Appendix 9

CONFIDENCE LIMITS

The record of annual peak flows at a site is a random sample of the underlying population of annual peaks and can be used to estimate the frequency curve of that population. If the same size random sample could be selected from a different period of time, a different estimate of the underlying population frequency curve probably would result. Thus, an estimated flood frequency curve can be only an approximation to the true frequency curve of the underlying population of annual flood peaks. To gauge the accuracy of this approximation, one may construct an interval or range of hypothetical frequency curves that, with a high degree of confidence, contains the population frequency curve. Such intervals are called confidence intervals and their end points are called confidence limits.

This appendix explains how to construct confidence intervals for flood discharges that have specified exceedance probabilities. To this end, let X_P^* denote the true or population logarithmic discharge that has exceedance probability P. Upper and lower confidence limits for X_P^*, with confidence level c, are defined to be numbers $U_{P,c}(X)$ and $L_{P,c}(X)$, based on the observed flood records, X, such that the upper confidence limit $U_{P,c}(X)$ lies above X_P^* with probability c and the lower limit $L_{P,c}(X)$ lies below X_P^* with probability c. That is, the confidence limits have the property that

$$\text{Probability} \quad \{U_{P,c}(X) \geq X_P^*\} = c \qquad (9\text{-}1a)$$

$$\text{Probability} \quad \{L_{P,c}(X) \leq X_P^*\} = c \qquad (9\text{-}1b)$$

*
Explicit formulas for computing the confidence limits are given below; the above formulas simply explain the statistical meaning of the confidence limits.

The confidence limits defined above are called <u>one-sided</u> confidence limits because each of them describes a bound or limit on just one side of the population p-probability discharge. A two-sided confidence interval can be formed from the overlap or union of the two one-sided intervals, as follows:

$$\text{Probability} \quad \left\{ L_{P,c}(X) \leq X_P^* \leq U_{P,c}(X) \right\} = 2c-1 \qquad (9\text{-}2)$$

Thus, the union of two one-sided 95-percent confidence intervals is a two-sided 90-percent interval. It should be noted that the two-sided interval so formed may not be the narrowest possible interval with that confidence level; nevertheless, it is considered satisfactory for use with these guidelines.

It may be noted in the above equations that $U_{P,c}(X)$ can lie above X_P^* if and only if $U_{P,c}(X)$ lies above a fraction (1-P) of all possible floods in the population. In quality control terminology, $U_{P,c}(X)$ would be called an upper <u>tolerance limit</u>, at confidence level c, for the proportion (1-P) of the population. Similarly, $L_{P,c}(X)$ would be a lower tolerance limit for the proportion (P). Because the tolerance limit terminology refers to proportions of the population, whereas the confidence-limit terminology refers directly to the discharge of interest, the confidence-limit terminology is adopted in these guidelines.

Explicit formulas for the confidence limits are derived by specifying the general form of the limits and making additional simplifying assumptions to analyze the relationships between sample statistics and population statistics. The general form of the confidence limits is specified as:

$$U_{P,c}(X) = \overline{X} + S\left(K_{P,c}^{U}\right) \qquad (9\text{-}3a)$$

$$L_{P,c}(X) = \overline{X} + S\left(K_{P,c}^{L}\right) \qquad (9\text{-}3b)$$

*

*
in which $\overline{X}$ and S are the logarithmic mean and standard deviation of the final estimated log Pearson Type III frequency curve and $K^{U}_{P,c}$ and $K^{L}_{P,c}$ are upper and lower confidence coefficients.

The confidence coefficients approximate the non-central t-distribution. The non-central t-variate can be obtained in tables (41, 32), although the process is cumbersome when G_w is non-zero. More convenient is the use of the following approximate formulas (32, pp. 2-15), based on a large sample approximation to the non-central t-distribution (42):

$$K^{U}_{P,c} = \frac{K_{G_w,P} + \sqrt{K^2_{G_w,P} - ab}}{a} \qquad (9\text{-}4a)$$

$$K^{L}_{P,c} = \frac{K_{G_w,P} - \sqrt{K^2_{G_w,P} - ab}}{a} \qquad (9\text{-}4b)$$

in which

$$a = 1 - \frac{z_c^2}{2(N-1)} \qquad (9\text{-}5)$$

$$b = K^2_{G_w,P} - \frac{z_c^2}{N} \qquad (9\text{-}6)$$

and z_c is the standard normal deviate (zero-skew Pearson Type III deviate) with cumulative probability c (exceedance probability 1-c). The systematic record length N is deemed to control the statistical reliability of the estimated frequency curve and is to be used for calculating confidence limits even when historic information has been used to estimate the frequency curve.

