Trace(J)w + Det(J) = 0$ *if either* a) $Det(J) \leq 0$, *or* b) $P(1) < 0$ *or* c) $P(-1) < 0$ *then the system displays real eigenvalues.*

Proof. First we recall from Azariadis (1993) that a sufficient condition for such a linearized system to have real eigenvalues is that $[Trace(J)]^2 - 4Det(J) \geq 0$. Then to prove a) is trivial.

To prove b) we start by noting that $\mathcal{P}(1) < 0$ means that $\mathcal{P}(1) = 1 - Trace(J) + Det(J) < 0$.

But this implies that $4Trace(J) - 4 > 4Det(J)$ that with the aforementioned sufficient condition in turn leads to $[Trace(J)]^2 - 4Det(J) > [Trace(J)]^2 - 4Trace(J) + 4 = [Trace(J) - 2]^2 \geq 0$. Hence the eigenvalues are real.

To prove c) we point out that $\mathcal{P}(-1) < 0$ means that $\mathcal{P}(1) = 1 + Trace(J) + Det(J) < 0$. But this implies that $-4Trace(J) - 4 > 4Det(J)$ that in turn leads to $[Trace(J)]^2 - 4Det(J) > [Trace(J)]^2 + 4Trace(J) + 4 = [Trace(J) + 2]^2 \geq 0$. Hence the eigenvalues are real. ■

7.1 Proof of Proposition 1

Proof. To prove all the parts of the proposition we use (37), (38), (39) and (40) to derive the following system

$$\begin{pmatrix} \hat{\pi}_t^N \\ \hat{e}_{t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}}_{J} \begin{pmatrix} \hat{\pi}_{t+1}^N \\ \hat{e}_t \end{pmatrix} + U\hat{\eta}_{\pi_{t+1}} + V\hat{z}_{t+1}^N$$

where

$$\begin{aligned} J_{11} &= \beta\left[1 + \frac{H\rho_e\bar{e}}{\Delta}\right] \qquad J_{12} = \beta\left[K - \frac{H\rho_e\bar{e}(1+\bar{\epsilon})}{\bar{\epsilon}\Delta}\right] \\ J_{21} &= \beta\frac{\bar{\epsilon}}{1+\bar{\epsilon}}\left[1 + \frac{H\rho_e\bar{e}}{\Delta}\right] \qquad J_{22} = \beta\left[\frac{K\bar{\epsilon}}{1+\bar{\epsilon}} - \frac{H\rho_e\bar{e}}{\Delta}\right] + \frac{\Delta}{1+\bar{\epsilon}} \end{aligned}$$

$\Delta = 1 + \bar{\epsilon} - \rho_e\bar{e}$, $\hat{\eta}_{\pi_{t+1}}$ is the forecast error for the non-traded goods inflation defined as $\hat{\eta}_{\pi_{t+1}} = \hat{\pi}_{t+1}^N - E_t(\hat{\pi}_{t+1}^N)$ and where the forms of U and V are omitted since they are not needed in what follows. For the the proof of the proposition it is convenient to pursue the determinacy analysis for the system written in the form $\hat{y}_t = J\hat{y}_{t+1} + U\hat{\eta}_{\pi_{t+1}} + V\hat{z}_{t+1}^N$ where $\hat{y}_t = (\hat{\pi}_t^N, \hat{e}_{t-1})'$ instead of $\hat{y}_{t+1} = J^{-1}\hat{y}_t - J^{-1}U\hat{\eta}_{\pi_{t+1}} - J^{-1}V\hat{z}_{t+1}^N$. It will reduce tremendously the number of cases that have to be analyzed.[37] For this linearized system we have that the trace of J, the determinant of J, and the characteristic polynomial associated with J correspond to

$$\begin{aligned} Trace(J) &= \beta\left[1 + \frac{K\bar{\epsilon}}{1+\bar{\epsilon}}\right] + \frac{1+\bar{\epsilon}-\rho_e\bar{e}}{1+\bar{\epsilon}} > 0 \\ Det\,(J) &= \beta\left[\frac{1+\bar{\epsilon}-(1-H)\rho_e\bar{e}}{1+\bar{\epsilon}}\right] \\ \mathcal{P}(\upsilon) &= \upsilon^2 - Trace(J)\upsilon + Det(J) = 0 \end{aligned}$$

[37] One can analyze either of the forms because the eigenvalues of J^{-1} correspond to υ_i^{-1}, where υ_i are the eigenvalues of J.

respectively. Using these expressions we obtain

$$\mathcal{P}(1) = \frac{\beta\rho_e\bar{e}\left(\frac{1}{\beta}+H-1\right)-\beta K\bar{\epsilon}}{1+\bar{\epsilon}} < 0$$

$$\mathcal{P}(-1) = \frac{\beta\bar{e}\left(1+\frac{1}{\beta}-H\right)}{1+\bar{\epsilon}}(\tilde{\rho}_e-\rho_e)$$

where $\tilde{\rho}_e = \frac{\left[2(1+\bar{\epsilon})\left(1+\frac{1}{\beta}\right)+K\bar{\epsilon}\right]}{\left[\bar{e}\left(1+\frac{1}{\beta}-H\right)\right]}$.

All these expressions together with $\rho_e < 0$, $K > 0$ and $H > 0$, and the assumptions about the structural parameters allows us to observe that $Trace(J) > 0$ and $\mathcal{P}(1) < 0$. By Lemma 1, this last inequality implies that the eigenvalues are real.

To prove part a) we proceed as follows. Observe that since $1+\frac{1}{\beta} < H$ then $\tilde{\rho}_e < 0$. Hence from $1+\frac{1}{\beta} < H$ and $\rho_e < \tilde{\rho}_e < 0$ we can conclude that $\mathcal{P}(-1) < 0$. This in conjunction with $\mathcal{P}(1) < 0$ and $Trace(J) > 0$ imply that the system has two explosive eigenvalues $|\upsilon_1| > 1$ and $|\upsilon_2| > 1$ (which means that $\left|\frac{1}{\upsilon_1}\right| < 1$ and $\left|\frac{1}{\upsilon_2}\right| < 1$). Thus the steady state is a source (See Azariadis, 1993). Since $\hat{\pi}_t^N$ is the only non-predetermined variable of the system then by Blanchard and Kahn (1980) we conclude that the model displays real indeterminacy.

To prove b) we note that from $1+\frac{1}{\beta} < H$ and $\tilde{\rho}_e < \rho_e < 0$ we can conclude that $\mathcal{P}(-1) > 0$. This in tandem with $\mathcal{P}(1) < 0$ and $Trace(J) > 0$ imply that the system has one explosive eigenvalue $|\upsilon_1| > 1$ and one non-explosive eigenvalue $|\upsilon_2| < 1$. Hence the steady state is a saddle path (See Azariadis, 1993). Since $\hat{\pi}_t^N$ is the only non-predetermined variable of the system then by Blanchard and Kahn (1980) we conclude that the model displays real determinacy.

