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Abstract

In this paper we present an extensive analysis of the consequences for global equilibrium determinacy of implementing active interest rate rules (i.e. monetary rules where the nominal interest rate responds more than proportionally to changes in inflation) in flexible-price open economies. We show that conditions under which these rules generate aggregate instability by inducing cyclical and chaotic equilibrium dynamics depend on particular characteristics of open economies such as the degree of (trade) openness and the degree of exchange rate pass-through implied by the presence of non-traded distribution costs. For instance, we find that a forward-looking rule is more prone to induce endogenous cyclical and chaotic dynamics the more open the economy and the higher the degree of exchange rate pass-through. The existence of these dynamics and their dependence on the degree of openness are in general robust to different timings of the rule (forward-looking versus contemporaneous rules), to the use of alternative measures of inflation in the rule (CPI versus Core inflation), as well as to changes in the timing of real money balances in liquidity services ("cash-when-I-am-done" timing versus "cash-in-advance" timing).

Keywords: Small Open Economy, Interest Rate Rules, Taylor Rules, Multiple Equilibria, Chaos and Endogenous Fluctuations.

JEL Classifications: E32, E52, F41

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1 Introduction

In recent years there has been a revival of theoretical and empirical literature aimed at understanding the macroeconomic consequences of implementing diverse monetary rules in the Small Open Economy (SOE).[1] In this literature the study of interest rate rules whose interest rate response coefficient to inflation is greater than one, generally referred to as Taylor rules or active rules, has received particular attention.[2] To some extent the importance given to these rules in the SOE literature is just a consequence of some of the benefits that the closed economy literature has claimed for them. For instance, Bernanke and Woodford (1997) and Clarida, Galí and Gertler (2000), among others, have argued that active rules are desirable because they guarantee a unique Rational Expectations Equilibrium (REE) whereas rules whose interest rate response coefficient to inflation is less than one, also called passive rules, induce aggregate instability in the economy by generating multiple equilibria. Despite these arguments supporting active rules in closed economies, Benhabib, Schmitt-Grohé and Uribe (2001b, 2002a,b) have pointed out that they are based on results that rely on the type of equilibrium analysis that is adopted. In fact these policy prescriptions are usually derived from a local determinacy of equilibrium analysis, i.e. identifying conditions for rules that guarantee equilibrium uniqueness in an arbitrarily small neighborhood of the target steady state. In contrast by pursuing a global equilibrium analysis in tandem with the observation that nominal interest rates are bounded below by zero, Benhabib et al. have shown that active rules can induce aggregate instability in closed economies through endogenous cycles, chaotic dynamics and liquidity traps.[3]

What motivates our paper is the fact that the open economy literature on interest rate rules has also restricted its attention to local dynamics and not to global dynamics, often disregarding the zero bound on the nominal interest rate. By doing this, the literature has gained in tractability but has also overlooked a possibly wider set of equilibrium dynamics.

To the best of our knowledge, our work is the first attempt in the open economy literature to understand how interest rate rules may lead to global endogenous fluctuations. We pursue a global and non-linear equilibrium analysis of a traditional flexible-price SOE model with traded and non-traded goods, whose government follows an active forward-looking rule by responding to the expected future CPI-inflation. We show that conditions under which active rules induce aggregate instability by generating cyclical and chaotic dynamics depend on some specific features of an open economy such as the degree of openness of the economy (measured as the share of traded goods in consumption) and the degree of exchange rate pass-through into import prices (implied by the presence of non-traded distribution services). For example, we find that a forward-looking rule is more prone to induce cyclical equilibria and chaotic dynamics the more open the

[1] See Ball (1999), Clarida, Galí and Gertler (1998, 2001), Galí and Monacelli (2004), Kollman (2002), Lubik and Schorfheide (2003), and Svensson (2000), among others.

[2] See Taylor (1993).

[3] In this context a liquidity trap is understood as a decelerating inflation dynamics where the economy is headed to a situation of low and possibly negative inflation and low and possibly zero interest rates and in which monetary policy is ineffective to stop this process.

economy and the higher the degree of exchange rate pass-through. If consumption and money are Edgeworth complements in utility these dynamics occur around an extremely low interest rate steady state. On the other hand if consumption and money are substitutes these dynamics appear around the high interest rate target set by the monetary authority.[4]

For a given specification of the rule, we show that the existence of these dynamics and their dependence on the degree of openness are in general robust to different timings of the rule (forward-looking versus contemporaneous rules), to the use of alternative measures of inflation in the rule (CPI versus Core inflation), as well as to changes in the timing of real money balances in liquidity services ("cash-when-I-am-done" timing versus "cash-in-advance" timing.)

Table 1:

Country	Degree of Openness (Imports/GDP)	Response Coefficient to Inflation (ρ_π)	Type of Interest Rate Rule
France	0.22	$1.13^{\dagger}$	Forward-Looking
Costa Rica	0.42	1.47^{*}	Forward-Looking
Colombia	0.20	$1.31^{\times}$	Forward-Looking
Chile	0.28	2.10°	Forward-Looking
United Kingdom	0.28	$1.84^{\ddagger}$	Contemporaneous
Australia	0.19	$2.10^{\ddagger}$	Contemporaneous
Canada	0.31	$2.24^{\ddagger}$	Contemporaneous
New Zealand	0.28	$2.49^{\ddagger}$	Contemporaneous

Note: Data from IFS were used to calculate the Imports/GDP share, while the significant estimates for the interest rate response coefficient to the CPI-inflation (ρ_π) come from : $^{\times}$Bernal (2003),†Clarida et al. (1998), *Corbo (2000), ‡Lubik and Schorfheide (2003), and $^{\circ}$Restrepo (1999). The degree of openness of the economy is the annual average of imports to GDP share for the period of time used for the estimation of ρ_π.

The relevance of our results stems from the fact that they point out the importance of considering particular features of the open economy in the design of monetary policy. Clearly both the degree of openness and the degree of exchange rate pass-through are open economy features that have been neglected by previous closed economy studies. Furthermore, both are characteristics that vary significantly among economies that follow (or followed) active interest rate rules. For instance Table 1 shows the diverse degrees of openness, measured as the share of imports to GDP, for some industrialized and developing economies that have been claimed to follow interest rate rules. In addition Campa and Goldberg (2004) and Frankel,

[4] The existence of two stationary equilibria is a consequence of combining active interest rate rules with the zero lower bound on nominal interest rates as shown by Benhabib et al. (2001b).

Parsley and Wei (2005), among others, provide empirical evidence suggesting that the degree of exchange rate pass-through into import prices not only varies across industrialized and developing economies; but it has also varied over time within these economies.

This paper is different from closed economy contributions such as Benhabib et al. (2002a) in some key aspects. First, our analysis shows that the assumption of money in the production function used by Benhabib et al. is not necessary to obtain cyclical and chaotic equilibrium dynamics under rules. We introduce money in the utility function and show that the existence of these global dynamics depends on whether consumption and money are Edgeworth complements or substitutes in the utility function. This observation was raised by Benhabib et al. (2001a) in the local determinacy of equilibrium analysis context but to the best of our knowledge it has not been raised in the context of a global equilibrium analysis.

Second, we show that if consumption and money are complements then it is possible to have "non-monotonic liquidity traps" featuring periodic and aperiodic oscillations around an extremely low interest rate steady state that is different from the target steady state. On the contrary if consumption and money are substitutes then cyclical and chaotic dynamics occur only around the target steady state. Although this case is reminiscent of the one in Benhabib et al. (2002a), it also presents a subtle difference. In their closed economy model period-3 cycles and therefore chaos always occur only for sufficiently low coefficients of relative risk aversion. Our results show that these dynamics can basically appear for any coefficient of relative risk aversion greater than one and provided that the economy is sufficiently open. In this sense open economies are more prone than closed economies to display these cyclical dynamics.

Third, we identify necessary and sufficient conditions for the design of active forward looking rules that do not generate cyclical and chaotic equilibria. For some given structural parameters, these conditions generally entail an appropriate choice of the rule's responsiveness to inflation. Although cyclical dynamics can be ruled out, liquidity traps and (hyper) inflationary paths remain viable equilibria. Previous works on monetary economics, however, have proposed solutions on how to deal with these equilibria.[5]

There are previous works in the SOE literature that have tried to identify conditions under which interest rate rules may lead to local multiple equilibria.[6] For instance, De Fiore and Liu (2003), Linnemann and Schabert (2002), and Zanna (2003), among others, discuss the importance of the degree of openness of the economy in the local equilibrium analysis. The last work also points out the key role played by the degree of the exchange rate pass-through. Although our work is related to these previous studies it is different from them in the type of equilibrium analysis that is pursued. To the extent that our work considers the zero lower bound for the interest rate and pursues a global equilibrium analysis, it is able to focus on a wider set of equilibrium dynamics.

In contrast to the above mentioned works in the SOE literature, this paper does not consider nominal price

[5] See Alstadheim and Henderson (2004), Benhabib et al. (2002b) or Christiano and Rostagno (2001) for the liquidity traps case and Obstfelf and Rogoff (1983) for the inflationary paths case.

[6] For two country models see Batini, Levine and Pearlman (2004) among others.

rigidities. In this sense it is similar to the closed economy works of Benhabib et al. (2002a,b), Carlstrom and Fuerst (2001), and Leeper (1991) among others. In Airaudo and Zanna (2005) we introduce price stickiness and study, through a Hopf bifurcation analysis, how rules can induce cyclical dynamics that never converge to the target steady state. As in the current paper, the existence of equilibrium cycles depends on some open economy features.

The remainder of this paper is organized as follows. In Section 2 we present a flexible-price model with its main assumptions. We define the open economy equilibrium and derive some basic steady state results. In Section 3 we pursue a local and a global equilibrium analyses for an active forward-looking interest rate rule. In Section 4 we study active forward-looking rules that can preclude the existence of cyclical equilibria. In Section 5 we investigate how the degree of exchange rate pass-through can affect the existence of cyclical dynamics for forward-looking rules. We pursue a sensitivity analysis to gauge the robustness of our main results in Section 6. Finally Section 7 concludes.

2 A Flexible-Price Model

2.1 The Household-Firm Unit

Consider a Small Open Economy (SOE) populated by a large number of infinitely lived household-firm units. They are identical. Each unit derives utility from consumption (c_t), real money balances (m_t^d), and not working ($1 - h_t^T - h_t^N$) according to

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left[(c_t)^{\gamma} \left(m_t^d \right)^{1-\gamma} \right]^{1-\sigma} - 1}{1-\sigma} + \psi(1 - h_t^T - h_t^N) \right\} \tag{1}$$

$$c_t = (c_t^T)^{\alpha} (c_t^N)^{(1-\alpha)} \tag{2}$$

where β, $\gamma \in (0,1)$, and $\psi, \sigma > 0$ but $\sigma \neq 1$;[7] E_0 is the expectations operator conditional on the set of information available at time 0; c_t^T and c_t^N denote the consumption of traded and non-traded goods in period t respectively; $m_t^d = \frac{M_t^d}{p_t}$ are real money balances (domestic currency money balances deflated by the Consumer Price Index, CPI, p_t, to be defined below); h_t^T and h_t^N stand for labor supplied to the production of traded and non-traded goods respectively and $\alpha \in (0,1)$ is the share of traded goods in the consumption aggregator (2). We interpret this share as a measure of the degree of (trade) openness of the economy. As α goes to zero, domestic agents do not value internationally traded goods for consumption. Then the economy is fundamentally closed. Whereas if α goes to one, non-traded goods are negligible in consumption. We refer

[7] The case of $\sigma = 1$ corresponds to the case of separability among consumption and money in the utility function. It implies no distortionary effects of transaction money demand. It can be easily shown that no equilibrium cycles occur in this case. Hence, we do not consider it in the analysis.

to this case as the completely open economy.

Although we use specific functional forms, they are general enough to convey the main message of this paper. They will allow us to show *analytically* how cyclical dynamics induced by the interest rate rule depend on the degree of openness α.[8] They also allow us to study how these dynamics are affected by whether consumption, c_t, and money, m_t, are either Edgeworth substitutes or complements. By defining $U = \frac{\left(c_t^{\gamma} m_t^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma}$ and noticing that the sign of the cross partial derivative U_{cm} satisfies $sign\left\{U_{cm}\right\} = sign\left\{(1-\sigma)\right\}$, then we can distinguish between the case of Edgeworth substitutes when $U_{cm} < 0$ $(\sigma > 1)$ or the case of complements when $U_{cm} > 0$ $(\sigma < 1)$. Moreover, given that $\gamma \in (0,1)$ and that the coefficient of relative risk aversion (CRRA) can be expressed as $\widetilde{\sigma} \equiv -\frac{U_{cc}c}{U_c} = 1 - \gamma\left(1-\sigma\right)$, then $\sigma \gtreqless 1$ implies $\widetilde{\sigma} \gtreqless 1$. As a result of this we will refer to σ as the "risk aversion parameter."

The representative unit produces traded and non-traded goods by employing labor according to the technologies

$$y_t^T = z_t^T \left(h_t^T\right)^{\theta_T} \qquad and \qquad y_t^N = z_t^N \left(h_t^N\right)^{\theta_N}, \tag{3}$$

where $\theta_T, \theta_N \in (0,1)$ and z_t^T and z_t^N are productivity shocks following stationary AR(1) stochastic processes. We assume that these shocks are the sole source of fundamental uncertainty.

As standard in the literature, we assume that the Law of One Price holds for traded goods and normalize the foreign price of the traded good to one.[9] Hence $P_t^T = \mathcal{E}_t$, where P_t^T is the domestic currency price of traded goods and $\mathcal{E}_t$ is the nominal exchange rate. This simplification together with (2) can be used to derive the Consumer Price Index (CPI)

$$p_t \equiv \frac{\left(\mathcal{E}_t\right)^{\alpha}\left(P_t^N\right)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}. \tag{4}$$

Using equation (4) and defining the gross nominal devaluation rate as $\epsilon_t \equiv \mathcal{E}_t/\mathcal{E}_{t-1}$ and the gross non-traded goods inflation rate as $\pi_t^N \equiv P_t^N/P_{t-1}^N$, we derive the gross CPI-inflation rate

$$\pi_t = \epsilon_t^{\alpha}(\pi_t^N)^{(1-\alpha)} \tag{5}$$

where $\pi_t \equiv \frac{p_t}{p_{t-1}}$. It is just a weighted average of different goods inflations whose weights are related to the degree of openness, α. The real exchange rate (e_t) is defined as the ratio of the price of traded goods (nominal exchange rate) and the price of non-traded goods

$$e_t \equiv \mathcal{E}_t/P_t^N. \tag{6}$$

[8] We could consider a CES function for aggregate consumption (2) to emphasize the fact that the intratemporal elasticity of substitution between the two types of consumption can be different from one. However, this would not affect our conclusions on the role of openness, but simply prevent us from obtaining analytical results.

[9] In Section 5 we will relax the assumption of the Law of One Price" at the consumption level by introducing non-traded distribution costs. This will allow us to model an imperfect degree of exchange rate pass-through into import prices.

Then the gross real exchange rate depreciation, $\frac{e_t}{e_{t-1}}$, can be written as

$$\frac{e_t}{e_{t-1}} = \frac{\epsilon_t}{\pi_t^N}. \tag{7}$$

As has become very common in the open economy literature such as Clarida et al. (2001) and Galí and Monacelli (2004) among others, we assume that the household-firm units have access to a complete set of internationally traded claims. In each period $t \geq 0$ the agents can purchase two types of financial assets: fiat money M_t^d and nominal state contingent claims, D_{t+1}. The latter pay one unit of (foreign) currency for a specific realization of the fundamental shocks in $t+1$. Although the existence of complete markets is a very strong assumption, it is well known that they can be approximated by a set of non-state contingent instruments featuring a wide range of maturities and indexations.[10] In this paper the assumption of complete markets serves the sole purpose of ruling out the unit root problem of the small open economy, allowing us to pursue a meaningful local determinacy of equilibrium analysis. In this way, we can compare the results from the global equilibrium analysis to the ones from the local equilibrium analysis.[11] Nevertheless our global results on the existence of cyclical and chaotic equilibrium dynamics will still hold if instead we assume incomplete markets, as we will show in Section 6.