The use of equations 9-3 through 9-6 is illustrated by calculating 95-percent confidence limits for $X^{*}_{0.01}$, the 0.01 exceedance probability flood, when the estimated frequency curve has logarithmic mean, standard deviation, and skewness of 3.00, 0.25, and 0.20, respectively based on 50 years of systematic record.

*

$z_c = 1.645$ $\qquad K_{G_W,P} = 2.4723$

$$a = 1 - \frac{(1.645)^2}{98} = 0.9724$$

$$b = (2.4723)^2 - \frac{(1.645)^2}{50} = 6.058$$

$$K^u_{0.01,\,0.95} = \frac{2.4723 + \sqrt{(2.4723)^2 - (0.9724)(6.058)}}{0.9724}$$

$$= 3.026$$

$$K^L_{0.01,\,0.95} = \frac{2.4723 - \sqrt{(2.4723)^2 - (0.9724)(6.058)}}{0.9724}$$

$$= 2.059$$

$$U_{0.01,\,0.95}(X) = 3.00 + (0.25)(3.026) = 3.756$$

$$L_{0.01,\,0.95}(X) = 3.00 + (0.25)(2.059) = 3.515$$

The corresponding limits in natural units (cubic feet per second) are 3270 and 5700; the estimated 0.01 exceedance probability flood is 4150 cubic feet per second.

Table 9-1 is a portion of the non-central t tables (43) for a skew of zero and can be used to compute $K^U_{P,c}$ and $K^L_{P,c}$ for selected values of P and c when the distribution of logarithms of the annual peaks is normal (i.e., $G_W=0$).

An example of using table 9-1 to compute confidence limits is as follows: Assume the 95-percent confidence limits are desired for $X^*_{0.01}$, the 0.01 exceedance probability flood for a frequency curve with logarithmic mean, standard deviation and skewness of 3.00, 0.25 and 0.00, respectively, based on 50 years of systematic record.

$*K^{U}_{0.01,\ 0.95} = 2.862$ Found by entering table 9-1 with confidence level 0.05, systematic record length 50 and exceedance probability 0.01.

$K^{L}_{0.01,\ 0.95} = 1.936$ Found by entering table 9-1 with confidence level 0.95,systematic record length 50 and exceedance probability 0.01.

$$U_{0.01,\ 0.95}(X) = 3.00 + 0.25(2.862) = 3.715$$

$$L_{0.01,\ 0.95}(X) = 3.00 + 0.25(1.936) = 3.484$$

The corresponding limits in natural units (cubic feet per second) are 3050 and 5190; the estimated 0.01 exceedance probability flood is 3820 cubic feet per second. *

*

Appendix 9 Notation

$U_{P,c}(X)$ = upper confidence limit in log units

$L_{P,c}(X)$ = lower confidence limit in log units

P = exceedance probability

c = confidence level

X_P^* = population logarithmic discharge for exceedance probability P

$\overline{X}$ = mean logarithm of peak flows

S = standard deviation of logarithms of annual peak discharges

$K_{G_W,P}$ = Pearson Type III coordinate expressed in number of standard deviations from the mean for weighted skew (G_W) and exceedance probability (P).