Finally to prove c) we use the fact that $H < 1+\frac{1}{\beta}$ implies that $\tilde{\rho}_e > 0$ and therefore $\mathcal{P}(-1) > 0$. Then the analysis pursued to prove part b) follows. ■

7.2 Proof of Corollary 1

Proof. First use the steady state description and the definition of K and H given by (41) to calculate

$$\frac{\partial\bar{e}}{\partial\alpha} = \frac{(1-\theta_N)\bar{e}}{(1-\alpha)} > 0 \qquad \frac{\partial\bar{c}^N}{\partial\alpha} = -\frac{\theta_N\bar{c}^N}{(1-\theta_N)\bar{e}}\frac{\partial\bar{e}}{\partial\alpha} < 0 \qquad \frac{\partial K}{\partial\alpha} = \frac{K}{\theta_N\bar{c}^N}\frac{\partial\bar{c}^N}{\partial\alpha} < 0$$

$$\frac{\partial\bar{c}^N}{\partial\mu} = \left(\frac{\lambda\theta_N}{\psi}\right)^{\frac{\theta_N}{1-\theta_N}}\frac{(1-\theta_N)}{\mu^2\bar{e}} > 0 \qquad \frac{\partial K}{\partial\mu} = K\left(\frac{1}{\mu}+\frac{1}{\theta_N\bar{c}^N}\frac{\partial\bar{c}^N}{\partial\mu}\right) > 0$$

$$\frac{\partial K}{\partial\phi} = -\frac{K}{\phi} < 0$$

where we use the assumptions about the structural parameters to determine the sign of the derivatives. Next we use these results and the definition of $\tilde{\rho}_e$ given in Proposition 1, together with the assumptions about the structural parameters to derive

$$\frac{\partial(\tilde{\rho}_e\bar{e})}{\partial\alpha} = \Xi\frac{\partial K}{\partial\alpha} < 0 \qquad \frac{\partial(\tilde{\rho}_e\bar{e})}{\partial\mu} = \Xi\frac{\partial K}{\partial\mu} > 0 \qquad \frac{\partial(\tilde{\rho}_e\bar{e})}{\partial\phi} = \Xi\frac{\partial K}{\partial\phi} < 0$$

where $\Xi = \frac{\left(1+\frac{1}{\beta}\right)\left[\bar{\epsilon}+2\frac{H}{K}(1+\bar{\epsilon})\right]}{\left(1+\frac{1}{\beta}-H\right)^2} > 0.$ ■

7.3 Proof of Proposition 2

Proof. To prove a) we proceed in the following way. First, we rewrite the first equation in (45) as

$$\left(1 - \hat{\beta}^{-1}L + \hat{\beta}^{-1}\hat{\delta}L^2\right)\hat{e}_t = \hat{\eta}_{e_t} - \frac{1}{\hat{\beta}}\hat{\zeta}_{t-1}^N \tag{59}$$

where L is the lag operator, $\hat{\zeta}_t^N = \hat{\kappa}\zeta_t^N$ and $\hat{\eta}_{e_{t+1}} = \hat{e}_{t+1} - E_t(\hat{e}_{t+1})$ is a martingale difference sequence. The associated characteristic equation of (59) is

$$\hat{b}^2 - \frac{1}{\hat{\beta}}\hat{b} + \frac{\hat{\delta}}{\hat{\beta}} = 0 \tag{60}$$

whose roots are denoted by $\hat{b}_i$ with $i = 1, 2$. Second we establish a relationship between the roots $\hat{b}_i$ and the roots υ_i in the proof of Proposition 1. In particular it is straightforward to show that $\hat{b}_i = \frac{1}{\upsilon_i}$ and that

$$\frac{1}{\hat{\beta}} = \frac{Trace(J)}{Det(J)} = Trace(J^{-1}) = \hat{b}_1 + \hat{b}_2 \qquad and \qquad \frac{\hat{\delta}}{\hat{\beta}} = \frac{1}{Det(J)} = Det(J^{-1}) = \hat{b}_1\hat{b}_2 \tag{61}$$

where J was defined in the proof of Proposition 1.

Third, using these relationships and the proof in Proposition 1, it is trivial to show that under real determinacy, the unique equilibrium of the model (45) characterized by the fundamental solution (47) with $\hat{b} \in (-1, 1)$ is in fact a solution of the model (45).

Fourth, we derive the E-stability conditions. Consider the model (45) and assume that the agents follow a perceived law of motion (PLM) that in this case of real determinacy corresponds to the fundamental solution

$$\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{z}_t^N$$

Iterating forward this law of motion and taking expectations we obtain $E_t\hat{e}_{t+1} = \hat{a} + \hat{b}\hat{e}_t + \hat{c}\varrho^N\hat{z}_t^N$; using this to eliminate all the forecasts in the model (45) and assuming that agents know $\hat{e}_t$ when they make their forecasts, we can derive the implied actual law of motion (ALM)

$$\hat{e}_t = \left(\frac{\hat{\beta}\hat{a} + \hat{\alpha}}{1 - \hat{\beta}\hat{b}}\right) + \left(\frac{\hat{\delta}}{1 - \hat{\beta}\hat{b}}\right)\hat{e}_{t-1} + \left(\frac{\hat{\beta}\hat{c}\varrho^N + \hat{\kappa}}{1 - \hat{\beta}\hat{b}}\right)\hat{z}_t^N$$