Under complete markets the representative agent's flow constraint for each period can be written as

$$M_t^d + E_t Q_{t,t+1} D_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \mathcal{E}_t c_t^T - P_t^N c_t^N \tag{8}$$

where $E_t Q_{t,t+1} D_{t+1}$ denotes the cost of all contingent claims bought at the beginning of period t and $Q_{t,t+1}$ refers to the period-t price of a claim to one unit of currency delivered in a particular state of period $t+1$, divided by the probability of occurrence of that state and conditional of information available in period t. Constraint (8) says that the total end-of-period nominal value of the financial assets can be worth no more than the value of the financial wealth brought into the period, W_t, plus non-financial income during the period net of the value of taxes, $\mathcal{E}_t \tau_t$, and the value of consumption spending.

To derive the period-by-period budget constraint, we use the definition of the total beginning-of-period wealth, in the following period, $W_{t+1} = M_t^d + D_{t+1}$ and the fact that the period-t price of a claim that pays one unit of currency in every state in period $t+1$ is equal to the inverse of the risk-free gross nominal interest rate; that is $E_t Q_{t,t+1} = \frac{1}{R_t}$. From this, the definition of W_{t+1} and (8) we obtain

$$E_t Q_{t,t+1} W_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \frac{R_t - 1}{R_t} M_t^d - \mathcal{E}_t c_t^T - P_t^N c_t^N. \tag{9}$$

[10] See Angeletos (2002).
[11] See Schmitt-Grohé and Uribe (2003).

The representative unit is also subject to a Non-Ponzi game condition

$$\lim_{j \to \infty} E_t q_{t+j} W_{t+j} \geq 0 \tag{10}$$

at all dates and under all contingencies where q_t represents the period-zero price of one unit of currency to be delivered in a particular state of period t divided by the probability of occurrence of that state and given information available at time 0. It satisfies $q_t = Q_1 Q_2 .....Q_t$ with $q_0 \equiv 1$.

The problem of the representative household-firm unit reduces to choosing the sequences $\{c_t^T,\ c_t^N,\ h_t^T,\ h_t^N,\ M_t^d,\ W_{t+1}\}_{t=0}^{\infty}$ in order to maximize (1) subject to (2), (3), (9) and (10), given W_0 and the time paths of R_t, $\mathcal{E}_t$, P_t^N, Q_{t+1} and τ_t. Note that since the utility function specified in (1) implies that the preferences of the agent display non-satiation then both constraints (9) and (10) hold with equality.

The first order conditions correspond to (9) and (10) with equality and

$$\alpha\gamma \left(c_t^T\right)^{\alpha\gamma(1-\sigma)-1} \left(c_t^N\right)^{(1-\alpha)\gamma(1-\sigma)} \left(m_t^d\right)^{(1-\gamma)(1-\sigma)} = \lambda_t \tag{11}$$

$$\frac{\alpha c_t^N}{(1-\alpha)c_t^T} = e_t \tag{12}$$

$$\frac{\lambda_t}{e_t}\theta_N \left(h_t^N\right)^{(\theta_N-1)} = \psi = \lambda_t \theta_T \left(h_t^T\right)^{(\theta_T-1)} \tag{13}$$

$$m_t^d = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\left(\frac{R_t}{R_t-1}\right) c_t^T e_t^{1-\alpha} \tag{14}$$

$$\frac{\lambda_t}{\mathcal{E}_t} Q_{t,t+1} = \beta \frac{\lambda_{t+1}}{\mathcal{E}_{t+1}} \tag{15}$$

where $\lambda_t/\mathcal{E}_t$ is the Lagrange multiplier of the flow budget constraint.

The interpretation of the first order conditions is straightforward. Equation (11) is the usual intertemporal envelope condition that makes the marginal utility of consumption of traded goods equal to the marginal utility of wealth measure in terms of traded goods (λ_t). Condition (12) implies that the marginal rate of substitution between traded and non-traded goods must be equal to the real exchange rate. Condition (13) equalizes the value of the marginal products of labor in both sectors. Equation (14) represents the demand for real money balances. And finally condition (15) describes a standard pricing equation for one-step-ahead nominal contingent claims for each period t and for each possible state of nature.

2.2 The Government

The government issues two nominal liabilities: money, M_t^s, and a one period risk-free domestic bond, B_t^s, that pays a gross risk-free nominal interest rate R_t. We assume that it cannot issue or hold state contingent claims. It also levies taxes, τ_t, pays interest on its debt, $(R_t - 1)B_t^s$, and receives revenues from seigniorage.

Then the government's budget constraint can be written as $L_t^s = R_{t-1}L_{t-1}^s - (R_{t-1}-1)M_{t-1}^s - \mathcal{E}_t\tau_t$, where $L_t^s = M_t^s + B_t^s$.

We proceed to describe the fiscal and monetary policies. The former corresponds to a generic Ricardian policy: the government picks the path of the lump-sum transfers, τ_t, in order to satisfy the intertemporal version of its budget constraint in conjunction with the transversality condition $\lim_{t\to\infty} \frac{L_t^s/\mathcal{E}_t}{\prod_{k=0}^{t}\left(\frac{R_k}{\epsilon_{k+1}}\right)} = 0$. The latter is described as an interest rate feedback rule whereby the government sets the nominal interest rate, R_t, as a continuous and increasing function of the deviation of the expected future CPI-inflation rate, $E_t(\pi_{t+1})$, from a target, π^*.[12] For analytical and computational purposes, as in Benhabib et al. (2002a) and Christiano and Rostagno (2001), we use the following specific non-linear rule[13]

$$R_t = \rho(E_t\pi_{t+1}) \equiv 1 + (R^* - 1)\left(\frac{E_t\pi_{t+1}}{\pi^*}\right)^{\frac{A}{R^*-1}} \tag{16}$$

where $R^* = \pi^*/\beta$ and R^* corresponds to the target interest rate. (16) always satisfies the zero bound on the nominal interest rate, i.e. $R_t = \rho(E_t\pi_{t+1}) > 1$. In addition we assume that the government responds aggressively to inflation. This means that at the inflation target, the rule's elasticity to inflation $\xi \equiv \frac{\rho'(\pi^*)\pi^*}{\rho(\pi^*)} = \frac{A}{R^*}$ is strictly bigger than 1. Following Leeper (1991) we call rules with this property active rules.

Assumption 0: $\xi = \frac{A}{R^*} > 1$. *That is, the rule is active.*

2.3 International Capital Markets

Besides complete markets there is free international capital mobility. Then the no-arbitrage condition $Q_{t,t+1}^w = Q_{t,t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}$ holds, where $Q_{t,t+1}^w$ refers to the period-t foreign currency price of a claim to one unit of foreign currency delivered in a particular state of period $t+1$ divided by the probability of occurrence of that state and conditional of information available in period t.

Furthermore under the assumption of complete markets a condition similar to (15) must hold from the maximization problem of the representative agent in the Rest of The World (ROW). That is, $\frac{\lambda_t^w}{P_t^{Tw}}Q_{t,t+1}^w = \frac{\lambda_{t+1}^w}{P_{t+1}^{Tw}}\beta^w$ where λ_t^w represents the marginal utility of nominal wealth in the ROW, β^w denotes the subjective discount rate of the ROW and P_t^{Tw} is the foreign price of traded goods. Since we normalize this price to one ($P_t^{Tw} = 1$) then assuming that $\beta^w = \beta$ leads to $\lambda_t^w Q_{t,t+1}^w = \lambda_{t+1}^w\beta$.

Combining this last equation, with condition (15) and the fact that $P_t^T = \mathcal{E}_t$ yields $\frac{\lambda_{t+1}}{\lambda_t} = \frac{\lambda_{t+1}^w}{\lambda_t^w}$, which holds at all dates and under all contingencies. This condition implies that the domestic marginal utility of wealth is proportional to its foreign counterpart: $\lambda_t = \Lambda\lambda_t^w$where Λ refers to a constant parameter that determines the wealth difference between the SOE and the ROW. From the perspective of a SOE, λ_t^w can be taken as an exogenous variable. For simplicity we assume that λ_t^w is constant and equal to λ^w. As a result

[12] In the sensitivity analysis presented in Section 6 we also study contemporaneous and backward-looking rules.
[13] Cycles and chaos would also occur if the interest rate rule was a linear function of inflation.

of this λ_t becomes a constant. Then

$$\lambda_t = \lambda = \Lambda\lambda^w. \tag{17}$$

This allows us to write condition (15) as $Q_{t,t+1} = \frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\beta = \frac{\beta}{\epsilon_{t+1}}$ that together with $E_t Q_{t,t+1} = \frac{1}{R_t}$ yields

$$R_t = \beta^{-1}\left[E_t \frac{1}{\epsilon_{t+1}}\right]^{-1} \tag{18}$$

which is similar to an uncovered interest parity condition.

2.4 The Definition of Equilibrium

In this paper we will focus on perfect foresight equilibria. In other words, we assume the all the agents in the economy, including the government, forecast correctly all the anticipated variables. Hence for any variable x_t we have that $E_t x_{t+j} = x_{t+j}$ with $j \geq 0$ implying that we can drop the expectation operator in the previous equations. For instance, under perfect foresight, condition (18) becomes

$$R_t = \beta^{-1}\epsilon_{t+1} \tag{19}$$

that corresponds to the typical uncovered interest parity condition as long as β^{-1} represents the foreign international interest rate.[14]

In order to provide a definition of the equilibrium dynamics subject of our study, we find a reduced non-linear form of the model. To do so we use the definitions (5) and (7) together with conditions (11)-(14), (17), (19), and the market clearing conditions for money and the non-traded good, $M_t^d = M_t^s = M_t$ and $y_t^N = \left(h_t^N\right)^{\theta_N} = c_t^N$, to obtain

$$\pi_{t+1}\left(\frac{R_{t+1}}{R_{t+1}-1}\right)^{\chi} = \left(\frac{R_t}{R_t-1}\right)^{\chi}\beta R_t \tag{20}$$

where

$$\chi = \frac{(\sigma-1)(1-\alpha)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N + \alpha(1-\theta_N)] + (1-\alpha)(1-\theta_N)}. \tag{21}$$

Combining (16) and (20) and dropping the expectation operator yields

$$\left(\frac{R_{t+1}}{R_{t+1}-1}\right)^{\chi} = \frac{R_t}{R^*}\left(\frac{R^*-1}{R_t-1}\right)^{\frac{R^*-1}{A}}\left(\frac{R_t}{R_t-1}\right)^{\chi} \tag{22}$$

which corresponds to the reduced non-linear form of the model that can be used to pursue the local and global determinacy of equilibrium analyses.[15] We use this equation in order to provide a definition of a

[14] This holds by the previous analysis since $E_t Q_{t,t+1}^w = \frac{1}{R_t^w} = \beta$.

[15] This equation abstracts from possible effects that fiscal policies can have on the equilibrium dynamics of the economy. The reason is that we have assumed a Ricardian fiscal policy under which the intertemporal version of the government's budget constraint and its transversality condition will be always satisfied.

Perfect Foresight Equilibrium (PFE).

Definition 1 *Given the target R^* and the initial condition R_0, a Perfect Foresight Equilibrium (PFE) is a deterministic process $\{R_t\}_{t=0}^{\infty}$, with $R_t > 1$ for any t, that satisfies equation (22) if the interest rate rule is forward-looking.*

Although Definition 1 is stated exclusively in terms of the nominal interest rate (R_t), it must be clear that multiple perfect foresight equilibrium solutions to (22) imply real local and/or global indeterminacy of all the endogenous variables.[16] In other words the indeterminacy of the nominal interest rate implies real indeterminacy in our model because of the non-separability in the utility function between money and consumption.[17]

In order to pursue the equilibrium analysis we need to identify the steady state(s) of the economy. From (5), (7) and (20) we obtain that at the steady-state(s), $\pi^{Nss} = \epsilon^{ss} = \pi^{ss}$, and $R^{ss} = \pi^{ss}/\beta$. Using these and the rule (16) we have that

$$(R^* - 1)^{\frac{R^*-1}{A}} R^{ss} = R^* (R^{ss} - 1)^{\frac{R^*-1}{A}} . \tag{23}$$

Clearly $R^{ss} = R^* > 1$ is a solution to (23), and therefore a feasible steady state. But if the rule is active at R^*, that is if $\xi = \frac{A}{R^*} > 1$, then another lower steady state $R^L \in (1, R^*)$ exists and it is unique. At this low steady state the elasticity of the rule to inflation satisfies $\xi = \frac{A}{R^L} < 1$. The following proposition formalizes the existence of the low steady state R^L.

Proposition 1 *If $\frac{A}{R^*} > 1$ (an active rule) and $R^{ss} > 1$ (the zero lower bound) then there exists a solution $R^{ss} = R^L \in (1, R^*)$ solving (23) besides the trivial solution $R^{ss} = R^*$.*

Proof. See the Appendix. ■

The existence of two steady states plays a crucial role in the derivation of our results as in the closed economy model of Benhabib et al. (2002a). As a matter of fact, the steady state equation (23) of our SOE is identical to theirs. It is independent of the non-policy structural parameters. Hence no fold bifurcation (i.e. appearance/disappearance of steady states) occurs because of changes in these parameters. What distinguishes our model from theirs are the equilibrium dynamics off the two steady states. This is a consequence of the following two features of our model. First, by introducing traded and non-traded goods we present an economy with two sectors that although homogeneous in terms of price setting behavior (both feature flexible prices), are fundamentally different in terms of the degree of openness to international trade. As we will see

[16] By indeterminacy we refer to a situation where one or more real variables are not pinned down by the model. We use the terms indeterminacy and multiple equlibria interchangeably. The same comment applies to determinacy and a unique equilibrium.

[17] In fact, for any given R_t, by simply manipulating equations (5), (7), (11)- (14), (17), (19), and the market clearing conditions for money and the non-traded good, we can obtain all the remaining real endogenous variables. For instance the labor allocated for the production of the non-traded good and the real exchange rate can be expressed as functions of the gross interest rate only, $h_t^N = h^N(R_t)$ and $e_t = e(R_t)$, respectively, whereas from the market clearing condition, $c_t^N = (h_t^N)^{\theta_N}$, we obtain $c_t^N = c^N(R_t)$.

below this degree, measured by α, will influence the equilibrium dynamics. Second, by considering money in the non-separable utility function we are able to study how the existence of cyclical dynamics depend on whether money and consumption are either Edgeworth complements ($\sigma < 1$) or substitutes ($\sigma > 1$).

In the analysis to follow we will study how α and σ affect the local and global equilibrium dynamics in our SOE model while keeping constant the other structural parameters (β, γ and θ_N) and the policy parameters (A and R^*). This will allow us to compare economies that implement the same monetary rule but differ in the degree of openness α and the risk aversion parameter σ. To accomplish this goal we will proceed in two steps. First we will analyze how these dynamics are affected by the composite parameter χ defined in (21). Second by taking into account the dependence of χ on both α and σ we will unveil the effect of the degree of openness and the risk aversion parameter on the existence of local and global dynamics (cycles and chaos). In this sense we will regard χ as a function of α and σ.[18] That is $\chi(\alpha, \sigma)$. For the second step we will use and refer to (21), and to Lemmata 4 and 5 in the Appendix. In turn, these Lemmata and subsequent propositions will use the following definitions

$$\chi_{\max} \equiv \frac{(1-\gamma)(1-\theta_N)}{\theta_N} \in (0, +\infty) \qquad \chi_{\min} \equiv -(1-\gamma) \in (-1, 0) \tag{24}$$

$$\mu(\sigma) = \frac{(\sigma-1)(1-\gamma)(1-\theta_N)}{\sigma\theta_N + (1-\theta_N)} \tag{25}$$

where $\chi_{\max}$ and $\chi_{\min}$ are considered scalars and $\mu(\sigma)$ is considered a function of σ.