G_W = weighted skew coefficient

$K_{P,c}^U$ = upper confidence coefficient

$K_{P,c}^L$ = lower confidence coefficient

N = systematic record length

z_c = is the standard normal deviate *

TABLE 9-1
CONFIDENCE LIMIT DEVIATE VALUES FOR NORMAL DISTRIBUTION

Confidence Level	Systematic Record Length	EXCEEDANCE PROBABILITY											
	N	.002	.005	.010	.020	.040	.100	.200	.500	.800	.900	.950	.990
.01	10	6.178	5.572	5.074	4.535	3.942	3.048	2.243	.892	-.107	-.508	-.804	-1.314
	15	5.147	4.639	4.222	3.770	3.274	2.521	1.841	.678	-.236	-.629	-.929	-1.458
	20	4.675	4.212	3.832	3.419	2.965	2.276	1.651	.568	-.313	-.705	-1.008	-1.550
	25	4.398	3.960	3.601	3.211	2.782	2.129	1.536	.498	-.364	-.757	-1.064	-1.616
	30	4.212	3.792	3.447	3.071	2.658	2.030	1.457	.450	-.403	-.797	-1.107	-1.667
	40	3.975	3.577	3.249	2.893	2.500	1.902	1.355	.384	-.457	-.854	-1.169	-1.741
	50	3.826	3.442	3.125	2.781	2.401	1.821	1.290	.340	-.496	-.894	-1.212	-1.793
	60	3.723	3.347	3.038	2.702	2.331	1.764	1.244	.309	-.524	-.924	-1.245	-1.833
	70	3.647	3.278	2.974	2.644	2.280	1.722	1.210	.285	-.545	-.948	-1.272	-1.865
	80	3.587	3.223	2.924	2.599	2.239	1.688	1.183	.265	-.563	-.968	-1.293	-1.891
	90	3.538	3.179	2.883	2.561	2.206	1.661	1.160	.250	-.578	-.984	-1.311	-1.913
	100	3.498	3.143	2.850	2.531	2.179	1.639	1.142	.236	-.591	-.998	-1.326	-1.932
.05	10	4.862	4.379	3.981	3.549	3.075	2.355	1.702	.580	-.317	-.712	-1.017	-1.563
	15	4.304	3.874	3.520	3.136	2.713	2.068	1.482	.455	-.406	-.802	-1.114	-1.677
	20	4.033	3.628	3.295	2.934	2.534	1.926	1.370	.387	-.460	-.858	-1.175	-1.749
	25	3.868	3.478	3.158	2.809	2.425	1.838	1.301	.342	-.497	-.898	-1.217	-1.801
	30	3.755	3.376	3.064	2.724	2.350	1.777	1.252	.310	-.525	-.928	-1.250	-1.840
	40	3.608	3.242	2.941	2.613	2.251	1.697	1.188	.266	-.565	-.970	-1.297	-1.896
	50	3.515	3.157	2.862	2.542	2.188	1.646	1.146	.237	-.592	-1.000	-1.329	-1.936
	60	3.448	3.096	2.807	2.492	2.143	1.609	1.116	.216	-.612	-1.022	-1.354	-1.966
	70	3.399	3.051	2.765	2.454	2.110	1.581	1.093	.199	-.629	-1.040	-1.374	-1.990
	80	3.360	3.016	2.733	2.425	2.083	1.559	1.076	.186	-.642	-1.054	-1.390	-2.010
	90	3.328	2.987	2.706	2.400	2.062	1.542	1.061	.175	-.652	-1.066	-1.403	-2.026
	100	3.301	2.963	2.684	2.380	2.044	1.527	1.049	.166	-.662	-1.077	-1.414	-2.040