Then we obtain the T-mapping $T(\hat{a},\hat{b},\hat{c}) = [T_1, T_2, T_3]' = \left[\frac{\hat{\beta}\hat{a}+\hat{\alpha}}{1-\hat{\beta}\hat{b}}, \frac{\hat{\delta}}{1-\hat{\beta}\hat{b}}, \frac{\hat{\beta}\hat{c}\varrho^N+\hat{\kappa}}{1-\hat{\beta}\hat{b}}\right]'$, whose fixed points correspond to the rational expectations equilibrium with $\hat{a}$ satisfying $\hat{a} = \frac{\hat{\beta}\hat{a}+\hat{\alpha}}{1-\hat{\beta}\hat{b}}$, $\hat{b}$ satisfying (60) and $\hat{c}$ satisfying $\hat{c} = \frac{\hat{\beta}\hat{c}\varrho^N+\hat{\kappa}}{1-\hat{\beta}\hat{b}}$. Note that since $\hat{\alpha} = 0$, then $\hat{a} = \frac{\hat{\alpha}}{1-\hat{\beta}\hat{b}-\hat{\beta}} = 0$. Moreover since the matrix of derivatives $DT(\hat{a},\hat{b},\hat{c})$ is block triangular then it is simple to show that its eigenvalues are $\frac{\partial T_1}{\partial \hat{a}} = \frac{\hat{\beta}}{1-\hat{\beta}\hat{b}}$, $\frac{\partial T_2}{\partial \hat{b}} = \frac{\hat{\beta}\hat{\delta}}{\left(1-\hat{\beta}\hat{b}\right)^2}$ and $\frac{\partial T_3}{\partial \hat{c}} = \frac{\hat{\beta}\varrho^N}{1-\hat{\beta}\hat{b}}$, which in turn mean that the E-stability conditions correspond to

$$\frac{\hat{\beta}}{1-\hat{\beta}\hat{b}} < 1 \qquad \frac{\hat{\beta}\hat{\delta}}{\left(1-\hat{\beta}\hat{b}\right)^2} < 1 \qquad \textit{and} \qquad \frac{\hat{\beta}\varrho^N}{1-\hat{\beta}\hat{b}} < 1$$

Moreover using (60) we can rewrite these conditions as

$$\frac{1}{\frac{1}{\hat{\beta}} - \hat{b}} < 1 \qquad \frac{\hat{b}^2}{\frac{\hat{\delta}}{\hat{\beta}}} < 1 \qquad \textit{and} \qquad \frac{\varrho^N}{\frac{1}{\hat{\beta}} - \hat{b}} < 1 \tag{62}$$

Finally recall that in the proof of Proposition 1 we derived that real determinacy is associated with one explosive eigenvalue $|\upsilon_1| > 1$ and one non-explosive eigenvalue $|\upsilon_2| < 1$. Then using this and $\hat{b}_i = \frac{1}{\upsilon_i}$ we can infer that $\hat{b}_1 \in (-1,1)$ and either $\hat{b}_2 \in (-\infty,-1)$ or $\hat{b}_2 \in (1,\infty)$. Moreover using (61) we can rewrite the conditions (62) as

$$\begin{aligned} \frac{1}{\frac{1}{\hat{\beta}} - \hat{b}} &= \frac{1}{Trace(J^{-1}) - \hat{b}} = \frac{1}{\hat{b}_1 + \hat{b}_2 - \hat{b}} < 1 \qquad \frac{\hat{b}^2}{\frac{\hat{\delta}}{\hat{\beta}}} = \frac{\hat{b}^2}{Det(J^{-1})} = \frac{\hat{b}^2}{\hat{b}_1\hat{b}_2} < 1 \quad \textit{and} \\ \frac{\varrho^N}{\frac{1}{\hat{\beta}} - \hat{b}} &= \frac{\varrho^N}{Trace(J^{-1}) - \hat{b}} = \frac{\varrho^N}{\hat{b}_1 + \hat{b}_2 - \hat{b}} < 1 \end{aligned} \tag{63}$$

For $\hat{b} = \hat{b}_1$ we have that the E-stability conditions (63) become

$$\frac{1}{\hat{b}_1 + \hat{b}_2 - \hat{b}_1} = \frac{1}{\hat{b}_2} < 1 \qquad \frac{\hat{b}_1^2}{\hat{b}_1\hat{b}_2} = \frac{\hat{b}_1}{\hat{b}_2} < 1 \qquad \textit{and} \qquad \frac{\varrho^N}{\hat{b}_1 + \hat{b}_2 - \hat{b}_1} = \frac{\varrho^N}{\hat{b}_2} < 1 \tag{64}$$

Then defining $\hat{b} = \hat{b}_1$ Proposition 2 follows since it is simple to see that the E-stability conditions are satisfied given $\varrho^N \in (0,1)$, $\hat{b}_1 \in (-1,1)$ and either $\hat{b}_2 \in (-\infty,-1)$ or $\hat{b}_2 \in (1,\infty)$. ■

7.4 Proof of Proposition 3

Proof. This proof builds on Evans and McGough (2003). First, we rewrite the model (45) as

$$\left(1 - \hat{\beta}^{-1}L + \hat{\beta}^{-1}\hat{\delta}L^2\right)\hat{e}_t = \hat{\eta}_{e_t} - \frac{1}{\hat{\beta}}\hat{\zeta}^N_{t-1} \tag{65}$$

where L is the lag operator, $\hat{\zeta}_t^N = \hat{\kappa}\zeta_t^N$and $\hat{\eta}_{e_{t+1}} = \hat{e}_{t+1} - E_t(\hat{e}_{t+1})$ is a martingale difference sequence. The associated characteristic equation of (65) is

$$\hat{b}^2 - \frac{1}{\hat{\beta}}\hat{b} + \frac{\hat{\delta}}{\hat{\beta}} = 0 \tag{66}$$

whose roots are denoted by $\hat{b}_i$ with $i = 1, 2$. Note that $\frac{\hat{\delta}}{\hat{\beta}} = \hat{b}_1\hat{b}_2$. Recall that in the proof of Proposition 1 we argued that real indeterminacy was associated with the two explosive eigenvalues $|\upsilon_1| > 1$ and $|\upsilon_2| > 1$. Then using this and $\hat{b}_i = \frac{1}{\upsilon_i}$ we can infer that $\hat{b}_1 \in (-1, 1)$ and $\hat{b}_2 \in (-1, 1)$. Moreover using (61) and the definitions $\hat{\beta}$ and $\hat{\delta}$ in (46) we obtain $\frac{\hat{\delta}}{\hat{\beta}} = \frac{1+\bar{\epsilon}}{\beta(1-H)\bar{e}\left(\frac{1+\bar{\epsilon}}{(1-H)\bar{e}} - \rho_e\right)} = \hat{b}_1\hat{b}_2$. Using the conditions of real indeterminacy, $1 + \frac{1}{\beta} < H$ and $\rho_e < \tilde{\rho}_e$, it is simple to derive that $\hat{b}_1\hat{b}_2 = \frac{\hat{\delta}}{\hat{\beta}} < 0$. Then using Lemma 1 we can conclude that the roots are real. In addition without loss of generality we can assume that the roots are $\hat{b}_1 \in (0, 1)$ and $\hat{b}_2 \in (-1, 0)$.

Second, we point out that following Propositions 3 and 4 in Evans and McGough (2003), it is simple to prove that the process $\hat{e}_t$ is a rational expectations equilibrium in (45) with $\varrho^N = 0$, if and only if there is a martingale difference sequence $\hat{\eta}_{e_{t+1}}$ such that $\hat{e}_t$ solves (48) with (49).