Definition 2 *Using (24) and (25) define the scalars* $\sigma^i \equiv \frac{1-\frac{\Upsilon^i}{\chi_{\min}}}{1-\frac{\Upsilon^i}{\chi_{\max}}}$ *and the functions* $\alpha^i(\sigma) \equiv \frac{1-\frac{\Upsilon^i}{\mu(\sigma)}}{1-\frac{\Upsilon^i}{\chi_{\min}}}$ *for* $i = w, k, f, d$ *where* $\Upsilon^w \equiv R^L\left(1 - \frac{R^*-1}{A}\right) - 1 < 0$, $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right)(R^*-1) > 0$, $\Upsilon^f \equiv \frac{\Upsilon^w}{2}$ *and* $\Upsilon^d \equiv \frac{\Upsilon^k}{2}$; *and the functions* $\alpha^i(\sigma)$ *are characterized in Lemma 4 when* $\sigma > 1$ *and in Lemma 5 when* $\sigma \in (0, 1)$.

3 Equilibrium Dynamics under Forward-Looking Rules

Our study of forward-looking rules is motivated by the evidence provided by Clarida et al. (1998) for industrialized economies and by Corbo (2000) for developing economies. Both works suggest that these economies have followed forward-looking rules.

In order to derive analytical results for both the local and the global equilibrium analyses we will assume that the constant parameters γ, θ_N, A and R^* satisfy the following assumptions.[19],[20]

[18] χ also depends on γ and θ_N. However we will not pursue any bifurcation analysis with respect to these parameters.

[19] Assumption 1 is necessary and sufficient for the existence of ranges of α and σ where local indeterminacy occurs, as well as for the existence of a flip bifurcation frontier in the case of $\sigma > 1$. Assumption 2 allows monotonic liquidity traps for low values of σ. It could therefore be dropped without affecting the possibility of cycles around the passive steady state. Assumption 3 - which basically requires enough separation between the two steady states - is useful in proving the existence of a flip bifurcation frontier in the case of $\sigma < 1$. Extensive algebra (not reported here for reasons of space) shows that Assumption 3 holds if, for a given target R^*, the rule is sufficiently active.

[20] The calibration exercise that we present below suggests that these assumptions are not unrealistic. For given monetary

Assumption 1: $\chi_{\max} > \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$.

Assumption 2: $\chi_{\min} < \frac{1-R^*}{A}$.

Assumption 3: $R^* - 1 > A\left(R^L - 1\right)$.

3.1 The Local Determinacy of Equilibrium Analysis

The local determinacy of equilibrium analysis for forward-looking rules is pursued by log-linearizing equation (22) around the target steady state R^*, yielding

$$\hat{R}_{t+1} = \left[1 + \frac{\frac{R^*}{A} - 1}{\frac{\chi}{R^*-1}}\right]\hat{R}_t. \tag{26}$$

Since R_t is a non-predetermined variable, studying local determinacy is equivalent to finding conditions that make the linear difference equation (26) explosive. The next Lemma shows how local equilibrium determinacy depends on χ.

Lemma 1 *Define* $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right) > 0$ *and consider* $\chi \in \mathbb{R}$. *Suppose the government follows an active forward-looking rule then: 1) the equilibrium is locally unique if* $\chi < \Upsilon^d$; *2) there exist locally multiple equilibria if* $\chi > \Upsilon^d$.

Proof. See the Appendix. ■

These simple determinacy of equilibrium conditions for χ can be reinterpreted in terms of the degree of openness α and the risk aversion parameter σ in the following Proposition.

Proposition 2 *Consider* σ^d *and* $\alpha^d(\sigma)$ *in Definition 2 where* $\sigma^d > 1$ *and* $\alpha^d : (1, +\infty) \rightarrow (-\infty, 1)$. *Suppose that the government follows an active forward-looking rule.*

1. *There exists a locally unique equilibrium*

   (a) *if consumption and money are Edgeworth complements, i.e.* $\sigma \in (0, 1)$, *and for any degree of openness, i.e.* $\alpha \in (0, 1)$;

   (b) *if consumption and money are Edgeworth substitutes, i.e.* $\sigma > 1$, *and the economy is sufficiently open satisfying* $\alpha > \alpha^d_{\min}$; *where* $\alpha^d_{\min} \equiv \max\left\{0, \alpha^d(\sigma)\right\}$ *is positive and strictly increasing for* $\sigma > \sigma^d$, *but constant and equal to zero for any* $\sigma \in \left(1, \sigma^d\right]$.

2. *There exist locally multiple equilibria if consumption and money are Edgeworth substitutes satisfying* $\sigma > \sigma^d$, *and the economy is sufficiently closed satisfying* $\alpha \in (0, \alpha^d(\sigma))$.

---

policy parameters and for given θ, both Assumptions 1 and 2 imply a minimum share of real balances in utility. We can in fact rewrite Assumption 1 as $(1 - \gamma) > \frac{1}{2}\frac{\theta_N}{(1-\theta_N)}\left(\frac{R^*-1}{A}\right)(A - R^*)$. For the calibration used in Table 2, the right hand side of this inequality is about 0.006, while we set $1 - \gamma = 0.03$ consistently with the literature. Similarly, Assumption 2 can be written as $1 - \gamma > \frac{R^*-1}{A}$, with the right hand side equal to 0.007 in our calibration.

Proof. See the Appendix. ■

The results of this proposition show the importance of α and σ in the local characterization of the equilibrium. In a nutshell, active forward-looking rules guarantee local uniqueness in the following cases: when regardless of the degree of openness the risk aversion parameter σ is sufficiently low; and when the economy is sufficiently open for high values of σ.[21] It is in this sense that an active rule might be viewed as stabilizing. Local equilibrium determinacy, however, does not guarantee global equilibrium determinacy. To see this we pursue a global characterization of the equilibrium dynamics in the following subsection.

3.2 The Global Determinacy of Equilibrium Analysis

To pursue the global equilibrium analysis we rewrite equation (22) as the forward mapping $R_{t+1} = f(R_t)$ where

$$f(R_t) \equiv \frac{1}{1 - J(R_t)^{\frac{1}{\chi}}} \tag{27}$$

and

$$J(R_t) \equiv \frac{R^*}{[R^* - 1]^{\frac{R^*-1}{A}}} \frac{[R_t - 1]^{\chi + \frac{R^*-1}{A}}}{R_t^{1+\chi}}. \tag{28}$$

Then the global analysis corresponds to studying the global PFE dynamics that satisfy $R_{t+1} = f(R_t)$ given an initial condition $R_0 > 1$ and subject to the zero-lower-bound condition $f^n(R_0) > 1$ for any $n \geq 1$. The types of cyclical and chaotic dynamics we will be referring to are those conforming to the following definitions.

Definition 3 ***Period-n cycle.*** *A value "R" is a point of a period-n cycle if it is a fixed point of the n-th iterate of the mapping $f(.)$, i.e. $R = f^n(R)$, but not a fixed point of an iterate of any lower order. If "R" is such, we call the sequence $\{R, f(R), f^2(R), ..., f^{n-1}(R)\}$ a period-n cycle.*

Definition 4 ***Topological chaos.*** *The mapping $f(.)$ is topologically chaotic if there exists a set "S" of uncountable many initial points, belonging to its domain, such that no orbit that starts in "S" will converge to one another or to any existing period orbit.*

The global analysis requires a full characterization of $f(.)$ in (27) not only around its stationary solutions, like in the local analysis, but over its entire domain. In this characterization it is useful to take into account that a necessary condition for the existence of cyclical dynamics in continuously differentiable maps is that the mapping $f(.)$ slopes negatively at either one of the two steady states.[22] Lemma 6 in the Appendix investigates the properties of the mapping $f(.)$ showing that they depend critically on χ. Here we only

[21] In fact this result is more general since a quick inspection of Proposition 2 suggests that if the the economy is very open, an active rule leads to a unique equilibrium regardless of the values of the other structural parameters.

[22] See Lorenz (1993).

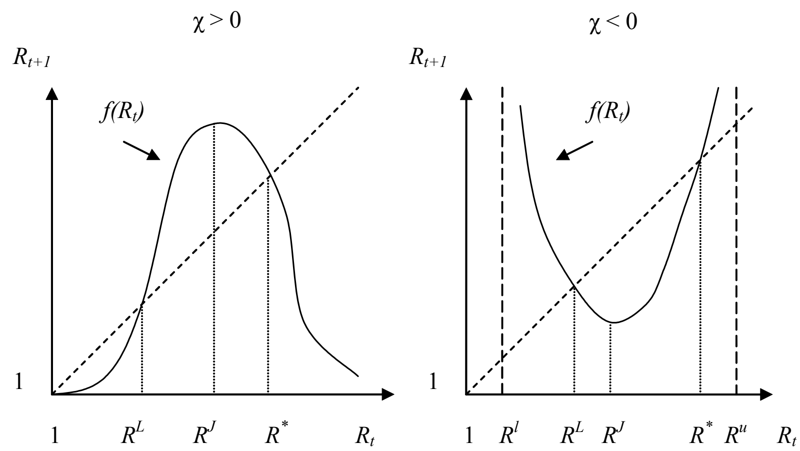

Figure 1: This Figure shows the mapping $R_{t+1} = f(R_t)$ for $\chi > 0$, and $\chi < 0$ but $\chi + \frac{R^*-1}{A} > 0$. R_t denotes the nominal interest rate. A formal characterization of this mapping is provided in Lemma 6 in the Appendix.

provide a big picture of the analysis. First of all, the Lemma specifies conditions under which $f(.)$ satisfies the zero-lower-bound requirement. Second, it makes use of the following conditions

$$sign\{f'(R_t)\} = sign\left\{\frac{J'(R_t)}{\chi}\right\} \qquad \text{and} \qquad sign\{J'(R_t)\} = sign\left\{\frac{1+\chi}{1-\frac{R^*-1}{A}} - R_t\right\} \tag{29}$$

which imply that for $\chi \neq 0$ then $R^J \equiv \frac{1+\chi}{1-\frac{R^*-1}{A}}$ is a critical point of $f(.)$ as long as $R^J > 1$. With these conditions the Lemma shows that the mapping $R_{t+1} = f(R_t)$ is always single-peaked for $\chi > 0$ whereas for $\chi < 0$ it is single-troughed only if $\chi + \frac{R^*-1}{A} > 0$.

Figure 1 displays a graphical representation of the cases where the equilibrium mapping $R_{t+1} = f(R_t)$ has a critical point between the two steady states. The right panel considers the case of $\chi < 0$ and $\chi + \frac{R^*-1}{A} > 0$, while the left panel the case of $\chi > 0$. In the left one, $f(.)$ always satisfies the equilibrium conditions for any $R_t \in (1, +\infty)$ and crosses the 45 degree line twice at $R^L > 1$ and $R^* > R^L$ (the two steady states). Furthermore, $\lim_{R_t \to 1} f(R_t) = \lim_{R_t \to +\infty} f(R_t) = 0$ and there is a maximum at $R^J \in (R^L, R^*)$. In the right panel, all equilibrium conditions are satisfied only within a subset $(R^l, R^u) \subset (1, +\infty)$ defined in Lemma 6. Within that set, $f(.)$ crosses the 45 degree line at $R^L > 1$ and $R^* > R^L$, as in the previous case, but now $R^J \in (R^L, R^*)$ is a minimum and $\lim_{R_t \to R^{l+}} f(R_t) = \lim_{R_t \to R^{u-}} f(R_t) = +\infty$. These are clearly the cases in which we are interested, as they imply a negative derivative of $f(.)$ at either one of the two steady states.

Figure 1 together with Lemma 6 suggest that depending on the sign of χ, cycles may appear around either the active steady state or the passive steady state. Hence we proceed to look for flip bifurcation thresholds for χ i.e. critical values of χ that determine a change in the stability properties of the steady state where the map $f(.)$ is negatively sloped. If the steady state is stable, any equilibrium orbit, that starts in a map invariant set centered around this state, will asymptotically converge to the steady state itself, monotonically or spirally. Thus equilibrium cycles are impossible. On the contrary if the steady state is unstable, such orbit will keep oscillating within the map invariant set and either it converge to a stable n-period cycle, or not converge at all displaying aperiodic but bounded dynamics (chaotic equilibrium paths). We first consider the case of $\chi \in \left(R^L\left(1-\frac{R^*-1}{A}\right)-1, 0\right)$ and show that endogenous cyclical dynamics of period 2 can occur around the passive steady state.

Lemma 2 *Let* $\Upsilon^w \equiv R^L\left(1-\frac{R^*-1}{A}\right)-1$, *and define the points* $\underline{R} \equiv f\left(R^J\right)$, *i.e. the image of* R^J *(the critical point of* $f(.)$*), and* $\widetilde{R} \equiv f^{-1}\left(R^*\right)$, *i.e. the inverse image of the high steady state. Consider* $\chi \in (\Upsilon^w, 0)$ *and assume that* $f_{\min} \equiv f\left(R^J\right) \geq \widetilde{R}$.[23] *Then:*

1. $R^L \in \left(\underline{R}, R^J\right)$ *and* $R^L > \widetilde{R}$;
2. *the set* $[\underline{R}, R^*]$ *is invariant under the mapping* $f(.)$ *and attractive for any* $R_t \in \left[\widetilde{R}, \underline{R}\right]$ *where* $\widetilde{R} < \underline{R}$;
3. *period-2 cycles within* $[\underline{R}, R^*]$ *and centered around the passive steady state occur when* $\chi \in \left(\frac{\Upsilon^w}{2}, 0\right)$.

Proof. See Appendix. ■

For the case of $\chi \in \left(0, \left(1-\frac{R^*}{A}\right)(R^*-1)\right)$ endogenous cyclical dynamics of period 2 exist around the active steady state, instead.

Lemma 3 *Let* $\Upsilon^k \equiv \left(1-\frac{R^*}{A}\right)(R^*-1)$, *and define the points* $\overline{R} \equiv f\left(R^J\right)$, *i.e. the image of* R^J *(the critical point of* $f(.)$*) and* $\widetilde{R} \equiv f^{-1}\left(R^*\right)$, *i.e. the inverse image of the high steady state. Consider* $\chi \in \left(0, \Upsilon^k\right)$ *and assume* $f_{\max} \equiv f\left(R^J\right) \leq \widetilde{R}$. *Then:*

1. $R^* \in \left(R^J, \overline{R}\right)$ *and* $\widetilde{R} > R^L$;
2. *the set* $\left[R^L, \overline{R}\right]$ *is invariant under the mapping* $f(.)$ *and attractive for any* $R_t \in (\overline{R}, \widetilde{R}]$ *where* $\overline{R} < \widetilde{R}$;
3. *period-2 cycles within* $\left[R^L, \overline{R}\right]$ *and centered around the active steady state occur when* $\chi \in \left(0, \frac{1}{2}\Upsilon^k\right)$.

Proof. See Appendix. ■

Similarly to the local determinacy analysis, the conditions for endogenous cycles derived in terms of χ can be easily translated into conditions described in terms of the degree of openness α and the risk aversion parameter σ. The next Proposition accomplishes this goal.

[23] This assumption rules out explosive paths for initial conditions between the two steady states. It is similar in flavor to the one used by Boldrin et al. (2001) and Matsuyama (1991). If $f_{\min} = f\left(R^J\right) < \widetilde{R}$ there would not be a non-trivial mapping-invariant set. This case displays a different type of multiplicity. It can be shown that there exists a set of points within the set $[\underline{R}, R^*]$ that leave such a set after a finite number of iterations, and settle to an exploding path diverging from the active steady state (see Matsuyama, 1991).