TABLE 9-1 (CONTINUED)
CONFIDENCE LIMIT DEVIATE VALUES FOR NORMAL DISTRIBUTION

Confidence Level	Systematic Record Length N	EXCEEDANCE PROBABILITY .002	.005	.010	.020	.040	.100	.200	.500	.800	.900	.950	.990
.10	10	4.324	3.889	3.532	3.144	2.716	2.066	1.474	.437	-.429	-.828	-1.144	-1.715
	15	3.936	3.539	3.212	2.857	2.465	1.867	1.320	.347	-.499	-.901	-1.222	-1.808
	20	3.743	3.364	3.052	2.712	2.338	1.765	1.240	.297	-.541	-.946	-1.271	-1.867
	25	3.623	3.255	2.952	2.623	2.258	1.702	1.190	.264	-.570	-.978	-1.306	-1.908
	30	3.541	3.181	2.884	2.561	2.204	1.657	1.154	.239	-.593	-1.002	-1.332	-1.940
	40	3.433	3.082	2.793	2.479	2.131	1.598	1.106	.206	-.624	-1.036	-1.369	-1.986
	50	3.363	3.019	2.735	2.426	2.084	1.559	1.075	.184	-.645	-1.059	-1.396	-2.018
	60	3.313	2.974	2.694	2.389	2.051	1.532	1.052	.167	-.662	-1.077	-1.415	-2.042
	70	3.276	2.940	2.662	2.360	2.025	1.511	1.035	.155	-.674	-1.091	-1.431	-2.061
	80	3.247	2.913	2.638	2.338	2.006	1.495	1.021	.144	-.684	-1.103	-1.444	-2.077
	90	3.223	2.891	2.618	2.319	1.989	1.481	1.010	.136	-.693	-1.112	-1.454	-2.090
	100	3.203	2.873	2.601	2.305	1.976	1.470	1.001	.129	-.701	-1.120	-1.463	-2.101
.25	10	3.599	3.231	2.927	2.596	2.231	1.671	1.155	.222	-.625	-1.043	-1.382	-2.008
	15	3.415	3.064	2.775	2.460	2.112	1.577	1.083	.179	-.661	-1.081	-1.422	-2.055
	20	3.320	2.978	2.697	2.390	2.050	1.528	1.045	.154	-.683	-1.104	-1.448	-2.085
	25	3.261	2.925	2.648	2.346	2.011	1.497	1.020	.137	-.699	-1.121	-1.466	-2.106
	30	3.220	2.888	2.614	2.315	1.984	1.475	1.002	.125	-.710	-1.133	-1.479	-2.123
	40	3.165	2.838	2.568	2.274	1.948	1.445	.978	.108	-.726	-1.151	-1.499	-2.147
	50	3.129	2.805	2.538	2.247	1.924	1.425	.962	.096	-.738	-1.164	-1.513	-2.163
	60	3.105	2.783	2.517	2.227	1.907	1.411	.950	.088	-.747	-1.173	-1.523	-2.176
	70	3.085	2.765	2.501	2.213	1.893	1.401	.942	.081	-.753	-1.181	-1.532	-2.186
	80	3.070	2.752	2.489	2.202	1.883	1.392	.935	.076	-.759	-1.187	-1.538	-2.194
	90	3.058	2.740	2.478	2.192	1.875	1.386	.929	.071	-.763	-1.192	-1.544	-2.201
	100	3.048	2.731	2.470	2.184	1.868	1.380	.925	.068	-.767	-1.196	-1.549	-2.207

TABLE 9-1 (CONTINUED)
CONFIDENCE LIMIT DEVIATE VALUES FOR NORMAL DISTRIBUTION

Confidence Level	Systematic Record Length	EXCEEDANCE PROBABILITY											
	N	.002	.005	.010	.020	.040	.100	.200	.500	.800	.900	.950	.990
.75	10	2.508	2.235	2.008	1.759	1.480	1.043	.625	-.222	-1.155	-1.671	-2.104	-2.927
	15	2.562	2.284	2.055	1.803	1.521	1.081	.661	-.179	-1.083	-1.577	-1.991	-2.775
	20	2.597	2.317	2.085	1.831	1.547	1.104	.683	-.154	-1.045	-1.528	-1.932	-2.697
	25	2.621	2.339	2.106	1.851	1.566	1.121	.699	-.137	-1.020	-1.497	-1.895	-2.648
	30	2.641	2.357	2.123	1.867	1.580	1.133	.710	-.125	-1.002	-1.475	-1.869	-2.614
	40	2.668	2.383	2.147	1.888	1.600	1.151	.726	-.108	-.978	-1.445	-1.834	-2.568
	50	2.688	2.400	2.163	1.903	1.614	1.164	.738	-.096	-.962	-1.425	-1.811	-2.538
	60	2.702	2.414	2.176	1.916	1.625	1.173	.747	-.088	-.950	-1.411	-1.795	-2.517
	70	2.714	2.425	2.186	1.925	1.634	1.181	.753	-.081	-.942	-1.401	-1.782	-2.501
	80	2.724	2.434	2.194	1.932	1.640	1.187	.759	-.076	-.935	-1.392	-1.772	-2.489
	90	2.731	2.441	2.201	1.938	1.646	1.192	.763	-.071	-.929	-1.386	-1.764	-2.478
	100	2.739	2.447	2.207	1.944	1.652	1.196	.767	-.068	-.925	-1.380	-1.758	-2.470
.90	10	2.165	1.919	1.715	1.489	1.234	.828	.429	-.437	-1.474	-2.066	-2.568	-3.532
	15	2.273	2.019	1.808	1.576	1.314	.901	.499	-.347	-1.320	-1.867	-2.329	-3.212
	20	2.342	2.082	1.867	1.630	1.364	.946	.541	-.297	-1.240	-1.765	-2.208	-3.052
	25	2.390	2.126	1.908	1.669	1.400	.978	.570	-.264	-1.190	-1.702	-2.132	-2.952
	30	2.426	2.160	1.940	1.698	1.427	1.002	.593	-.239	-1.154	-1.657	-2.080	-2.884
	40	2.479	2.209	1.986	1.740	1.465	1.036	.624	-.206	-1.106	-1.598	-2.010	-2.793
	50	2.517	2.244	2.018	1.770	1.493	1.059	.645	-.184	-1.075	-1.559	-1.965	-2.735
	60	2.544	2.269	2.042	1.792	1.513	1.077	.662	-.167	-1.052	-1.532	-1.933	-2.694
	70	2.567	2.290	2.061	1.810	1.529	1.091	.674	-.155	-1.035	-1.511	-1.909	-2.662
	80	2.585	2.307	2.077	1.824	1.543	1.103	.684	-.144	-1.021	-1.495	-1.890	-2.638
	90	2.600	2.321	2.090	1.836	1.553	1.112	.693	-.136	-1.010	-1.481	-1.874	-2.618
	100	2.613	2.333	2.101	1.847	1.563	1.120	.701	-.129	-1.001	-1.470	-1.861	-2.601