Third, we derive the E-stability conditions adapting the analysis of Evans and McGough (2003). In particular, note that we assume that agents knows $\hat{e}_t$ when making the forecast $E_t\hat{e}_{t+1}$. Consider the stochastic model (45) and suppose that the agents follow a perceived law of motion (PLM) such as

$$\begin{aligned} \hat{e}_t &= \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{e}_{t-2} + \hat{d}\hat{\xi}_t + \hat{k}\hat{\zeta}_t^N + \hat{l}\hat{\zeta}_{t-1}^N \\ \hat{\xi}_t &= \hat{\chi}\hat{\xi}_{t-1} + \hat{\eta}_{e_t} \end{aligned}$$

where $\hat{\eta}_{e_{t+1}} = \hat{e}_{t+1} - E_t\hat{e}_{t+1}$ is a martingale difference sequence. Iterating forward these laws of motion and taking expectations we obtain $E_t\hat{e}_{t+1} = \hat{a} + \hat{b}\hat{e}_t + \hat{c}\hat{e}_{t-1} + \hat{d}\hat{\chi}\hat{\xi}_t + \hat{l}\hat{\zeta}_t^N$; using this to eliminate all the forecasts in the model (45) and assuming that agents know $\hat{e}_t$ when they make their forecasts, we can derive the implied actual law of motion (ALM)

$$\hat{e}_t = \left(\frac{\hat{\beta}\hat{a} + \hat{\alpha}}{1 - \hat{\beta}\hat{b}}\right) + \left(\frac{\hat{\beta}\hat{c} + \hat{\delta}}{1 - \hat{\beta}\hat{b}}\right)\hat{e}_{t-1} + \left(\frac{\hat{\beta}\hat{d}\hat{\chi}}{1 - \hat{\beta}\hat{b}}\right)\hat{\xi}_t + \left(\frac{\hat{\beta}\hat{l} + 1}{1 - \hat{\beta}\hat{b}}\right)\hat{\zeta}_t^N$$

Then we obtain the T-mapping

$$T(\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{k}, \hat{l}) = [T_1, T_2, T_3, T_4, T_5, T_6]' = \left[\frac{\hat{\beta}\hat{a} + \hat{\alpha}}{1 - \hat{\beta}\hat{b}}, \frac{\hat{\beta}\hat{c} + \hat{\delta}}{1 - \hat{\beta}\hat{b}}, 0, \frac{\hat{\beta}\hat{d}\hat{\chi}}{1 - \hat{\beta}\hat{b}}, \frac{\hat{\beta}\hat{l} + 1}{1 - \hat{\beta}\hat{b}}, 0\right]'$$

whose fixed points correspond to the rational expectations equilibrium with $\hat{b}$, $\hat{d}$ and $\hat{k}$ satisfying (66), $\hat{d} = \frac{\hat{\beta}\hat{d}\hat{\chi}}{1-\hat{\beta}\hat{b}}$, $\hat{k} = \frac{\hat{\beta}\hat{l}+1}{1-\hat{\beta}\hat{b}}$, respectively and $\hat{a} = \hat{c} = \hat{l} = 0$.[38] Since the matrix of derivatives

[38] Note that $\hat{a} = \frac{\hat{\alpha}}{1-\hat{\beta}\hat{b}-\hat{\beta}} = 0$ since $\hat{\alpha} = 0$.

$DT(\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{k}, \hat{l})$ is block triangular then it is simple to show that the eigenvalues correspond to $\frac{\partial T_1}{\partial \hat{a}} = \frac{\hat{\beta}}{1-\hat{\beta}\hat{b}}$, $\frac{\partial T_2}{\partial \hat{b}} = \frac{\hat{\beta}\hat{\delta}}{\left(1-\hat{\beta}\hat{b}\right)^2}$, $\frac{\partial T_3}{\partial \hat{c}} = 0$, $\frac{\partial T_4}{\partial \hat{d}} = \frac{\hat{\beta}\hat{\chi}}{1-\hat{\beta}\hat{b}}$, $\frac{\partial T_5}{\partial \hat{k}} = 0$ and $\frac{\partial T_6}{\partial \hat{l}} = 0$, which in turn means that the E-stability conditions reduce to

$$\frac{\hat{\beta}}{1-\hat{\beta}\hat{b}} < 1 \qquad \frac{\hat{\beta}\hat{\delta}}{\left(1-\hat{\beta}\hat{b}\right)^2} < 1 \qquad and \qquad \frac{\hat{\beta}\hat{\chi}}{1-\hat{\beta}\hat{b}} < 1$$

Moreover using (66) and noting that $\frac{\hat{\delta}}{\hat{\beta}} = \hat{b}_1\hat{b}_2$ and $\frac{1}{\hat{\beta}} = \hat{b}_1 + \hat{b}_2$ we can rewrite these conditions as

$$\frac{1}{\hat{b}_1 + \hat{b}_2 - \hat{b}} < 1 \qquad \frac{\hat{b}^2}{\hat{b}_1\hat{b}_2} < 1 \qquad and \qquad \frac{\hat{\chi}}{\hat{b}_1 + \hat{b}_2 - \hat{b}} < 1 \tag{67}$$

However note that for the last E-stability condition, it is always true that for the common factor representation we have that either $\hat{b} = \hat{b}_1$ and therefore $\hat{\chi} = \hat{b}_2$ implying $\frac{\hat{\chi}}{\hat{b}_1+\hat{b}_2-\hat{b}} = 1$, or, $\hat{b} = \hat{b}_2$ and therefore $\hat{\chi} = \hat{b}_1$ implying $\frac{\hat{\chi}}{\hat{b}_1+\hat{b}_2-\hat{b}} = 1$. Hence the differential equation for $\hat{d}(\tau)$ is $\frac{d\hat{d}(\tau)}{d\tau} = \left(\frac{\hat{\chi}}{\hat{b}_1+\hat{b}_2-\hat{b}} - 1\right)\hat{d}(\tau)$. Using a similar argument to the one developed in Evans and Honkapohja (1992) it is possible to show that as either $\hat{b} \to \hat{b}_1$ or $\hat{b} \to \hat{b}_2$ then $\hat{d}(\tau)$ converges to a finite value. This means that the only stability conditions that are required to be checked are $\frac{1}{\hat{b}_1+\hat{b}_2-\hat{b}} < 1$ and $\frac{\hat{b}^2}{\hat{b}_1\hat{b}_2} < 1$.

Fourth, we recall our result from the beginning of this proof that states that under real indeterminacy the roots are $\hat{b}_1 \in (0, 1)$ and $\hat{b}_2 \in (-1, 0)$.