Proposition 3 *Suppose that the government follows an active forward-looking rule.*

1. *Consider* σ^f *and* $\alpha^f(\sigma)$ *in Definition 2 where* $\sigma^f \in (0,1)$ *and* $\alpha^f : (0,1) \to (-\infty, 1)$ *and assume that consumption and money are Edgeworth complements, i.e.* $\sigma \in (0,1)$. *Then period-2 equilibrium cycles exist around the passive steady state if the economy is sufficiently open satisfying* $\alpha > \alpha^f_{\min}$, *where* $\alpha^f_{\min} \equiv \max\left\{0, \alpha^f(\sigma)\right\}$ *is positive and strictly decreasing for* $\sigma \in \left(0, \sigma^f\right)$, *but constant and equal to zero for any* $\sigma \in \left[\sigma^f, 1\right)$.

2. *Consider* σ^d *and* $\alpha^d(\sigma)$ *in Definition 2 where* $\sigma^d > 1$ *and* $\alpha^d : (1, +\infty) \to (-\infty, 1)$ *and assume that consumption and money are Edgeworth substitutes, i.e.* $\sigma > 1$. *Then period-2 equilibrium cycles exist around the active steady state if the economy is sufficiently open satisfying* $\alpha > \alpha^d_{\min}$, *where* $\alpha^d_{\min} \equiv \max\left\{0, \alpha^d(\sigma)\right\}$ *is positive and strictly increasing for* $\sigma > \sigma^d$, *but constant and equal to zero for any* $\sigma \in \left(1, \sigma^d\right]$.

Proof. See the Appendix. ■

Proposition 3 is one of the main contributions of our paper. It states that at either sufficiently low or sufficiently high risk aversion coefficients (σ), forward looking rules are more prone to induce endogenous cyclical dynamics the more open the economy; while for σ sufficiently close to 1, but different from it, forward looking rules will lead to those dynamics regardless of the degree of openness.[24]

The second point of this Proposition is also useful to make the following interesting argument. The sufficient condition for the existence of period-2 cycles when $\sigma > 1$ and the local determinacy condition stated in Point 1b) of Proposition 2 are exactly the same. This is a clear example of why local analysis can be misleading. By log-linearizing around the steady state, local analysis implicitly assumes that any path starting arbitrarily close to it and diverging cannot be part of an equilibrium since it will eventually explode and thus violate some transversality condition. This is not the case here as the global analysis proves that the true non-linear map features a bounded map-invariant and attractive set around the active steady state. It is then possible to have equilibrium paths that starting arbitrarily close to the target steady state will converge to a stable deterministic cycle.

Given the functional form of $f(.)$ in (27) it is very difficult to derive analytical conditions for α and σ under which forward-looking rules induce either cycles of period higher than 2 or chaotic dynamics. Therefore in order to shed some light on the role of both α and σ in delivering these dynamics, as well as to find some empirical confirmation of our analytical results, we pursue a simple calibration-simulation exercise.

[24] As mentioned before, the discountinuity of the model with respect to $\sigma = 1$ corresponds to a utility function that is separable in money and consumption. In this case it is possible to show that no cyclical dynamics can occur.

Table 2: Parametrization

θ_N	β	π^*	R^*	$1-\gamma$	$\frac{A}{R^*}$
0.56	0.99	$1.031^{\frac{1}{4}}$	$1.072^{\frac{1}{4}}$	0.03	2.24

We set the time unit to be a quarter and use Canada as the representative economy. From Mendoza (1995) we borrow the labor income shares for the non-traded sector and set $\theta_N = 0.56$. The steady-state inflation, π^*, and the steady state nominal interest rate, R^*, are calculated as the average of the CPI-inflation and the Central Bank discount rate between 1983-2002. This yields $\pi^* = 1.031^{\frac{1}{4}}$ and $R^* = 1.072^{\frac{1}{4}}$. Then the subjective discount rate is determined by $\beta = \pi^*/R^*$. We use the estimate of Lubik and Schorfheide (2003) for the Canadian interest rate response coefficient to inflation which corresponds to $\frac{A}{R^*} = 2.24$. Estimates for the share of expenditures on real money balances, $1-\gamma$, for Canada are not available. For the US, estimates of this parameter vary from 0.0146 to 0.039 depending on the specification of the utility function and method of estimation. We set $1-\gamma$ equal to 0.03 that is in line with the estimates provided by previous works.[25] Table 2 gathers the parametrization.

As in the analytical study, in the simulation exercise we vary α and σ keeping the remaining parameters as in Table 2. Nevertheless, an estimate of α for Canada can be obtained from the average imports to GDP share during 1983-2002, yielding $\alpha = 0.31$. In contrast, obtaining an estimate of σ is more difficult. As explained before, σ is related to the CRRA coefficient $\widetilde{\sigma}$ through $\widetilde{\sigma} = 1 - \gamma(1-\sigma)$ which spans over a wide range. The RBC literature usually sets $\widetilde{\sigma} = 2$. This value and $1-\gamma = 0.03$ imply $\sigma = 2.03$. Since the value of σ determines whether consumption and real money balances are either Edgeworth substitutes or complements we will use different values for the CRRA $\widetilde{\sigma}$. For instance, we let $\widetilde{\sigma} \in \{0.8, 1.5, 2, 2.5\}$ which in tandem with $1-\gamma = 0.03$ leads to $\sigma \in \{0.79, 1.51, 2.03, 2.55\}$ respectively.

Given $\sigma \in \{0.79, 2.03\}$ which corresponds to CRRA of $\tilde{\sigma} \in \{0.8, 2\}$ we construct Figure 2. It presents the bifurcation (or orbit) diagrams for the degree of openness α. The left panel considers the case when money and consumption are complements by setting $\tilde{\sigma} = 0.8$. The right panel corresponds to the case when they are substitutes as $\tilde{\sigma} = 2$. With $\alpha \in (0,1)$ on the horizontal axis and $R_t > 1$ on the vertical axis, the solid lines in the diagram correspond to stable solutions of period n. The left and right panels of the figure show how by increasing α an active forward-looking rule can drive the economy into period-2 cycles, period-4 cycles,...period-n cycles and eventually chaotic dynamics. Starting from $\alpha = 0$, both panels show that for low degrees of openness the economy, that is described by the mapping $R_{t+1} = f(R_t)$ in (27), always settles on a stable steady state equilibrium after a long enough series of iterations. It settles on the passive steady state, R^L, for $\tilde{\sigma} = 0.8$ and on the active steady state, R^*, for $\tilde{\sigma} = 2$. Once α reaches some threshold a

[25] See Holman (1998) among others.

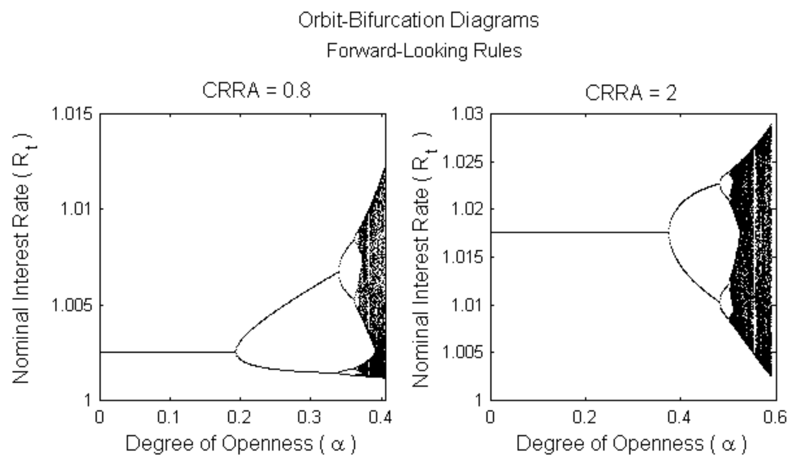

Figure 2: Orbit-bifurcation diagrams for the degree of openness, α. R_t denotes the nominal interest rate. The diagrams show the set of limit points as a function of α, under two different coefficients of risk aversion (CRRA) $\tilde{\sigma} = 0.8$ and $\tilde{\sigma} = 2$, and under an active forward-looking rule. Depending on α, an active forward-looking rule may drive the economy into period-2 cycles, period-4 cycles,...period-n cycles and even chaotic dynamics.

stable period-2 cycle appears, as indicated by the first split into two branches in both panels. As we increase α further in both panels, both branches split again yielding a period-4 stable cycle. A cascade of further period doubling occurs as we keep increasing α, yielding cycles of period-8, period-16 and so on. Finally for sufficiently high α values, the rule produces aperiodic chaotic dynamics, i.e. the attractor of the map (27) changes from a finite to an infinite set of points.

From Figure 2 we also see that when consumption and money are complements then cyclical and chaotic dynamics occur around the passive steady state; whereas if they are substitutes then these dynamics appear around the target active steady state. Nevertheless for both cases, forward-looking rules are more prone to induce cycles and chaos the more open the economy.

In order to summarize and compare the results of the local and global determinacy of equilibrium analyses we construct Figure 3. It shows the combinations of the degree of openness and the risk aversion parameter, α and σ, for which there is local and/or global (in)determinacy. For $\sigma \geq 0$ and $\alpha \in [0, 1]$ we plot two threshold frontiers: the flip bifurcation frontier for period-2 cycles around the passive steady state, $\alpha^f(\sigma)$ and the frontier $\alpha^d(\sigma)$ for both local determinacy and period-2 cycles around the active steady state. Regions featuring a locally unique equilibrium are labeled with a "U", while those featuring locally multiple equilibria are labeled with an "M". Clearly, "U" appears everywhere but below the curve $\alpha^d(\sigma)$ implying that local

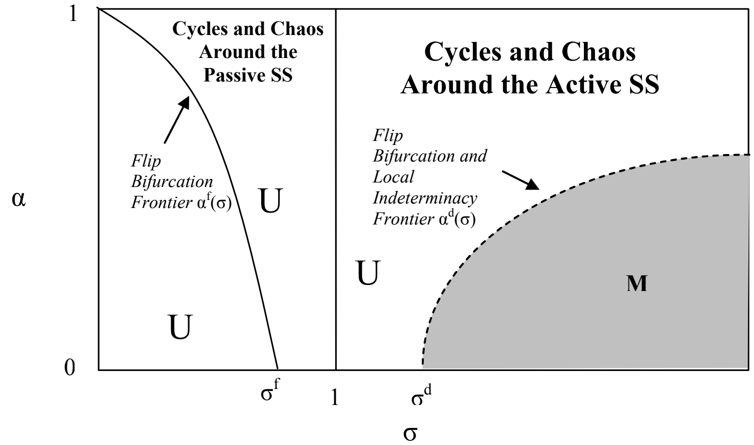

Figure 3: Equilibrium analysis for an active forward-looking interest rate rule. This figure shows a comparison between the local equilibrium analysis and the global equilibrium analysis as the degree of openness α and the coefficient of risk aversion σ vary. "M" stands for local multiple equilibria and "U" stands for a local unique equilibrium.

determinacy occurs for a wide range of (α, σ) combinations. In fact note how local determinacy coexists with global indeterminacy.

It is also interesting to compare our results with the ones in Benhabib et al. (2002a). There are some important differences. First our results derived in a money-in-the-utility-function set-up point out that it is not necessary to assume a productive role for money to obtain cyclical and chaotic equilibria. Second if consumption and money are complements then it is possible to have liquidity traps as in Benhabib et al. (2002a). But some of them may be "non-monotonic" and converge to a cycle around an extremely low interest rate steady state. On the contrary if consumption and money are substitutes then cyclical and chaotic dynamics occur only around the active steady state. Although this case is reminiscent of the one in Benhabib et al. (2002a), it also presents a subtle difference. In their closed economy model period-3 cycles always occur only for sufficiently low σ, while our results show that they can basically appear for any $\sigma > 1$ provided that there is enough degree of openness in the economy. In this sense and with respect to closed economies, open economies are more prone to display these cyclical dynamics.

4 Stabilizing Endogenous Fluctuations

The rule's elasticity to inflation was treated as given in the previous analysis, since the objective was to compare the performance of a particular rule across economies differing in trade openness and risk aversion. But this parameter is actually a policy choice. Recognizing this poses the following question. Given all the non-policy structural parameters, in particular, given the degree of openness α and the risk aversion parameter σ, what elasticity to inflation will eliminate cyclical and chaotic dynamics? To answer this question we can do the following simple exercise.[26]

The bifurcation thresholds that determine the existence of cyclical dynamics can be implicitly represented by $\chi = \Upsilon$ where χ depends on α, σ, γ, and θ_N and Υ depends on R^* and $\frac{A}{R^*}$. Then we can keep R^* fixed as well as α, σ, γ, and θ_N (that determine χ) and use $\chi = \Upsilon$ to solve for the bifurcation thresholds in terms of the elasticity $\xi \equiv \frac{A}{R^*}$, subject to $\xi > 1$. This will help us to find values of ξ that preclude the existence of cycles.

It is simple to show that, for the case of $\chi < 0$, there cannot exist cycles when[27]

$$\chi \leq R^L \left(1 - \frac{R^* - 1}{A}\right) - 1 \tag{30}$$

and for the case of $\chi > 0$ when

$$\chi \geq \left(1 - \frac{R^*}{A}\right)(R^* - 1) \tag{31}$$

Let's consider the case of $\chi > 0$. We notice that if the interest rate target is set such that $R^* < 1 + \chi$, then inequality (31) always holds for any active interest rate rule. If instead, $R^* > 1 + \chi$, then inequality (31) is equivalent to $\xi \leq \frac{1}{1 - \frac{\chi}{R^* - 1}}$, which means that cycles are ruled out if the interest rate rule is not too active. To illustrate this we use the calibration in Table 2 and set $\alpha = 0.4$ and CRRA $= 2$ (or equivalently $\sigma = 2.03$). Under this parametrization $\chi > 0$ and the right panel of Figure 2 suggests that there are period-2 cycles around the active steady state. In order to rule out them the elasticity to inflation should be below 1.35.

In the case of $\chi < 0$, inequality (30) can be written as $R^L \geq \frac{1+\chi}{1-\Upsilon}$, for $\Upsilon \equiv \frac{R^* - 1}{A} = \frac{R^* - 1}{R^* \xi}$. First of all we notice that, since in equilibrium $R^L > 1$, then the last inequality always holds when $\frac{1+\chi}{1-\Upsilon} \leq 1$, i.e. $\chi + \Upsilon \leq 0$. Given the definition of elasticity, the latter can also be written as $\xi \geq \xi^H$ where $\xi^H \equiv \frac{R^* - 1}{R^*(-\chi)}$. Hence a sufficiently active rule satisfying $\xi > \xi^H$ does not allow for equilibrium cycles. Nevertheless, even if we had $\xi < \frac{R^* - 1}{R^*(-\chi)}$, still $R^L \geq \frac{1+\chi}{1-\Upsilon}$ could hold. From the implicit definition of R^L in equation (23) and the related proof in Proposition 1, the last inequality is equivalent to $\frac{1}{1-\Upsilon}\left[\frac{R^* - 1}{1-\Upsilon}(\chi + \Upsilon)\right]^{\Upsilon} \geq \frac{R^*}{1+\chi}$, which depends solely on χ, R^* and ξ. This inequality defines implicitly a threshold ξ^L below which cycles are ruled out. Summarizing when $\chi < 0$ then rules with elasticities $\xi > 1$ that satisfy either $\xi > \xi^H$ or $\xi < \xi^L$ will preclude the existence of cyclical dynamics. For instance, using the calibration in Table 2 and setting $\alpha = 0.4$ and CRRA$= 0.8$ (or equivalently $\sigma = 0.79$), we know that the left panel of Figure 2 suggests that there are chaotic dynamics around the passive steady state. But for this parametrization we obtain $\xi^H = 8.75$ and $\xi^L = 1.22$. Thus these complex dynamics can be ruled out as long as the elasticity to inflation is either

[26] Assumptions 0-3 will not be violated by the following analysis.
[27] The formal derivation is available upon request.

$\xi < 1.22$ or $\xi > 8.75$.