TABLE 9-1 (CONTINUED)
CONFIDENCE LIMIT DEVIATE VALUES FOR NORMAL DISTRIBUTION

EXCEEDANCE PROBABILITY

Confidence Level	Systematic Record Length N	.002	.005	.010	.020	.040	.100	.200	.500	.800	.900	.950	.990
.95	10	1.989	1.757	1.563	1.348	1.104	.712	.317	-.580	-1.702	-2.355	-2.911	-3.981
	15	2.121	1.878	1.677	1.454	1.203	.802	.406	-.455	-1.482	-2.068	-2.566	-3.520
	20	2.204	1.955	1.749	1.522	1.266	.858	.460	-.387	-1.370	-1.926	-2.396	-3.295
	25	2.264	2.011	1.801	1.569	1.309	.898	.497	-.342	-1.301	-1.838	-2.292	-3.158
	30	2.310	2.053	1.840	1.605	1.342	.928	.525	-.310	-1.252	-1.777	-2.220	-3.064
	40	2.375	2.113	1.896	1.657	1.391	.970	.565	-.266	-1.188	-1.697	-2.125	-2.941
	50	2.421	2.156	1.936	1.694	1.424	1.000	.592	-.237	-1.146	-1.646	-2.065	-2.862
	60	2.456	2.188	1.966	1.722	1.450	1.022	.612	-.216	-1.116	-1.609	-2.022	-2.807
	70	2.484	2.214	1.990	1.745	1.470	1.040	.629	-.199	-1.093	-1.581	-1.990	-2.765
	80	2.507	2.235	2.010	1.762	1.487	1.054	.642	-.186	-1.076	-1.559	-1.964	-2.733
	90	2.526	2.252	2.026	1.778	1.500	1.066	.652	-.175	-1.061	-1.542	-1.944	-2.706
	100	2.542	2.267	2.040	1.791	1.512	1.077	.662	-.166	-1.049	-1.527	-1.927	-2.684
.99	10	1.704	1.492	1.314	1.115	.886	.508	.107	-.892	-2.243	-3.048	-3.738	-5.074
	15	1.868	1.645	1.458	1.251	1.014	.629	.236	-.678	-1.841	-2.521	-3.102	-4.222
	20	1.974	1.743	1.550	1.336	1.094	.705	.313	-.568	-1.651	-2.276	-2.808	-3.832
	25	2.050	1.813	1.616	1.399	1.152	.757	.364	-.498	-1.536	-2.129	-2.633	-3.601
	30	2.109	1.867	1.667	1.446	1.196	.797	.403	-.450	-1.457	-2.030	-2.515	-3.447
	40	2.194	1.946	1.741	1.515	1.259	.854	.457	-.384	-1.355	-1.902	-2.364	-3.249
	50	2.255	2.002	1.793	1.563	1.304	.894	.496	-.340	-1.290	-1.821	-2.269	-3.125
	60	2.301	2.045	1.833	1.600	1.337	.924	.524	-.309	-1.244	-1.764	-2.202	-3.038
	70	2.338	2.079	1.865	1.630	1.365	.948	.545	-.285	-1.210	-1.722	-2.153	-2.974
	80	2.368	2.107	1.891	1.653	1.387	.968	.563	-.265	-1.183	-1.688	-2.114	-2.924
	90	2.394	2.131	1.913	1.674	1.405	.984	.578	-.250	-1.160	-1.661	-2.082	-2.883
	100	2.416	2.151	1.932	1.691	1.421	.998	.591	-.236	-1.142	-1.639	-2.056	-2.850