To prove part a) we use the fact that for $\hat{b} = \hat{b}_1$ we have that the E-stability conditions $\frac{1}{\hat{b}_1+\hat{b}_2-\hat{b}} < 1$ and $\frac{\hat{b}^2}{\hat{b}_1\hat{b}_2} < 1$ become $\frac{1}{\hat{b}_2} < 1$ and $\frac{\hat{b}_1}{\hat{b}_2} < 1$. Since $\hat{b}_1 \in (0, 1)$ and $\hat{b}_2 \in (-1, 0)$ it is clear that these E-stability conditions are satisfied. Hence the common factor representation (48) and (49) with $\hat{b}_i = \hat{b}_1 \in (0, 1)$ and $\hat{b}_j = \hat{b}_2 \in (-1, 0)$ is learnable in the E-stability sense.

To prove part b) we utilize the fact that for $\hat{b} = \hat{b}_2$ we have that the E-stability condition $\frac{1}{\hat{b}_1+\hat{b}_2-\hat{b}} < 1$ is clearly not satisfied for $\hat{b} = \hat{b}_2 \in (-1, 0)$, given that $\hat{b}_1 \in (0, 1)$ implies $\frac{1}{\hat{b}_1} > 1$. ■

7.5 Proof of Proposition 4

Proof. To prove a) first we write the characteristic equation associated with (50) as

$$\mathcal{P}(\hat{b}) = \hat{b}^2 - \frac{1}{\hat{\beta}_p}\hat{b} + \frac{\hat{\delta}_p}{\hat{\beta}_p} = 0$$

where $\hat{\beta}_p$, and $\hat{\delta}_p$ are defined in (51). Second using this, and $\rho_e < 0$, $K > 0$ and $H > 0$, and the assumptions about the structural parameters we can derive that

$$\begin{aligned}\mathcal{P}(1) &= 1 - \frac{1}{\hat{\beta}_p} + \frac{\hat{\delta}_p}{\hat{\beta}_p} = \left(\frac{1}{\beta} - 1\right)\left(\frac{\rho_e \bar{e}}{1+\bar{\epsilon}}\right) - \frac{K\bar{\epsilon}}{1+\bar{\epsilon}} + \frac{H\rho_e\bar{e}}{1+\bar{\epsilon}} < 0 \\ \mathcal{P}(-1) &= 1 + \frac{1}{\hat{\beta}_p} + \frac{\hat{\delta}_p}{\hat{\beta}_p} = \frac{\bar{e}\left(H - 1 - \frac{1}{\beta}\right)}{1+\bar{\epsilon}}\left(\breve{\rho}_e - \rho_e\right)\end{aligned}$$

where $\breve{\rho}_e = \frac{2\left(\frac{1}{\beta}+1\right)(1+\bar{\epsilon})+K\bar{\epsilon}}{\bar{e}\left(H-1-\frac{1}{\beta}\right)}$. Observe that since $\mathcal{P}(1) < 0$ by Lemma 1 we know that the eigenvalues are real.

Third note that if either $H > 1 + \frac{1}{\beta}$ and $\rho_e < 0$, or $H < 1 + \frac{1}{\beta}$ and $\breve{\rho}_e < \rho_e < 0$, then $\mathcal{P}(-1) > 0$. Using this and the previous results that $\mathcal{P}(1) < 0$ and that the eigenvalues are real we may infer that the steady state is a saddle with one of the eigenvalues falling in $(-1, 1)$ and the other one falling in $(1, \infty)$ as explained in Azariadis (1993). Given that $\hat{e}_t$ is the only predetermined variable then by Blanchard and Kahn (1980) we conclude that the model displays real determinacy.

Fourth it is straightforward to prove that the fundamental solution $\hat{e}_t = \hat{a} + \hat{b}\hat{e}_{t-1} + \hat{c}\hat{z}_t^N$ is in fact a solution of (50). Fifth we prove that this fundamental solution is E-stable. In order to do so we need to derive the E-stability conditions. The procedure to derive them is exactly the same as the procedure followed in the proof of Proposition 2. In fact the conditions are the same as the ones previously derived. Here we present only the conditions. The E-stability conditions are

$$\frac{1}{\hat{b}_1 + \hat{b}_2 - \hat{b}_1} = \frac{1}{\hat{b}_2} < 1 \qquad \frac{\hat{b}_1^2}{\hat{b}_1\hat{b}_2} = \frac{\hat{b}_1}{\hat{b}_2} < 1 \qquad \textit{and} \qquad \frac{\varrho^N}{\hat{b}_1 + \hat{b}_2 - \hat{b}_1} = \frac{\varrho^N}{\hat{b}_2} < 1$$

It is simple to see that the E-stability conditions are satisfied given $\varrho^N \in (0, 1)$, $\hat{b}_1 \in (-1, 1)$ and $\hat{b}_2 \in (1, \infty)$. Then the fundamental solution is learnable in the E-stability sense and statement a) of this proposition follows.

To prove b) it is enough to note that if $H < 1 + \frac{1}{\beta}$ and $\breve{\rho}_e < \rho_e < 0$ then we can derive that $\mathcal{P}(-1) < 0$. Using this and the previous results that $\mathcal{P}(1) < 0$ and that the eigenvalues are real we may infer from Azariadis (1993) that the steady state is a source with one eigenvalue falling in $(-\infty, -1)$ and the other in $(1, \infty)$. Since $\hat{e}_t$ is the only predetermined variable then by Blanchard and Kahn (1980) we conclude that the model displays no equilibrium for this case. ■

7.6 Proof of Proposition 5

Proof. To prove all the parts of the proposition we use (37), (38), (39) and the log-linearized versions of (8) and $\epsilon_t = \rho(\pi_t)$ to derive the following system

$$\begin{pmatrix} \hat{\pi}_t^N \\ \hat{e}_{t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}}_{J} \begin{pmatrix} \hat{\pi}_{t+1}^N \\ \hat{e}_t \end{pmatrix} + U\hat{\eta}_{\pi_{t+1}} + V\hat{z}_{t+1}^N$$

where

$$J_{11} = \frac{\beta\left\{1 - \left[\alpha + (1-\alpha)H\right]\rho_\pi\right\}}{1 - \alpha\rho_\pi} \qquad J_{12} = \beta K$$

$$J_{21} = \frac{\beta\bar{\epsilon}\left(1 - \rho_\pi\right)\left\{1 - \left[\alpha + (1-\alpha)H\right]\rho_\pi\right\}}{(1+\bar{\epsilon})(1 - \alpha\rho_\pi)^2} \qquad J_{22} = 1 + \frac{\beta K\bar{\epsilon}(1-\rho_\pi)}{(1+\bar{\epsilon})(1-\alpha\rho_\pi)}$$