According to these results, the Taylor principle $\xi \equiv \frac{A}{R^*} > 1$ can be still a viable policy recommendation for equilibrium determinacy, even from a global equilibrium dynamics perspective. But there is an additional requirement: an upper bound on the rule's elasticity to inflation.[28]

5 Distribution Costs and Imperfect Exchange Rate Pass-Through

The previous results were derived assuming that the Law of One Price held at the consumption level for traded goods and normalizing their foreign price to one. This in turn implied that there was a perfect exchange rate pass-through into import prices. In this Section we relax this assumption by introducing non-traded distribution costs which in turn implies imperfect exchange rate pass-through into import prices. Our goal is to understand how this can affect the previous global results of Section 3.

We follow Burnstein, Neves and Rebelo (2003) by assuming that the traded good needs to be combined with some non-traded distribution services before it is consumed. In order to consume one unit of the traded good it is required η units of the non-traded good. Let $\tilde{P}_t^T$ and P_t^T be the prices in the domestic currency of the small open economy that producers of traded goods receive and that consumers pay, respectively. Hence the consumer price of the traded good can be written as

$$P_t^T = \tilde{P}_t^T + \eta P_t^N. \tag{32}$$

To simplify the analysis we assume that the Law of One Price holds for traded goods at the production level and normalize the foreign price of the traded good to one ($\tilde{P}_t^{Tw} = 1$). Thus $\tilde{P}_t^T = \mathcal{E}_t \tilde{P}_t^{Tw} = \mathcal{E}_t$.

The presence of distribution services leads to imperfect exchange rate pass-through into import prices. To see this we combine (32), $\tilde{P}_t^T = \mathcal{E}_t$ and $e_t = \mathcal{E}_t / P_t^N$ to obtain

$$\pi_t^T = \left(\frac{e_{t-1}}{e_{t-1} + \eta}\right) \epsilon_t + \left(\frac{\eta}{e_{t-1} + \eta}\right) \pi_t^N$$

where $\pi_t^T \equiv \frac{P_t^T}{P_{t-1}^T}$, $\epsilon_t \equiv \frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}$ and $\pi_t^N \equiv \frac{P_t^N}{P_{t-1}^N}$ correspond to the (gross) inflation of import prices, the (gross) nominal depreciation rate and the (gross) non-traded inflation respectively. Clearly if $\eta = 0$ then there is perfect pass-through of the nominal depreciation rate into the inflation of import prices and it is measured by $\frac{d\pi_t^T}{d\epsilon_t} = 1$. This is the case that we already studied. But if $\eta > 0$ then we obtain imperfect exchange rate pass-through measured by $\frac{d\pi_t^T}{d\epsilon_t} = \left(\frac{e_{t-1}}{e_{t-1}+\eta}\right) \in (0, 1)$. As the parameter of distribution costs, η, increases then the degree of exchange rate pass-through decreases.

In order to pursue a determinacy of equilibrium analysis we proceed as before. We obtain a reduced non-

[28]Note that although it is possible to rule out cyclical and chaotic dynamics, liquidity traps are still feasible. To eliminate them it is necessary to implement some of the fiscal-monetary regimes proposed by Benhabib et al. (2002a).

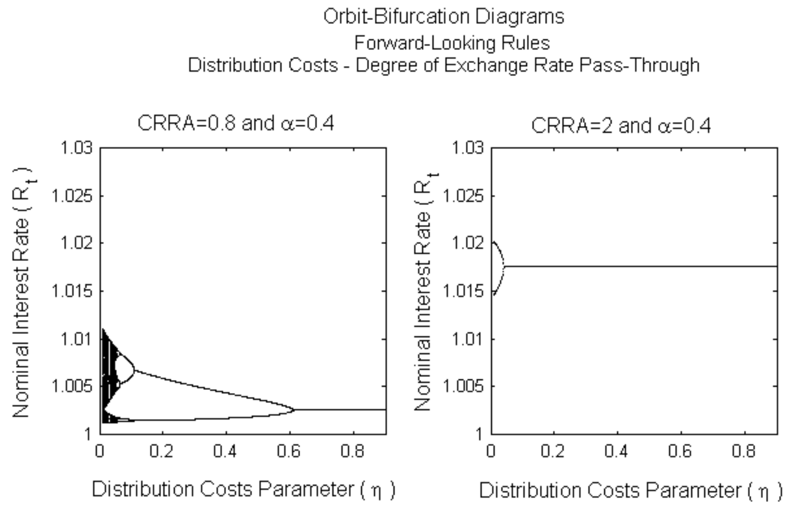

Figure 4: Orbit-bifurcation diagrams for the distribution costs parameter (η). R_t denotes the nominal interest rate. The diagrams show the set of limit points as a function of η, under two different sets of values of the degree of openness and the CRRA, $(\tilde{\sigma}, \alpha) = (0.8, 0.4)$ and $(\tilde{\sigma}, \alpha) = (2, 0.4)$, and under an active forward-looking rule. Depending on η, which is related to the degree of exchange rate pass-through, an active forward-looking rule may lead the economy into cyclical and chaotic dynamics.

linear form of the model that describes the dynamics of this economy. Nevertheless in contrast to the case of perfect exchange rate pass-through it is not possible to derive explicitly a difference equation similar to (22) that depends exclusively on the nominal interest rate R_t. Under imperfect exchange rate pass-through the dynamics of the economy are determined by the system

$$\left(\frac{e_{t+1}+\eta}{e_t+\eta}\right)^{\alpha}\left(\frac{e_t}{e_{t+1}}\right) = \left(\frac{R^*}{R_t}\right)\left(\frac{R_t-1}{R^*-1}\right)^{\frac{R^*-1}{A}}$$

$$\left(\frac{e_t+\eta}{e_{t+1}+\eta}\right)^{[\alpha(1-\sigma)+\sigma]}\left(\frac{(1-\alpha)e_{t+1}+\eta}{(1-\alpha)e_t+\eta}\right)^{\sigma}\left(\frac{e_{t+1}}{e_t}\right)^{\left(1+\frac{\sigma\theta_N}{1-\theta_N}\right)} = \left[\left(\frac{R_{t+1}-1}{R_{t+1}}\right)\left(\frac{R_t}{R_t-1}\right)\right]^{(1-\gamma)(1-\sigma)}.$$

Since it is not possible to derive analytical results we rely on numerical simulations. The objective of the simulations is to assess the impact of varying the distribution costs parameter η (and therefore the degree of exchange rate pass-through) on the previous results about cyclical and chaotic dynamics. To do so we use the parametrization of Table 2 and construct Figure 4.

We use two pairs of values for the degree of openness and the CRRA (that depends on σ). The first pair sets $\alpha = 0.4$ and CRRA=0.8. According to the left panel of Figure 2 for theses values there are chaotic dynamics around the passive steady state. The second pair sets $\alpha = 0.4$ and CRRA=2. For these values the right panel of Figure 2 suggests that there are period-2 cycles around the active steady state. In the context of the present discussion these results associated with the selected pairs correspond to perfect exchange rate pass-through or $\eta = 0$. Increasing η, or in other words decreasing the degree of exchange rate pass-through, has a non-trivial impact on these results as can be confirmed by Figure 4. Consider the left panel first, where $\alpha = 0.4$ and CRRA=0.8. Starting from $\eta = 0$ as η increases (as the degree of exchange rate pass-through declines) then the economy moves from displaying chaotic dynamics into displaying periodic cyclical dynamics. As η continues increasing the period of cycles decrease. Beyond $\eta \approx 0.6$ cycles disappear and the only fixed point that subsists corresponds to the passive steady state. On the other hand, the right panel of Figure 4 shows the results for $\alpha = 0.4$ and CRRA=2. For $\eta = 0$ the economy presents period-2 cycles around the active steady state as mentioned before. However as η increases (as the degree of exchange rate pass-through declines) cycles also disappear and the only attractor that subsists is the active steady state.

The results of this analysis can be summarized in the following Proposition.

Proposition 4 *Forward-looking rules are more prone to induce cyclical and chaotic dynamics the higher the degree of exchange rate pass-through.*

6 Sensitivity Analysis

Throughout the whole paper, the analysis was pursued under the following assumptions: 1) international complete financial markets, 2) the measure of inflation in the rule was the CPI-inflation, 3) a forward-looking rule and 4) real money balances entered into utility via what Carlstrom and Fuerst (2001) call a "cash-when-I'm-done" timing. In this section, we study the consequences of relaxing these assumptions. To simplify the analysis we still assume perfect exchange rate pass-through. We will show that our previous global findings of Section 3 hold even if we consider incomplete financial markets or if we introduce a forward-looking rule that responds to a different measure of inflation such as the non-traded inflation. That is, the degree of openness still plays an important role for the existence of cyclical and chaotic dynamics.

We also argue that cyclical and chaotic dynamics are less likely to occur under backward-looking rules while they can still appear under contemporaneous rules depending on the degrees of trade openness and risk aversion. And finally we prove that the alternative timing for real money balances, known as the "Cash-in-Advance" timing, affects the previously derived bifurcation thresholds, but does not preclude the existence of cyclical and chaotic dynamics.

6.1 Incomplete Markets

The assumption about complete markets is not essential for the results derived in Section 3 about cyclical and chaotic dynamics. Assuming incomplete markets leads to the same global dynamics results. To see this assume that the agent is blessed with perfect foresight and has access to an international bond b_t^w and a domestic bond B_t issued by the government. The former pays a constant international interest rate, R^w, and the latter pays an interest rate, R_t. Using this and the assumptions in the Subsection 2.1 we can rewrite the agent's budget constraint as

$$B_t + \mathcal{E}_t b_t^w + M_t = R^w \mathcal{E}_t b_{t-1}^w + M_{t-1} + R_{t-1} B_{t-1} + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \mathcal{E}_t c_t^T - P_t^N c_t^N$$

The agent chooses the sequences $\{c_t^T,\, c_t^N,\, h_t^T,\, h_t^N,\, M_t^d,\, b_t^w,\, B_t\}_{t=0}^{\infty}$ in order to maximize (1) subject to (2) and (3), the previously mentioned budget constraint and corresponding transversality conditions, given M_{-1}^d, b_{-1}^w, and B_{-1} and the time paths of R_t, $\mathcal{E}_t$, P_t^N, R^w and τ_t. The first order conditions of this problem for b_t^w and B_t correspond to

$$\lambda_t = \beta R^w \lambda_{t+1} \tag{33}$$

$$\lambda_t = \frac{\beta R_t \lambda_{t+1}}{\epsilon_t} \tag{34}$$

whereas the conditions for $\{c_t^T, c_t^N, h_t^T, h_t^N, M_t^d\}$ can be written as (11)-(14).

As is common in the small open economy literature, we assume $\beta R^w = 1$, which implies by (33) that $\lambda_t = \lambda_{t+1}$. This in turn means that there is a unit root in the system of equations that describe the dynamics of the economy. This prevents us from using (log)linearization techniques in order to pursue a meaningful local determinacy of equilibrium analysis. Hence we cannot derive local results similar to the ones in the Subsection 3.1.

But this does not change the previous results from the global determinacy of equilibrium analysis, since the global dynamics are still governed by (22). To see this use $\lambda_t = \lambda_{t+1}$ and (34) to obtain $\epsilon_t = \beta R_t$ which is identical to (19). This condition together with $\lambda_t = \lambda_{t+1}$, (5), (7), (11)-(14), and market clearing conditions can be used to derive an identical equation to (20) which in tandem with a forward-looking rule allows us to find an identical difference equation to (22). Then the cyclical and chaotic dynamics results that we derived before still hold under incomplete markets.

Nevertheless, introducing incomplete markets has an interesting consequence for the behavior of the current account. To explore this, recall that indeterminacy of the nominal interest rate in our model implies real indeterminacy because of the non-separability in the utility function between consumption and money. That is, one can obtain all the remaining real endogenous variables as a functions of R_t. In particular $h_t^N = h^N(R_t)$, $c_t^T = \frac{\alpha}{1-\alpha}\frac{\psi}{\lambda\theta_N} h_t^N$ and $h_t^T = h^T$. Using these and the definition of the current account

$$b_t^w - b_{t-1}^w = (R^w - 1) b_{t-1}^w + \left(h_t^T\right)^{\theta_T} - c_t^T$$

we can deduce that cycles of the nominal interest rate may cause cycles of the accumulation of foreign bonds and therefore cycles of the current account.

6.2 Choosing A Different Measure of Inflation: The Core Inflation

In the specification of the rule, the government can react to a measure of inflation different from the CPI-inflation. For instance it can respond to the core-inflation which can be defined as either the inflation of the traded goods or the inflation of the non-traded goods. These two cases can be considered as extreme cases of cases of a rule that reacts to the full-inflation $\pi_t^f \equiv \epsilon_t^\omega \left(\pi_t^N\right)^{1-\omega}$ when the government picks the weight $\omega \in [0,1]$.[29] In the former case it picks $\omega = 1$, whereas in the latter case it chooses $\omega = 0$.

To analyze these cases we derive a difference equation similar to (22) where the only difference is that the parameter χ is substituted by

$$\chi^f \equiv \frac{(\sigma-1)(1-\omega)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N + \alpha(1-\theta_N)] + (1-\alpha)(1-\theta_N)}.$$

Using this difference equation and setting sequentially $\omega = 1$ and $\omega = 0$ in χ^f we obtain the following. Cyclical and chaotic dynamics are not present under a rule that reacts to the future traded inflation, ϵ_{t+1} ($\omega = 1$); only liquidity traps are possible in this case. Nevertheless by reacting to the future non-traded inflation π_{t+1}^N, forward-looking rules can still induce cycles and their existence can depend on the degree of openness of the economy.

To illustrate this point we rely on simulations and using the parametrization in Table 2 we construct Figure 5. It presents the orbit-diagram for a forward-looking rule that reacts exclusively to the non-traded inflation where the relative risk aversion coefficient CRRA is $\tilde{\sigma} = 1.5$ (or equivalently $\sigma = 1.51$). From this figure it is clear that for low degrees of openness (α), the previously mentioned rule leads the economy to the active steady state. But as the degree of openness increases the same rule can induce period-2 cycles in the economy. Hence the appearance of these cycles depend on the degree of openness.

6.3 Contemporaneous and Backward-Looking Rules

In this subsection we consider varying the timing of the rule. We will study contemporaneous and backward-looking rules that still respond to the CPI inflation.

6.3.1 Contemporaneous Rules

Motivated by the recent estimations by Lubik and Schorfheide (2003) for the United kingdom, Canada, Australia and New Zealand, we study the determinacy of equilibrium for rules that respond to current CPI inflation: $R_t = 1 + (R^* - 1)\left(\frac{\pi_t}{\pi^*}\right)^{\frac{A}{R^*-1}}$ with $R^* = \pi^*/\beta$ and $R_t > 1$. Under these rules the equilibrium

[29] CPI inflation is clearly a specific case of full inflation for $\omega = \alpha$.