$\hat{\eta}_{\pi_{t+1}}$ is the forecast error for the non-traded goods inflation defined as $\hat{\eta}_{\pi_{t+1}} = \hat{\pi}_{t+1}^N - E_t(\hat{\pi}_{t+1}^N)$ and where the forms of U and V are omitted since they are not needed in what follows. For the the proof of the proposition it is convenient to pursue the determinacy analysis for the system written in the form $\hat{y}_t = J\hat{y}_{t+1} + U\hat{\eta}_{\pi_{t+1}} + V\hat{z}_{t+1}^N$ where $\hat{y}_t = (\hat{\pi}_t^N, \hat{e}_{t-1})'$ instead of $\hat{y}_{t+1} = J^{-1}\hat{y}_t - J^{-1}U\hat{\eta}_{\pi_{t+1}} - J^{-1}V\hat{z}_{t+1}^N$. It will reduce tremendously the number of cases that have to be analyzed.[39] For this linearized system we have that the trace of J, the determinant of J, and the characteristic polynomial associated with J correspond to

$$Trace(J) = 1 + \frac{\beta K\bar{\epsilon}(1-\rho_\pi)}{(1+\bar{\epsilon})(1-\alpha\rho_\pi)} + \frac{\beta\left\{1 - \left[\alpha + (1-\alpha)H\right]\rho_\pi\right\}}{1 - \alpha\rho_\pi}$$

$$Det\,(J) = \frac{\beta\left\{1 - \left[\alpha + (1-\alpha)H\right]\rho_\pi\right\}}{1 - \alpha\rho_\pi}$$

$$\mathcal{P}(\upsilon) = \upsilon^2 - Trace(J)\upsilon + Det(J) = 0$$

respectively. Using these expressions we obtain

$$\mathcal{P}(1) = -\frac{\beta K\bar{\epsilon}(1-\rho_\pi)}{(1+\bar{\epsilon})(1-\alpha\rho_\pi)}$$

$$\mathcal{P}(-1) = \frac{\beta\left[2\alpha\left(1+\frac{1}{\beta}\right) + 2(1-\alpha)H + \frac{K\bar{\epsilon}}{1+\bar{\epsilon}}\right]}{1-\alpha\rho_\pi}(\tilde{\rho}_\pi - \rho_\pi)$$

where $\tilde{\rho}_\pi = \frac{2\left(1+\frac{1}{\beta}\right)+\frac{K\bar{\epsilon}}{1+\bar{\epsilon}}}{2\alpha\left(1+\frac{1}{\beta}\right)+2(1-\alpha)H+\frac{K\bar{\epsilon}}{1+\bar{\epsilon}}} > 0$. In this proof we will use the facts that $K > 0$ and $H > 0$, as well as the constraints imposed on the structural parameters listed in the description of the model.

[39] One can analyze either of the forms because the eigenvalues of J^{-1} correspond to υ_i^{-1}, where υ_i are the eigenvalues of J.

To prove part a) we proceed as follows. Observe that since $\rho_\pi < 1$ and $\alpha \in (0,1)$ then $\rho_\pi < \frac{1}{\alpha}$ and therefore $\mathcal{P}(1) < 0$. By Lemma 1, this last inequality implies that the eigenvalues are real. In addition since by assumption $\tilde{\rho}_\pi < \rho_\pi$ (and $\rho_\pi < \frac{1}{\alpha}$), then we can infer that $\mathcal{P}(-1) < 0$. It is straightforward to show that $\mathcal{P}(1) < 0$ and $\mathcal{P}(-1) < 0$ imply that $Det\,(J) < -1$, a result we will use later. Hence $\mathcal{P}(-1) < 0$ in tandem with $\mathcal{P}(1) < 0$ help us to conclude that the system has two explosive eigenvalues $|\upsilon_1| > 1$ and $|\upsilon_2| > 1$ and that the steady state is a source as explained in Azariadis (1993). This means that $\left|\frac{1}{\upsilon_1}\right| < 1$ and $\left|\frac{1}{\upsilon_2}\right| < 1$. Since $\hat{\pi}_t^N$ is the only non-predetermined variable of the system then by Blanchard and Kahn (1980) we conclude that the model displays real indeterminacy.

To prove b) consider the assumption $\rho_\pi > \max\{1, \tilde{\rho}_\pi\}$. We have to take into account two cases: $\rho_\pi > \frac{1}{\alpha}$ and $\rho_\pi < \frac{1}{\alpha}$. For $\rho_\pi > \frac{1}{\alpha}$ we have that since by assumption $\rho_\pi > 1$ then $\mathcal{P}(1) < 0$. By Lemma 1, this last inequality implies that the eigenvalues are real. In addition since by assumption $\rho_\pi > \tilde{\rho}_\pi$ then we can infer that $\mathcal{P}(-1) > 0$. This in tandem with $\mathcal{P}(1) < 0$ imply that the steady state is a saddle point with one explosive eigenvalue $\upsilon_1 \in (1, \infty)$ and one non-explosive eigenvalue $\upsilon_2 \in (-1, 1)$, as shown in Azariadis (1993).

For $\rho_\pi < \frac{1}{\alpha}$ we have that since by assumption $\rho_\pi > 1$ then $\mathcal{P}(1) > 0$. In addition since by assumption $\rho_\pi > \tilde{\rho}_\pi$ then we can infer that $\mathcal{P}(-1) < 0$ which in turn implies, by Lemma 1, that the eigenvalues are real. Since $\mathcal{P}(1) > 0$ and $\mathcal{P}(-1) < 0$, then the steady state is a saddle point and the system has one explosive eigenvalue $\upsilon_1 \in (-\infty, -1)$ and one non-explosive eigenvalue $\upsilon_2 \in (-1, 1)$ as explained in Azariadis (1993).

Now consider the assumption $\rho_\pi < \min\{1, \tilde{\rho}_\pi\}$. Using this we can infer that in any case we have that since $\alpha \in (0,1)$ then $\rho_\pi < 1 < \frac{1}{\alpha}$. But this implies that $\mathcal{P}(1) < 0$. By Lemma 1, we derive that the eigenvalues are real. In addition since by assumption $\rho_\pi < \tilde{\rho}_\pi$ then we can infer that $\mathcal{P}(-1) > 0$. This in tandem with $\mathcal{P}(1) < 0$ imply that the steady state is a saddle point with one explosive eigenvalue $\upsilon_1 \in (1, \infty)$ and one non-explosive eigenvalue $\upsilon_2 \in (-1, 1)$, as shown in Azariadis (1993).

Therefore under either $\rho_\pi > \max\{1, \tilde{\rho}_\pi\}$ or $\rho_\pi < \min\{1, \tilde{\rho}_\pi\}$, the steady state is a saddle path. Since $\hat{\pi}_t^N$ is the only non-predetermined variable of the system then by Blanchard and Kahn (1980) we conclude that the model displays real determinacy.

Finally to prove c) we start by observing that $\tilde{\rho}_\pi < \frac{1}{\alpha}$. Hence $1 < \rho_\pi < \tilde{\rho}_\pi < \frac{1}{\alpha}$. These inequalities imply that $\mathcal{P}(1) > 0$ and $\mathcal{P}(-1) > 0$. Then we have to consider two cases: $\rho_\pi > \frac{1}{\alpha+(1-\alpha)H}$ and $\rho_\pi < \frac{1}{\alpha+(1-\alpha)H}$. For the first case if $\rho_\pi > \frac{1}{\alpha+(1-\alpha)H}$ then since $\rho_\pi < \frac{1}{\alpha}$ we have that $Det\,(J) < 0$. By Lemma 1 this implies that the roots are real. Moreover utilizing this and $\mathcal{P}(1) > 0$ and $\mathcal{P}(-1) > 0$ we can infer that the steady state is a sink with two non explosive eigenvalues $\upsilon_1 \in (0, 1)$ and $\upsilon_2 \in (-1, 0)$, as explained in Azariadis(1993).

On the other hand, for the second case we have that since $\rho_\pi < \frac{1}{\alpha+(1-\alpha)H}$ and $\rho_\pi < \frac{1}{\alpha+(1-\alpha)H} <$

$\frac{1}{\alpha}$ then $0 < Det\,(J) < 1$ (provided that $\beta \in (0,1)$). Using this and $\mathcal{P}(1) > 0$ and $\mathcal{P}(-1) > 0$ we can conclude from Azariadis (1993) that regardless of whether the eigenvalues are real or complex the steady state is a source with two non explosive eigenvalues $\upsilon_1 \in (-1,1)$ and $\upsilon_2 \in (-1,1)$.

Therefore in both cases we have concluded that the steady-state is a source. Hence $\left|\frac{1}{\upsilon_1}\right| > 1$ and $\left|\frac{1}{\upsilon_2}\right| > 1$.Since $\hat{\pi}_t^N$ is the only non-predetermined variable of the system then by Blanchard and Kahn (1980) we conclude that the model displays no equilibrium. ■

7.7 Proof of Corollary 2

Proof. First, observe that $\tilde{\rho}_\pi < 1$ implies that $H > 1 + \frac{1}{\beta}$. Second, from the Proof of Corollary 1 we have that $\frac{\partial K}{\partial \alpha} < 0$, $\frac{\partial K}{\partial \mu} > 0$ and $\frac{\partial K}{\partial \phi} < 0$. Use all these inequalities in tandem with

$$\begin{aligned}
\frac{\partial \tilde{\rho}_\pi}{\partial \alpha} &= \frac{2\left(H - 1 - \frac{1}{\beta}\right)\left[2\left(1 + \frac{1}{\beta}\right) + \frac{K\bar{\epsilon}}{1+\bar{\epsilon}}\right] - \Theta\frac{\partial K}{\partial \alpha}}{\Phi^2} \\
\frac{\partial \tilde{\rho}_\pi}{\partial \mu} &= -\frac{\Theta}{\Phi^2}\frac{\partial K}{\partial \mu} \qquad \frac{\partial \tilde{\rho}_\pi}{\partial \phi} = -\frac{\Theta}{\Phi^2}\frac{\partial K}{\partial \phi}
\end{aligned}$$

where $\Theta = 2\left(1 + \frac{1}{\beta}\right)(1-\alpha)\left(\frac{\bar{\epsilon}}{1+\bar{\epsilon}} + 2\frac{H}{K}\right) > 0$ and $\Phi = \alpha\left(1 + \frac{1}{\beta}\right) + 2(1-\alpha)H + \frac{K\bar{\epsilon}}{1+\bar{\epsilon}} > 0$, to derive a), b) and c). ■

7.8 Proof of Proposition 6

Proof. First, observe that the characteristic equation associated with (54) is

$$\hat{b}^2 - \frac{1}{\hat{\beta}_q}\hat{b} + \frac{\hat{\delta}_q}{\hat{\beta}_q} = 0 \tag{68}$$

whose roots are denoted by $\hat{b}_i$ with $i = 1, 2$. Second, we establish a relationship between these roots $\hat{b}_i$ and the roots υ_i in the proof of Proposition 5. In particular it is straightforward to show that $\hat{b}_i = \frac{1}{\upsilon_i}$ and that

$$\frac{1}{\hat{\beta}_q} = \frac{Trace(J)}{Det(J)} = Trace(J^{-1}) = \hat{b}_1 + \hat{b}_2 \qquad and \qquad \frac{\hat{\delta}_q}{\hat{\beta}_q} = \frac{1}{Det(J)} = Det(J^{-1}) = \hat{b}_1\hat{b}_2$$

where J was defined in Proposition 5.

Third we use these relationships to prove a). Since under real determinacy in the proof of Proposition 5 we have that either $\upsilon_1 \in (-\infty,-1)$ and $\upsilon_2 \in (-1,1)$ or $\upsilon_1 \in (1,\infty)$ and $\upsilon_2 \in (-1,1)$ then we can conclude that $\hat{b}_1 \in (-1,1)$ and either $\hat{b}_2 \in (-\infty,-1)$ or $\hat{b}_2 \in (1,\infty)$.

Fourth, it is simple to show that under real determinacy, the unique equilibrium of the model (54) characterized by the fundamental solution (56) with $\hat{b} \in (-1,1)$ is in fact a solution of the model (54).

Fifth, we derive the E-stability conditions. However the procedure is the same as the one explained in the proof of Proposition 2. We only rename $\hat{\beta}$ and $\hat{\delta}$ as $\hat{\beta}_q$ and $\hat{\delta}_q$ respectively. Then following the proof of Proposition 2, it is straightforward to prove that the unique equilibrium of the model (54) characterized by the fundamental solution (56) with $\hat{b} \in (-1, 1)$ and either $\hat{b}_2 \in (-\infty, -1)$ or $\hat{b}_2 \in (1, \infty)$ is E-stable.