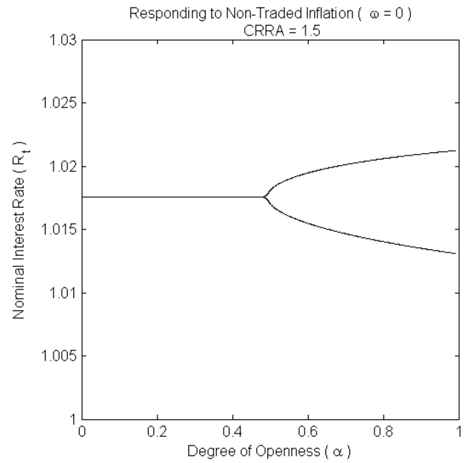

Figure 5: Orbit-bifurcation diagram for the degree of openness, α. R_t denotes the nominal interest rate. The diagram shows the set of limit points as a function of α, when the coefficient of relative risk aversion (CRRA) $\tilde{\sigma} = 1.5$ and when the active forward-looking rule responds exclusively to the non-traded inflation. Depending on α this rule can induce cyclical dynamics in the economy.

dynamics are described by

$$\left[\frac{R_{t+1}-1}{R_{t+1}}\right]^{\chi}\left[\frac{R^*-1}{R_{t+1}-1}\right]^{\frac{R^*-1}{A}} = \left[\frac{R_t-1}{R_t}\right]^{\chi}\frac{R^*}{R_t} \tag{35}$$

where χ defined in (21) depends on α and σ. An explicit representation for either the forward or the backward dynamics of (35) is not available in this case. Although it is feasible to derive some analytical results as before, for reasons of space, we only present some numerical simulations. These are sufficient to make the point that the degree of openness still affects the appearance of complex dynamics.[30]

In addition in the subsequent analysis we will focus on characterizing the existence of cycles of period 2 and 3 because of the following reasons. By Sarkovskii (1964)'s Theorem the existence of period-2 cycles is a necessary (but not a sufficient) condition for the existence of cycles of any higher order, while the existence of period-3 cycles implies the existence of cycles of every possible period. Furthermore, by Li and Yorke (1975), if a map possesses a period-3 cycle then it also features topologically chaotic trajectories.[31]

Using the parametrization of Table 2 we construct Figure 6. This Figure presents some qualitative properties of the global dynamics of the model for contemporaneous rules for different degrees of openness while setting $\sigma = 2.55$ (when consumption and money are Edgeworth substitutes). Each row of panels in Figure 6 contains plots of the map $R_{t+1} = f(R_t)$ implicitly defined in the difference equation (35), its second iterate $R_{t+2} = f^2(R_t)$ and its third iterate $R_{t+3} = f^3(R_t)$, respectively, for a given degree of openness α.

Going from the bottom to the top row we increase openness by considering $\alpha = \{0.01, 0.38, 0.90\}$. In

[30] The analytical results of this part of the paper are available upon request.
[31] See Lorenz (1993) for a precise statement of these two theorems.

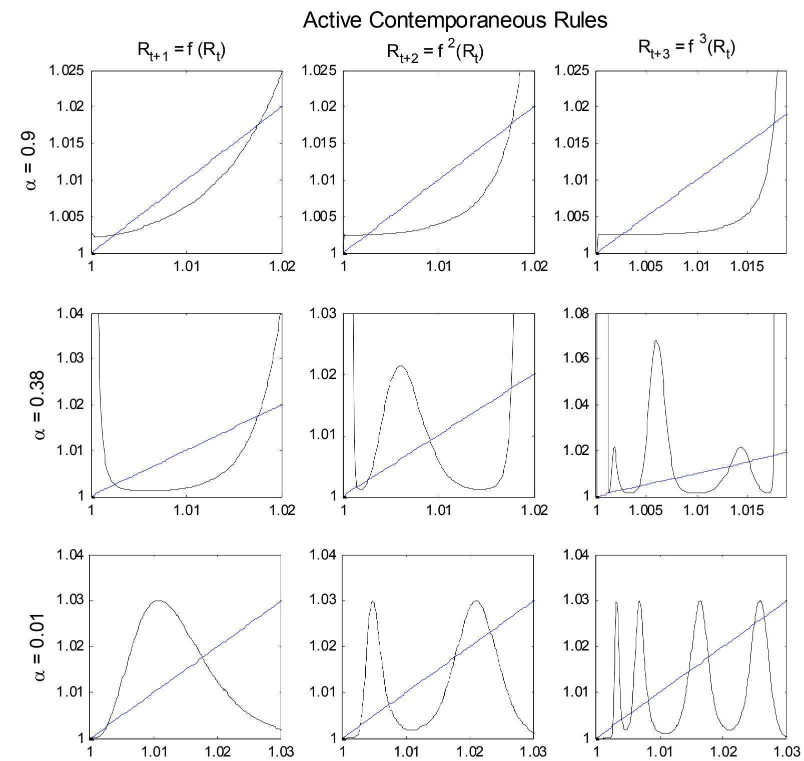

Figure 6: This graph shows the first, the second and the third iterates of the implicit mapping $R_{t+1} = f(R_t)$ defined in (35) for different degrees of openness of the economy (α). From the figure it is possible to infer that depending on α, two-period cycles and three-period cycles around the active and the passive steady states may arise.

all the panels the straight line corresponds to the 45^o degree line. The first result to notice is that for $\alpha = \{0.01, 0.38\}$, namely an almost closed and moderately open economy respectively, the second and third iterates, $R_{t+2} = f^2(R_t)$ and $R_{t+3} = f^3(R_t)$, have fixed points different from the steady state values R^* and R^L, implying the existence of cycles of period-2 and 3. Then by the Sarkovskii's (1964) Theorem and the Li and Yorke's (1975) Theorem, $f(R_t)$ features cycles of any order, as well as aperiodic cyclical dynamics (chaos). But there is an important difference between the dynamics for $\alpha = 0.01$ and those for $\alpha = 0.38$. In the former case, cycles and chaos appear around the active steady state, while in the latter they occur around the passive steady state. Finally, for very open economies ($\alpha = 0.90$), no cycles and chaotic dynamics appear at all. The monotonicity of the map $f(R_t)$ implies that liquidity traps are the only type of global equilibrium multiplicity in this case.

Given $\sigma = 2.55$ (or equivalently $\widetilde{\sigma} = 2.5$) it is possible to find the exact numerical values of the α thresholds triggering a qualitative switch in dynamics. We find that period-2 cycles appear around the active steady state when $\alpha \in (0.001, 0.22)$; and period-3 cycles occur around the active steady state when $\alpha \in (0.001, 0.16)$. Pushing α up, period-2 cycles appear around the passive steady state for $\alpha \in (0.25, 0.43)$, whereas period-3 cycles (and therefore chaos) exist for $\alpha \in (0.33, 0.38)$. Finally for $\alpha > 0.39$ only liquidity traps exist.

Overall the existence of cyclical and chaotic dynamics is robust to changing the timing of the rule from forward-looking to contemporaneous. The key difference is that now endogenous global fluctuations can only occur for the case when consumption and money are Edgeworth substitutes. Nevertheless for this case conclusions are quite different in the following sense: very open economies seem to be the less prone to endogenous cycles. Furthermore the qualitative changes induced by a variation in the degree of openness, for a fixed $\sigma > 1$, can be even more dramatic under a contemporaneous rule since they might even trigger a switch from one steady state to the other as the focus of fluctuations.

The following proposition summarizes these results.

Proposition 5 *Suppose the government follows an active contemporaneous rule. Then:*

1. *if consumption and money are Edgeworth complements, i.e.* $\sigma \in (0, 1)$*, there cannot be equilibrium cycles of any periodicity for any degree of openness* $\alpha \in (0, 1)$ *;*

2. *if consumption and money are Edgeworth substitutes, i.e.* $\sigma > 1$,

   (a) *there cannot be equilibrium cycles of any periodicity if the economy is sufficiently open;*

   (b) *cyclical and chaotic dynamics occur around the passive steady state for intermediate degrees of openness;*

   (c) *cyclical and chaotic dynamics appear around the active steady state occur if the economy is sufficiently closed.*

6.3.2 Backward-Looking Rules

We conclude the analysis of different timings for the rule by studying backward-looking rules defined as $R_t \equiv 1 + (R^* - 1)\left(\frac{\pi_{t-1}}{\pi^*}\right)^{\frac{A}{R^*-1}}$ with $\xi \equiv \frac{A}{R^*} > 1$. This specification in tandem with (20) conform a system of two first-order difference equations that can be used to pursue the global determinacy of equilibrium analysis. Since it is very difficult to derive analytical results, we rely on simulations in order to assess whether for different values of α and σ, the system presents cycles or chaos. The simulation results show that these types of dynamics are not present regardless of the degree of openness α when $\sigma \in \{0.79, 1, 51, 2.03, 2.55\}$ (or equivalently when the CRRA coefficient $\widetilde{\sigma} \in \{0.8, 1.5, 2, 2.5\}$). The interest rate converges to either the active or the passive steady-state.

Nevertheless this is not sufficient evidence to conclude that backward-looking rules will preclude the existence of cyclical dynamics. In fact in Airaudo and Zanna (2005) we show that these rules may still lead to these dynamics in open economies that face nominal price rigidities.

6.4 Real money balances timing and cycles

In our set-up the real money balances that provide transaction services, and therefore utility, are those left after leaving the goods market. This is the traditional timing adopted in money-in-utility-function models. Carlstrom and Fuerst (2001) call this timing the "Cash-When-I'm-Done" (CWID) timing and argue that is counterintuitive.[32] They suggest an alternative timing: the standard "Cash-In-Advance" (CIA) timing, where the real money balances entering the agent's utility are those left after leaving the bond market but before entering the goods market.

In this Subsection we construct a simple example showing that the results presented in Section 3 for active forward-looking rules are not driven by the CWID timing. Following Woodford (2003), we introduce a CIA timing by altering the budget constraint of the representative agent as follows

$$E_t Q_{t,t+1} W_{t+1} \leq W_t - \mathcal{E}_t \tau_t - \frac{R_t - 1}{R_t} M_t^d + \frac{1}{R_t}\left(\mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t c_t^T - P_t^N c_t^N\right)$$

while leaving unchanged all the remaining elements of the model presented in Section 2. By the same analysis we pursued in that section we obtain the non-linear reduced form

$$\left[\frac{R_{t+1} - 1}{R_{t+1}^{\Psi}}\right]^{\chi} = \frac{R^*}{(R^* - 1)^{\frac{R^*-1}{A}}} \frac{(R_t - 1)^{\chi + \frac{R^*-1}{A}}}{R_t^{1+\Psi\chi}} \tag{36}$$

that describes the equilibrium dynamics of the economy under forward-looking rules. The difference between

[32] A discussion on money-in-utility timing and cycles is already present in the papers of Fukuda (1993, 1995), pointing out that, under money growth rules, non-separability between consumption and real balances is not a necessary condition for cycles to exist (as one might have conjectured from the previous works of Matsuyama (1991) and Obstfeld and Rogoff (1983)). Under CIA timing cycles occur even under full separability.

(36) and its CWID counterpart (22) is the coefficient $\Psi \equiv \frac{1-\theta_N(1-\sigma)}{(1-\sigma)(1-\gamma)(1-\theta_N)}$, which is positive for $\sigma \in (0,1)$ and negative for $\sigma > 1$.[33] The structural parameter χ is unchanged and still defined by (21). The following Proposition suggests that under the CIA timing, cycles are still possible and their existence depends on the degree of openness α. Nevertheless this different timing implies different bifurcation thresholds for α from the ones derived in Section 3.

Proposition 6 *Consider the definition of χ in (21) and define $\Upsilon^{cia} \equiv \frac{\left(1-\frac{R^*-1}{A}\right)R^*-1}{2R^*}$. Suppose $\Upsilon^{cia} > 0$ and that the government follows an active forward-looking rule. If consumption and money are Edgeworth substitutes, i.e. $\sigma > 1$, then there exist period-2 cycles when $\chi < \Upsilon^{cia}$, a condition that is satisfied for sufficiently open economies, i.e $\alpha > \hat{\alpha}$.*

Proof. See the Appendix. ■

7 Conclusions

In this paper we show that active interest rate rules can have perverse effects in a small open economy by inducing endogenous cyclical and chaotic dynamics. Our main contribution is to show that the existence of these dynamics depends on some particular features of open economies such as the degree of openness (measured by the share of tradable goods in consumers' preferences) and the degree of exchange rate pass-through (implied by the presence of non-traded distribution services). In our model a forward-looking rule that responds to CPI-inflation is more prone to lead to cyclical and chaotic dynamics the more open the economy and the higher the degree of exchange rate pass-through. If consumption and money are Edgeworth complements these dynamics occur around an extremely low interest rate steady state. On the other hand if consumption and money are substitutes these dynamics appear around the interest rate target set by the monetary authority.

We also discuss how the government could design a forward-looking rule in order to rule out complex dynamics, while still preserving the Taylor principle at the target steady state. The analysis implies a "modified Taylor principle": for a given interest rate target, the interest rate rule responsiveness to CPI inflation should satisfy some lower (upper) bound depending on whether consumption and real balances are Edgeworth complements or substitutes.

The existence of cyclical and chaotic dynamics and their dependence on the degree of openness are in general robust to different inflation timings in the rule (forward-looking versus contemporaneous rules), to the use of alternative measures of inflation (CPI versus Core inflation), as well as to changes in the timing of real money balances in liquidity services ("Cash-When-I'm-Done" timing versus "Cash-In-Advance" timing).

Clearly the analytical tractability of our model comes from one of its limitations: we are considering a flexible price and perfectly competitive economy. Without any sort of nominal rigidity, monetary policy

[33] One could think of the CWID set up as one where $\Psi = 1$.

does not have any role for active stabilization. Nevertheless as we show in Airaudo and Zanna (2005), the existence of cycles induced by interest rate rules and the important role played by some open economy features in the characterization of the equilibrium are not precluded by introducing sticky-prices.

This paper leaves open some interesting research questions. First as standard in the literature, we model the government as an automaton that always follows the same monetary rule. That is, it implements an active interest rate rule with a given parametrization, independently from the evolution of inflation itself and other monetary variables. One might consider the possibility of an endogenous monetary policy regime switching featuring a change in the monetary instrument (from an interest rate rule to money growth and viceversa) or a change in the responsiveness to inflation triggered by an endogenous variable monitored by the central bank.

Finally we have implicitly assumed that agents in the economy can coordinate their actions and learn the particular equilibria that we studied. Relaxing this assumption and introducing learning as in Evans and Honkapojha (2001) can be another avenue to explore in future research.

A Appendix

This first part of the appendix presents the statements of some Lemmata and their proofs as well as some of the proofs of the Lemmata stated in the paper. The second part presents the proofs of the main Propositions of the paper.

A.1 Lemmata and Proofs

A.1.1 Proof of Lemma 1

Proof. Let $\phi^f \equiv 1 + \frac{\frac{R^*}{A}-1}{\frac{\chi}{R^*-1}}$ be the slope of (26) and define $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right) > 0$. Since R_t is a non-predetermined variable, the equilibrium is locally unique if and only if $\left|\phi^f\right| > 1$, i.e. the linearized mapping (26) is explosive and therefore the target steady state is the unique bounded PFE.

First suppose that $\chi < 0$. This together with Assumption 0 ($\frac{A}{R^*} > 1$) and the zero lower bound $R_t > 1$, implies that $\phi^f > 1$. Hence the equilibrium is locally unique. Now suppose that $\chi > 0$. This in tandem with Assumption 0 and the zero lower bound leads to $\phi^f < 1$, meaning that in order to have a unique equilibrium we need $\phi^f < -1$. Simple algebra shows that $\chi < \Upsilon^d$ implies $\phi^f < -1$. Then the map is explosive and the equilibrium is unique.

On the other hand, it is simple to show that if $\chi > \Upsilon^d$ then $-1 < \phi^f < 1$. Hence the map is non-explosive, i.e. from any initial condition R_0 off the target steady state R^*, R_t will eventually converge to R^*. This continuity of PFE paths is the source of local multiple equilibria. ■

A.1.2 Proof of Lemma 2

Proof. From point 2 of Lemma 6, the assumption $\chi \in (\Upsilon^w, 0)$ implies that the mapping $f(.)$ satisfies the zero-lower-bound condition only for $R_t \in \left(R^l, R^u\right) \subset (1, +\infty)$. Moreover within $\left(R^l, R^u\right)$ the mapping

$f(.)$ looks like an inverted logistic mapping with a minimum at R^J and $f'(R^L) < 0$ (see the right panel of Figure 1). With these properties of $f(.)$, the proof of Point 1 follows from the fact that $f'(R_t) < 0$ for any $R_t \in (R^l, R^J)$.