Sixth to prove b) we start by noting that in (68) we have that $\frac{\hat{\delta}_q}{\hat{\beta}_q} = \hat{b}_1 \hat{b}_2$. Next we recall that in the proof of a) in Proposition 5 we argued that real indeterminacy is associated with two non-explosive eigenvalues $|\upsilon_1| < 1$ and $|\upsilon_2| < 1$. Then using this and $\hat{b}_i = \frac{1}{\upsilon_i}$ help us to infer that $\hat{b}_1 \in (-1, 1)$ and $\hat{b}_2 \in (-1, 1)$. Furthermore under real indeterminacy it is easy to show that $\frac{\hat{\delta}_q}{\hat{\beta}_q} < -1$. Hence $\hat{b}_1 \hat{b}_2 = \frac{\hat{\delta}_q}{\hat{\beta}_q} < 0$ which in turn means that the roots are real. Then without loss of generality we can assume that the roots are $\hat{b}_1 \in (0, 1)$ and $\hat{b}_2 \in (-1, 0)$.

Seventh, we point out that following Propositions 3 and 4 in Evans and McGough (2003), it is simple to prove that the process $\hat{e}_t$ is a rational expectations equilibrium of (54) with $\varrho^N = 0$, if and only if there is a martingale difference sequence $\hat{\eta}_{e_{t+1}}$ such that $\hat{e}_t$ solves (57) with (58). Henceforth we focus on proving the learnability of the common factor representation. However the proof is similar to the proof for Proposition 3. We just have to rename $\hat{\beta}$ and $\hat{\delta}$ as $\hat{\beta}_q$ and $\hat{\delta}_q$ respectively, and follow the steps. Then statement b) follows. ■

8 References

Adams, C. and D. Gros (1986), "The Consequences of Real Exchange Rate Rules for Inflation: Some Illustrative Examples," *IMF Staff Papers*, 33, 439–476.

Azariadis, C. (1993), *Intertemporal Macroeconomics*, Blackwell, Cambridge, Massachusetts.

Benhabib, J., S. Schmitt-Grohé and M. Uribe (2001), "Monetary Policy Rules and Multiple Equilibria," *American Economic Review*, 91, 167-184.

Blanchard, O. and C. Kahn (1980), "The Solution of Linear Difference Models Under Rational Expectations," *Econometrica,* 48, 1305-1312.

Bullard, J. and K. Mitra, (2002), "Learning About Monetary Policy Rules," *Journal of Monetary Economics,* 49, 1105-1129.

Calvo, G. (1983), "Staggering Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics,* 12, 383-398.

Calvo, G., Reinhart, C. and C. Vegh (1995), "Targeting the Real Exchange Rate: Theory and Evidence," *Journal of Development Economics*, 47:1, 97-134.

Carlstrom, C. and T. Fuerst (2001), “Timing and Real Indeterminacy in Monetary Models,” *Journal of Monetary Economics*, 47:2, 285-298.

Clarida, J., J. Gali and M. Gertler (2000), “Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory,” *Quarterly Journal of Economics*, 115, 147-180.

Clarida, J., J. Gali and M. Gertler (2001), ”Optimal Monetary Policy in Open Versus Closed Economies: An Integrated Approach,” *American Economic Review*, 91 (2), 253-257.

Dornbusch, R.(1980), *Open Economy Macroeconomics*, New York: Basic.

Dornbusch, R.(1982), “PPP Exchange Rate Rules and Macroeconomic Stability,” *Journal of Political Economy*, 90:3, 158-165.

Dib, A. (2001), “An Estimated Canadian DSGE Model with Nominal and Real Rigidities,” Bank of Canada, Working Paper, 26.

Dupor, B. (2001), “Investment and Interest Rate Policy,” *Journal of Economic Theory,* 98, 85-113.

Evans, G, and S. Honkapohja (1986), “A Complete Characterization of ARMA Solutions to Linear Rational Expectations Models,” *Review of Economic Studies*, 53, 227–239.

Evans, G, and S. Honkapohja (1992), “On theRobustness of Bubbles in Linear RE Models,” *International of Economic Review*, 33, 1-14.

Evans, G, and S. Honkapohja (1999), “Learning Dynamics,” in *Handbook of Macroeconomics*, eds. J. Taylor and M. Woodford, Elsevier, Amsterdam, Vol. 1, 449-542.

Evans, G, and S. Honkapohja (2001), *Learning and Expectations*, Princeton University Press, New Jersey.

Evans, G, and B. McGough (2003), “Stable Sunspot Solutions in Model with Predetermined Variables,” Manuscript, University of Oregon, Forthcoming *Journal of Economic Dynamics and Control.*

Gali, J. and T. Monacelli (2004), “Monetary Policy and Exchange Rate Volatility in a Small Open Economy,” CREI, Manuscript.

Keynes, J. M. (1936), *The General Theory of Employment, Interest and Money*, Macmillan, London.

Kimbrough, K. (1986), “The Optimal Quantity of Money Rule in the Theory of Public Finance,” *Journal of Monetary Economics,* 18, 277-284.

Lizondo, J. (1991), “Real Exchange Rate Targets, Nominal Exchange Rate Policies, and Inflation,” *Revista de Análisis Económico*, 6, 5-22.

Marcet, A., and T. Sargent (1989), “Convergence of Least-Square Learning Mechanisms in Self-Referential Linear Stochastic Models,” *Journal of Economic Theory*, 48, 337–368.

McCallum, B. (1983), “On Nonuniqueness in Linear Rational Expectations Moldels: An Attempt at Perspective,” *Journal of Monetary Economics,* 11, 134-168.

Montiel, A. and J. Ostry (1991), “Macroeconomic Consequences of Real Exchange Rate Targeting in Developing Economies,” *IMF Staff Papers,* 38, 872-900.

Preston, B. (2003), “Learning About Monetary Policy Rules when Long-Horizon Expectations Matter,” Manuscript, Columbia University.

Rotemberg, J. (1982), ”Sticky Prices in the United States,“*Journal of Political Economy,* 90(6), 1187-1211.

Rotemberg, J. and M. Woodford (1999), “Interest-Rate Rules in an Estimated Sticky-Price Model,” in *Monetary Policy Rules*, edited by John B. Taylor, Chicago, National Bureau of Economic Research.

Schmitt-Grohé, S. and M. Uribe (2001), “Stabilization Policy and the Costs of Dollarization,” *Journal of Money, Credit and Banking*, 33, 482-509.

Uribe, M. (2003), “Real Exchange Rate Targeting and Macroeconomic Instability,” University of Pennsylvania, Manuscript, *Journal of International Economics*, Forthcoming.

Woodford, M. (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy,* Princeton University Press, New Jersey.

Zanna, L. F. (2003a), “Interest Rate Rules and Multiple Equilibria in the Small Open Economy: The Role of Non-Traded Goods,” IFDP #785, Board of Governors of the Federal Reserve System.

Zanna, L. F. (2003b), “International Economics, Monetary Rules and Multiple Equilibria” Ph.D. Dissertation, University of Pennsylvania.