To prove Point 2, take any $R_t \in [\underline{R}, R^*]$ whose first iterate is $R_{t+1} = \left[1 - J(R_t)^{\frac{1}{\chi(\alpha,\sigma)}}\right]^{-1}$. If $R_t \in [\underline{R}, R^J]$, then $R_{t+1} = f(R_t) > f(R^J) \equiv \underline{R}$ since $f'(R_t) < 0$ for any $R_t < R^J$; moreover $R_{t+1} < R^*$ if $f_{\min} \geq \widetilde{R}$. Similarly, if $R_t \in (R^J, R^*]$ then $R_{t+1} = f(R_t) < f(R^*) = R^*$ since $f'(R_t) > 0$ for $R_t > R^J$, as well as $R_{t+1} = f(R_t) > f(R^J) = \underline{R}$. Hence we have shown that for any $R_t \in [\underline{R}, R^*]$, $R_{t+1} = f(R_t) \in [\underline{R}, R^*]$. Then $f : [\underline{R}, R^*] \to [\underline{R}, R^*]$. The attractive property of the set is straightforward to show and therefore omitted.

Point 3 is based on the fact that a sufficient condition for the existence of period-2 cycles is $f'(R^L) < -1$ which implies an unstable passive steady state. To show this we define an auxiliary function $g(R_t) \equiv R_t - f^2(R_t)$ such that period-2 cycles are the zeros of $g(R_t)$. The assumption $f_{\min} \equiv f(R^J) \geq \widetilde{R}$ requires a distinction between the two cases: **a)** $f(R^J) > \widetilde{R}$ and **b)** $f(R^J) = \widetilde{R}$.

In case **a),** the map-invariant set is $[\underline{R}, R^*]$. Clearly $g(R^L) = g(R^*) = 0$ and $g(\underline{R}) \leq 0$. Also, by the chain rule, $g'(R^{ss}) = 1 - [f'(R^{ss})]^2$, with R^{ss} being either one of the two steady states. But then $f'(R^*) = \frac{A}{R^*} > 1$ implies that $g'(R^*) < 0$. What is ambiguous is the sign of $g(R^J)$. If $g(R^J) = 0$ then the period-2 cycle is obviously $\{R^J, \underline{R}\}$. If $g(R^J) < 0$, by continuity of $f(R_t)$ on $[\underline{R}, R^*]$, then there exists a point $R^c \in (R^J, R^*)$ such that $g(R^c) = 0$. Since the set $[\underline{R}, R^*]$ is map invariant the second point of the period-2 cycle belongs to the same set.[34] If instead $g(R^J) > 0$, nothing guarantees that the function $g(.)$ has zeros other than the two steady states. A sufficient condition for having another zero is that $g'(R^L) = 1 - [f'(R^L)]^2 < 0$, i.e. $f'(R^L) < -1$. If this holds, then $g(R) = 0$ at some point between R^L and R^J. From the definition of $f(.)$ it is simple to show that $f'(R^L) < -1$ is equivalent to $\chi > \frac{1}{2}\Upsilon^w$ where $\Upsilon^w \equiv R^L\left(1 - \frac{R^*-1}{A}\right) - 1$. Since we are considering $\chi > \Upsilon^w$, in case **a)** period-2 cycles around the passive steady state occur when $\chi \in \left(\frac{1}{2}\Upsilon^w, 0\right)$. In case **b)** all conditions of case **a)** hold, although in this case $g(R^J) < 0$ is always true. Therefore by a similar argument there are always period-2 cycles. ■

A.1.3 Proof of Lemma 3

Proof. Let $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right)(R^* - 1)$. The restriction $\chi \in (0, \Upsilon^k)$ implies, by Point 1 of Lemma 6, that the mapping $f(.)$ looks like a logistic map with a maximum at R^J, $f'(R^L) > 1$ and $f'(R^*) < 0$ (see the left panel of Figure 1).

The proofs of Points 1 and 2 are similar to their counterparts in Lemma 2, so both are omitted.

Point 3 involves searching for a flip bifurcation at the active steady state. Let's define an auxiliary function $g(R_t) \equiv R_t - f^2(R_t)$. Period 2 cycles are then zeros of $g(R_t)$. The assumption $f_{\max} \equiv f(R^J) \leq \widetilde{R}$ requires a distinction between the two cases**:** $f(R^J) = \widetilde{R}$ and $f(R^J) < \widetilde{R}$.

If $f(R^J) < \widetilde{R}$, the set invariant under mapping $f(.)$ is $[R^L, \overline{R}]$. Clearly $g(R^L) = g(R^*) = 0$, and $g'(R^L) < 0$, but $g(R^J) \gtreqless 0$. If $g(R^J) = 0$, a period-2 cycle is, by construction, $\{R^J, \overline{R}\}$. If $g(R^J) > 0$, by continuity of the function $g(.)$ there must exist a point $R^c \in (R^L, R^J)$ such that $g(R^c) = 0$. As R^c belongs to the map invariant set $[R^L, \overline{R}]$, $f^2(R^c) \in [R^L, \overline{R}]$ as well: a period-2 cycle exists. If instead $g(R^J) < 0$, by continuity, a sufficient condition for period-2 cycles is that $g'(R^*) = 1 - [f'(R^*)]^2 < 0$, i.e. $f'(R^*) < -1$

[34] We could actually show that this second point belong to the intervak $(\underline{R}, R^L)$, meaning that the equilibrium path jumps deterministically from the left to the right neighborhood of the passive steady state.

given that over these parametric ranges $f'(R^*) < 0$ always. Using the definition of $f(.)$ in (27) and by simple differentiation, this is in fact equivalent to $\chi < \frac{\Upsilon^k}{2}$.

The case of $f(R^J) = \widetilde{R}$ is trivial. All the properties of the previous case hold, but now $g(R^J) = R^J - R^L > 0$ always. So by similar arguments, a period-2 cycle always exists. ■

A.1.4 Lemma 4

Lemma 4 *Keep γ and θ_N constant and let $\chi(\alpha,\sigma)$, $\chi_{\max}$, $\chi_{\min}$ and $\mu(\sigma)$ be defined as in (21), (24) and (25). Consider any real number $\Upsilon^i \in (0, \chi_{\max})$ and define a function $\alpha^i(\sigma) \equiv \frac{\left[1-\frac{\Upsilon^i}{\mu(\sigma)}\right]}{\left[1-\frac{\Upsilon^i}{\chi_{\min}}\right]}$ for $\sigma > 1$.*

1. *Over the domain $(1, +\infty)$, the function $\alpha^i(\sigma)$ satisfies the following properties:*
   - *(a) it is continuously differentiable, strictly increasing and strictly concave;*
   - *(b) $\lim_{\sigma\to 1^+} \alpha^i(\sigma) = -\infty$ and $\lim_{\sigma\to+\infty} \alpha^i(\sigma) = \left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1}\left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right] \in (0,1)$*
   - *(c) $\alpha^i(\sigma) \gtreqless 0$ for $\sigma \gtreqless \sigma^i$ where $\sigma^i \equiv \frac{\left[1-\frac{\Upsilon^i}{\chi_{\min}}\right]}{\left[1-\frac{\Upsilon^i}{\chi_{\max}}\right]} > 1$;*
2. *For any given $\sigma > 1$, $\chi(\alpha,\sigma) \gtreqless \Upsilon^i$ if and only if $\alpha \lesseqgtr \alpha^i(\sigma)$.*
3. *For $\alpha \in (0,1)$ and $\sigma > 1$ we have that $\chi(\alpha,\sigma) < \Upsilon^i$ if and only if $\alpha > \max\{0, \alpha^i(\sigma)\}$.*

Proof. First of all recall the definitions of $\chi(\alpha,\sigma)$, $\chi_{\max}$, $\chi_{\min}$ and $\mu(\sigma)$ in (21), (24) and (25). Then take any real number $\Upsilon^i \in (0, \chi_{\max})$ and any $\sigma > 1$. If we solve explicitly the equation $\chi(\alpha,\sigma) = \Upsilon^i$ with respect to α, for a given σ, after some simple algebra, the solution is a function $\alpha^i(\sigma) \equiv \left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1}\left[1 - \frac{\Upsilon^i}{\mu(\sigma)}\right]$. We first prove point 1 for $\sigma \in (1, +\infty)$.

The statement that $\alpha^i(\sigma)$ is continuously differentiable, strictly increasing and strictly concave over the domain $(1, +\infty)$, follows from its definition, $\chi_{\max} \in (0, +\infty)$, $\chi_{\min} \in (-1, 0)$, $\Upsilon^i \in (0, \chi_{\max})$, and the fact that for $\sigma > 1$, the function $\mu(\sigma)$, defined in (25) satisfies the following. $\mu : (1, \infty) \to \mathbb{R}$, it is continuously differentiable, strictly increasing and strictly concave with respect to σ and satisfies $\lim_{\sigma\to 1^+} \mu(\sigma) = 0$ and $\lim_{\sigma\to+\infty} \mu(\sigma) = \chi_{\max} \in (0, +\infty)$. This proves Point 1(a).

Given that $\lim_{\sigma\to 1^+} \mu(\sigma) = 0$ and $\lim_{\sigma\to+\infty} \mu(\sigma) = \chi_{\max}$ then for $\Upsilon^i \in (0, \chi_{\max})$ and $\chi_{\min} \in (-1, 0)$ we have that $\lim_{\sigma\to 1^+} \alpha^i(\sigma) = -\infty$ and $\lim_{\sigma\to+\infty} \alpha^i(\sigma) = \left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1}\left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right]$. The assumption of $\Upsilon^i \in (0, \chi_{\max})$ together with $\chi_{\min} \in (-1, 0)$ guarantees that $\left[1 - \frac{\Upsilon^i}{\chi_{\min}}\right]^{-1}\left[1 - \frac{\Upsilon^i}{\chi_{\max}}\right] \in (0, 1)$. This proves Point 1(b).

Point 1(c) follows from solving $\mu(\sigma) = \Upsilon^i$ with respect to σ and obtaining $\sigma^i = \frac{\left[1-\frac{\Upsilon^i}{\chi_{\min}}\right]}{\left[1-\frac{\Upsilon^i}{\chi_{\max}}\right]}$. As $\Upsilon^i \in (0, \chi_{\max})$ and $\chi_{\min} \in (-1, 0)$, it must be that $\sigma^i > 1$. Using the definition of σ^i and the previous two points 1(a) and 1(b) then the rest of 1(c) follows.

Since for $\sigma > 1$ and $\alpha \in (0, 1)$, the function $\chi : (0,1) \times (1, \infty) \to \mathbb{R}$ is continuously differentiable, strictly decreasing with respect to α and satisfies $\chi(\alpha,\sigma) > 0$ then it follows that $\chi(\alpha,\sigma) \gtreqless \Upsilon^i$ if and only if $\alpha \lesseqgtr \alpha^i(\sigma)$. This proves Point 2.

To prove Point 3 we proceed as follows. From point 2 of the present Lemma, we know that $\chi(\alpha,\sigma) < \Upsilon^i$ if and only if $\alpha > \alpha^i(\sigma)$. From point 1(c) of the same Lemma we have also that $\alpha^i(\sigma) \gtreqless 0$ for $\sigma \gtreqless \sigma^i$. Since

$\alpha \in (0,1)$ then $\chi(\alpha,\sigma) < \Upsilon^i$ for any $\alpha \in (0,1)$ when $\sigma \leq \sigma^i$, as in this case $\alpha^i(\sigma) \leq 0$; while $\chi(\alpha,\sigma) < \Upsilon^i$ only for $\alpha > \alpha^i(\sigma) > 0$ if $\sigma > \sigma^i$. If we write this compactly, we have that $\chi(\alpha,\sigma) < \Upsilon^i$ if and only if $\alpha > \max\left\{0, \alpha^i(\sigma)\right\}$. ■

A.1.5 Lemma 5

Lemma 5 *Keep γ and θ_N constant and let $\chi(\alpha,\sigma)$, $\chi_{\max}$, $\chi_{\min}$ and $\mu(\sigma)$ be defined as in (21), (24) and (25). Consider any real number $\Upsilon^i \in (\chi_{\min}, 0)$ and define a function $\alpha^i(\sigma) \equiv \frac{\left[1-\frac{\Upsilon^i}{\mu(\sigma)}\right]}{\left[1-\frac{\Upsilon^i}{\chi_{\min}}\right]}$ for $\sigma \in (0,1)$.*

1. *Over the domain $(0,1)$, the function $\alpha^i(\sigma)$ satisfies the following properties:*
   - *(a) it is continuously differentiable, strictly decreasing and strictly concave;*
   - *(b) $\lim_{\sigma\to 0} \alpha^i(\sigma) = 1$ and $\lim_{\sigma\to 1^-} \alpha^i(\sigma) = -\infty$;*
   - *(c) $\alpha^i(\sigma) \gtreqless 0$ for $\sigma \lesseqgtr \sigma^i$ where $\sigma^i \equiv \frac{\left[1-\frac{\Upsilon^i}{\chi_{\min}}\right]}{\left[1-\frac{\Upsilon^i}{\chi_{\max}}\right]} \in (0,1)$.*
2. *For any given $\sigma \in (0,1)$, $\chi(\alpha,\sigma) \gtreqless \Upsilon^i$ if and only if $\alpha \gtreqless \alpha^i(\sigma)$.*
3. *For $\alpha \in (0,1)$ and $\sigma \in (0,1)$ we have that $\chi(\alpha,\sigma) > \Upsilon^i$ if and only if $\alpha > \max\left\{0, \alpha^i(\sigma)\right\}$.*

Proof. The proof is omitted since it is very similar to the proof of Lemma 4. ■

A.1.6 Lemma 6

Lemma 6 *Define the scalars $\Upsilon^p \equiv \frac{R^*-1}{A} > 0$, $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right)(R^*-1) > 0$ and $R^J \equiv \frac{1+\chi}{1-\Upsilon^p}$. Recall the definition of $f(R_t)$ in (27)-(28) for $R_t \in (1,+\infty)$.*

1. *Suppose that $\chi > 0$. Then a) $f : (1,+\infty) \to (1,+\infty)$ and it is continuously differentiable; b) $f(.)$ has a global maximum at R^J with $R^J > R^L$; c) $\lim_{R_t\to 1^+} f(R_t) = \lim_{R_t\to\infty} f(R_t) = 0$; d) $f'\left(R^L\right) > 1$ always, while $f'(R^*) \gtreqless 0$ depending on $\chi \gtreqless \Upsilon^k$.*
2. *Suppose that $\chi \in (-\Upsilon^p, 0)$. Let R^l and R^u be the solutions to $J(R_t) = 1$ with $R^l \in \left(1, R^L\right)$ and $R^u > R^*$. Then a) $f : \left(R^l, R^u\right) \to (1,+\infty)$ and it is continuously differentiable ; b) within $\left(R^l, R^u\right)$, $f(.)$ has a unique minimum at R^J; c) $\lim_{R_t\to R^{l+}} f(R_t) = \lim_{R_t\to R^{u-}} f(R_t) = +\infty$; d) $f'(R^*) > 1$ always, while $f'\left(R^L\right) \gtreqless 0$ depending on $\chi \lesseqgtr \Upsilon^w$ with $\Upsilon^w \equiv R^L(1-\Upsilon^p) - 1 \in (-\Upsilon^p, 0)$.*

Proof. In order to simplify the notation in the proof and the statements, we define the following scalars: $\Upsilon^p \equiv \frac{R^*-1}{A}$, $\Upsilon^w \equiv R^L(1-\Upsilon^p) - 1$ and $\Upsilon^k \equiv \left(1 - \frac{R^*}{A}\right)(R^*-1)$. From the zero-lower-bound restriction and the assumption of an active rule, it is clear that $\Upsilon^p > 0$ and $\Upsilon^k > 0$.

Recall the definition of $f(R_t)$ in (27). From (28), $J(R_t)$ is always positive and continuously differentiable for any $R_t \in (1,+\infty)$ and any $\chi \in \mathbb{R}$. Furthermore, since $\frac{R^*}{[R^*-1]^{\frac{R^*-1}{A}}} > 1$ and $\Upsilon^p \in (0,1)$ (as $A > R^* > 1$) then

$$\lim_{R_t\to+\infty} J(R_t) = 0 \text{ for any } \chi \in \mathbb{R} \tag{37}$$

and

$$\lim_{R_t \to 1^+} J(R_t) = \begin{cases} 0 & \text{if } \chi > -\Upsilon^p \\ +\infty & \text{if } \chi < -\Upsilon^p \end{cases} \tag{38}$$

From (29) we also observe that $J'(R_t) \gtreqless 0$ if and only if $R_t \lesseqgtr \frac{1+\chi}{1-\Upsilon^p}$, as well as that $\frac{1+\chi}{1-\Upsilon^p} \gtreqless 1$ if and only if $\chi + \Upsilon^p \gtreqless 0$. As a consequence of these inequalities, when $\chi + \Upsilon^p < 0$ we have $\frac{1+\chi}{1-\Upsilon^p} < 1$, and therefore $J'(R_t) < 0$ for any $R_t > 1$.

Next we prove point 1. Suppose that $\chi > 0$. As $\chi + \Upsilon^p > 0$ then from (29) and (37) we infer that $J(R_t)$ is single-peaked at $R^J \equiv \frac{1+\chi}{1-\Upsilon^p} > 1$, with $\lim_{R_t \to 1^+} J(R_t) = \lim_{R_t \to +\infty} J(R_t) = 0$. An inspection of (27) shows that for $\chi > 0$, $f(R_t) > 1$ and it is continuous over the domain $(1, +\infty)$ if and only if $J(R_t) \in (0,1)$. But $J(R_t) > 0$ and with a maximum at R^J when $\chi > 0$. Then since $J(R^J) < 1$ holds, we have that $J(R_t) \in (0,1)$ for any $R_t \in (1,+\infty)$.[35] This in turn implies that $f : (1,+\infty) \to (1,+\infty)$. Furthermore, $f(.)$ is continuously differentiable since $J(.)$ is continuously differentiable as well. This proves Point 1(a). Point 1(b) follows from (29) together with $\chi > 0$; while Point 1(c) is obtained from the definition of $f(.)$ in (27) and (37)-(38). The complete characterization of $f(.)$ for $\chi > 0$ is achieved by noticing that being continuous and crossing the 45 degree line twice within its domain (see Proposition 1) it must necessarily be that $R^J > R^L$. But then $f(.)$ must cross the 45 degree line at the low steady state R^L with a slope bigger than 1. That is $f'(R^L) > 1$.[36] Nevertheless the sign of $f'(R^*)$ remains ambiguous because $sign\{f'(R^*)\} = sign\{J'(R^*)\} = sign\{R^J - R^*\}$ and $R^J \gtreqless R^*$. Simple algebra shows that this last inequality is equivalent to $\chi \gtreqless \Upsilon^k$. This proves Point 1(d).

Suppose that $\chi \in (-\Upsilon^p, 0)$ then the function $J(.)$ has the same properties we derived at the beginning of the proof for the case of $\chi > 0$. Thus $J(.)$ is single-peaked at $R^J > 1$, with $\lim_{R_t \to 1^+} J(R_t) = \lim_{R_t \to +\infty} J(R_t) = 0$. However the fact that $R^L < R^*$ (see Proposition 1) combined with $\chi < 0$ imply that although $R^J \in (R^L, R^*)$ we now have that $J(R^L) > J(R^*) > 1$. As R^J is a global maximum, $J(R^J) > 1$ as well. These properties together with $\lim_{R_t \to 1^+} J(R_t) = \lim_{R_t \to +\infty} J(R_t) = 0$ imply that there exist two values, R^l and R^u, such that $J(R^l) = J(R^u) = 1$ where $R^l \in (1, R^L)$ and $R^u > R^*$[37] Given the definition (27) of f (.) we can see that it is not well-defined at both R^l and R^u. Moreover, as $\lim_{R_t \to R^{l-}} f(R_t) = \lim_{R_t \to R^{u+}} f(R_t) = -\infty$ and $f(R_t) < 0$ for any $R_t > R^u$ and any $R_t \in (1, R^l)$, then any initial condition $R_0 \notin (R^l, R^u)$ would violate the zero-lower-bound condition and therefore cannot be part of a PFE of our model. The range $R_t \in (R^l, R^u)$ is therefore necessary for an iterated sequence to be supported as an equilibrium. Within the interval (R^l, R^u), $J(.)$ satisfies $J(R_t) > 1$ and is continuously differentiable, implying $J(R_t)^{\frac{1}{\chi}} \in (0,1)$ for any $\chi < 0$. It follows that $f(R_t) > 1$ and $f(.)$ is continuously differentiable for any $R_t \in (R^l, R^u)$. This proves Point 2(a). Point 2(b) follows from (29) together with $\chi < 0$, while Point 2(c) is obtained from the definition of $f(.)$ in (27) and (37)-(38). Since $f(.)$ crosses the 45 degree line at both R^L and R^* (see Proposition 1) it must be that $f'(R^*) > 1$. Nevertheless the sign of $f'(R^L)$ remains ambiguous as $sign\{f'(R^L)\} = -sign\{J'(R^L)\} = sign\{R^L - R^J\}$and $R^L \gtreqless R^J$. Simple algebra shows that this last inequality is equivalent to $\chi \lesseqgtr R^L(1-\Upsilon^p) - 1$. Finally Observe that an active rule implies $-\Upsilon^p < R^L(1-\Upsilon^p) - 1 = \Upsilon^w$ and that, from Assumption 3, $\frac{R^L - 1}{R^L} < R^L - 1 < \Upsilon^p$. Then $R^L(1-\Upsilon^p) - 1 < 0$ such

[35] Proving that $J(R^J) < 1$ involes some algebra. It is available from us upon request.

[36] Simple algebra also proves that $\lim_{R_t \to 1} f'(R_t) \in (0,1)$.

[37] This should be clear from drwaing the function $J(.)$, which is continous, hump-shaped, with maximum bigger than 1 and limits equal to zero.

that $\Upsilon^w < \Upsilon^f < 0$. Using this and $-\Upsilon^p < R^L(1-\Upsilon^p) - 1 = \Upsilon^w$ we conclude that $\Upsilon^w \in (-\Upsilon^p, 0)$. This completes the proof for Point 2(d).

For $\chi \leq -\Upsilon^p < 0$, it can be shown that the mapping $f(.)$ features no critical point and it is strictly increasing. ■

A.2 Proofs of the Propositions

A.2.1 Proof of Proposition 1

Proof. Let the left hand side and the right hand side of equation (23) be denoted as $LHS(R^{ss})$ and $RHS(R^{ss})$ respectively. Since $R^* > 1$ we have that

$$\lim_{R^{ss}\to 1} LHS(R^{ss}) = (R^* - 1)^{\frac{R^*-1}{A}} > 0 \qquad \textit{and} \qquad \lim_{R^{ss}\to 1} RHS(R^{ss}) = 0.$$

It is simple to verify that given $R^* > 1$ and $R^{ss} > 1$ then $LHS(R^{ss})$ is linear in R^{ss} with slope $(R^* - 1)^{\frac{R^*-1}{A}} > 0$ and $RHS(R^{ss})$ slopes upwards as well,

$$\frac{\partial RHS(R^{ss})}{\partial R^{ss}} = R^* \frac{R^* - 1}{A} (R^{ss} - 1)^{\frac{R^*-1}{A}-1} > 0.$$

Moreover $\frac{A}{R^*} > 1$, $R^* > 1$ and $R^{ss} > 1$ are sufficient conditions for $RHS(R^{SS})$ to be strictly concave. That is $\frac{\partial^2 RHS(R^{ss})}{\partial (R^{ss})^2} = R^* \frac{R^*-1}{A} (R^{ss} - 1)^{\frac{R^*-1}{A}-2} \left(\frac{R^*-1}{A} - 1\right) < 0$.

Given that $LHS(R^{ss})$ is continuous and linear and $RHS(R^{ss})$ is continuous and strictly concave for any $R^{ss} > 1$, a sufficient condition for a second solution $R^{ss} = R^L \in (1, R^*)$ to exist is that the slope of the $RHS(R^{ss})$ at R^* is smaller than the slope of $LHS(R^{ss})$ at R^*. In other words, $R^* \frac{R^*-1}{A} (R^* - 1)^{\frac{R^*-1}{A}-1} < (R^* - 1)^{\frac{R^*-1}{A}}$, or equivalently $\frac{A}{R^*} > 1$. Moreover since $RHS(R^{ss})$ is strictly concave then no other solution to (23) different from R^* and R^L exists. ■

A.2.2 Proof of Proposition 2

Proof. Recall the definition of $\chi(\alpha, \sigma)$ in (21). For $\sigma \in (0,1)$ and $\alpha \in (0,1)$ we have that $\chi : (0,1) \times (0,1) \to \mathbb{R}$. Then Point 1(a) follows from point 1 of Lemma 1 and the fact that $\chi(\alpha, \sigma) < 0$ if $\sigma \in (0,1)$ and $\alpha \in (0,1)$.

Next we prove point 1(b). For $\sigma \in (1,\infty)$ and $\alpha \in (0,1)$ we have that $\chi : (0,1) \times (1,\infty) \to \mathbb{R}$. Define $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$. The proof of point 1(a) of Lemma 1 makes clear that $\chi = \Upsilon^d$ is the threshold value of χ that differentiates between active rules leading to either a unique equilibrium or multiple equilibria. Since we are interested in how the pair (α, σ) affects local equilibrium determinacy, we look for values of α that, for given σ, solve $\chi(\alpha, \sigma) = \Upsilon^d$. Simple algebra shows that this solution is $\alpha^d(\sigma) = \left[1 - \frac{\Upsilon^d}{\chi_{\min}}\right]^{-1} \left[1 - \frac{\Upsilon^d}{\mu(\sigma)}\right]$, a function relating α to σ. Since Assumption 1 holds, that is $\Upsilon^d \in (0, \chi_{\max})$, then we can use Lemma 4 to deduce that the function $\alpha^d(\sigma)$ is equivalent to the function $\alpha^i(\sigma)$, defined in that Lemma, for $\Upsilon^i = \Upsilon^d$. Therefore, $\alpha^d(\sigma)$ inherits all the properties of $\alpha^i(\sigma)$.

From Point 3 of Lemma 4, we can conclude that for any $\sigma > 1$ then $\alpha > \alpha^d_{\min}$ with $\alpha^d_{\min} \equiv \max\left\{0, \alpha^d(\sigma)\right\}$. This is a necessary and sufficient condition for $\chi(\alpha, \sigma) < \Upsilon^d$ to hold. Which, by Lemma 1, implies local equilibrium determinacy.

To prove point 2 we follow the same steps of proving 1(b) taking into account that from point 2 of Lemma 4 we can deduce that $\sigma > \sigma^d$ and $\alpha \in (0, \alpha^d(\sigma))$ leads to $\chi(\alpha, \sigma) > \Upsilon^d$. Which by point 2 of Lemma 1 implies multiple equilibria. ■

A.2.3 Proof of Proposition 3

Proof. The proof of Point 1 combines the results of Point 3 in Lemma 2 with Lemma 5.

Define $\Upsilon^p \equiv \frac{R^*-1}{A}$, $\Upsilon^w \equiv R^L(1-\Upsilon^p)-1$ and $\Upsilon^f \equiv \frac{1}{2}\Upsilon^w$. From Point 3 of Lemma 2, the flip bifurcation threshold is $\chi = \Upsilon^f$. Since χ is a function of α and σ then this threshold can be expressed in the (α, σ)-space as the solution to $\chi(\alpha, \sigma) = \Upsilon^f$ with respect to α, for any arbitrary $\sigma \in (0, 1)$. Simple algebra shows that the solution is a function $\alpha^f(\sigma) \equiv \left[1 - \frac{\Upsilon^f}{\chi_{\min}}\right]^{-1}\left[1 - \frac{\Upsilon^f}{\mu(\sigma)}\right]$. Observe that an active rule implies $-\Upsilon^p < R^L(1-\Upsilon^p)-1 = \Upsilon^w$ and that, from Assumption 3, $\frac{R^L-1}{R^L} < R^L - 1 < \Upsilon^p$. Then $R^L(1-\Upsilon^p)-1 < 0$ which implies that $\Upsilon^w < \Upsilon^f < 0$. Using this and $-\Upsilon^p < R^L(1-\Upsilon^p)-1 = \Upsilon^w$ we conclude that $\Upsilon^w \in (-\Upsilon^p, 0)$. From this and Assumption 2 we can see that $\Upsilon^f \in (\chi_{\min}, 0)$ satisfying the assumptions in Lemma 5. Hence $\alpha^f(\sigma)$ shares all the properties of the function $\alpha^i(\sigma)$ for $\Upsilon^i = \Upsilon^f$ stated in that Lemma. By Point 3 of Lemma 5, for an arbitrary $\sigma \in (0, 1)$, the flip bifurcation condition $\chi(\alpha, \sigma) > \Upsilon^f$ in Lemma 2 is therefore equivalent to $\alpha > \alpha^f_{\min}$ where $\alpha^f_{\min} \equiv \max\{0, \alpha^f(\sigma)\}$. From an application of Point 1 of Lemma 5 we can also see that such minimum degree of openness, $\alpha^f_{\min}$, is positive and decreasing in σ, for $\sigma \in (0, \sigma^f)$, but constant and equal to zero for $\sigma \in [\sigma^f, 1)$.

The proof of Point 2 is omitted since it is very similar to the proof of Point 1. But it uses the fact that $\sigma > 1$ and Point 3 of Lemma 3 together with Lemma 4 instead. ■

A.2.4 Proof of Proposition 6

Proof. We provide a sketch of the proof. Let $K(R_{t+1})$ and $J(R_t)$ be the left and the right hand sides of (36) respectively. First of all notice that for $\sigma > 1$, we have $\Psi < 0$ and $\chi > 0$. To simplify the notation let $\Gamma \equiv (1-\Psi)\chi$.

Then consider the function $K(.)$. Since $\Gamma > 0$, it is simple to show that $\lim_{R_{t+1}\to 1^+} K(R_{t+1}) = 0$ and $\lim_{R_{t+1}\to +\infty} K(R_{t+1}) = +\infty$. Furthermore simple algebra shows that $K(.)$ is strictly increasing over the entire domain $(1, +\infty)$. But then $K(.)$ is also globally invertible such that there exist a well defined function $f(R_t) = K^{-1}(J(R_t))$ describing the forward equilibrium dynamics.

Then consider the function $J(R_t)$. Since $\chi + \frac{R^*-1}{A} > 0$ for any $\alpha \in (0, 1)$, then $\lim_{R_t\to 1^+} J(R_t) = 0$, while:

$$\lim_{R_t\to +\infty} J(R_t) = \begin{cases} +\infty \text{ if } \Gamma > \left(1 - \frac{R^*-1}{A}\right) \\ 0 \quad \text{if } \Gamma < \left(1 - \frac{R^*-1}{A}\right) \end{cases}$$

Using the definition of Γ, let's consider the case of $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)}{(1-\Psi)}$ such that $\lim_{R_t\to +\infty} J(R_t) = 0$. It is possible to show that $J(.)$ has a critical point at $R^J = \frac{1+\Psi\chi}{\left(1-\frac{R^*-1}{A}\right)} > 1$. Therefore if $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)}{(1-\Psi)}$ the function $J(.)$ is single-peaked. This in turn in tandem with the fact that $K(.)$ is strictly increasing over the entire domain $(1, +\infty)$ implies that the equilibrium mapping $f(.)$ looks like a logistic map. Then if cycles exist they have to be centered around the active steady state. To show this we can use a similar argument to the one developed in Lemma 3. That is a sufficient condition for period-2 cycles to exist is $f'(R^*) < -1$. By computing this derivative at the target steady state and after some algebra this condition is equivalent to $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)R^*-1}{2R^*}$. Given $\sigma > 1$ it is possible to show that there exists $\hat{\alpha} \in (0,1)$ such that $\chi = \frac{\left(1-\frac{R^*-1}{A}\right)R^*-1}{2R^*}$ and that for $\alpha > \hat{\alpha}$, $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)R^*-1}{2R^*}$. ■

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