Computing the Trajectory of Free Jets

Tony L. Wahl, P.E., M.ASCE[1]; Kathleen H. Frizell[2]; and Elisabeth A. Cohen, P.E., M.ASCE[3]

Abstract: In recent years, design floods have increased beyond spillway capacity at numerous large dams. When additional spillway capacity is difficult or expensive to develop, designers may consider allowing the overtopping of a dam during extreme events. For concrete arch dams, this often raises issues of potential erosion and scour downstream from the dam, where the free jet initiating at the dam crest impacts the abutments and the downstream river channel. A recent review has shown that a commonly cited equation for predicting the trajectory of free jets is flawed, producing jet trajectories that are much too flat in this application. This could lead analysts to underestimate the amount of scour that could occur near a dam foundation, or conversely to overestimate the extent of scour protection required. This technical note presents the correct and incorrect jet trajectory equations, quantifies the errors associated with the flawed equation, and summarizes practical information needed to model the trajectory of free jets overtopping dam crests.

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Introduction

In recent years, the increasing magnitude of design floods has prompted reevaluations of spillway capacity and operational scenarios for large dams throughout the world. Many of these investigations have shown that current spillway capacity is inadequate, raising the possibility that dams might be overtopped during extreme events. Creating additional spillway capacity is often expensive and sometimes technically infeasible, and in these cases, dam owners sometimes consider accepting overtopping as a planned operation during extreme events. This creates new loading scenarios for the dam and raises questions about erosion and scour downstream from the dam. For concrete arch dams, scour may occur along the abutments and in the downstream river channel, where the jet overtopping the dam impacts upon materials that provide the foundation for the dam. To evaluate the need for protection of these areas, a comparison of the potential hydraulic attack and erosion resistance of these materials is needed (Annandale 2006). The first stage of this analysis is to define the jet trajectory and impact zones. A recent review of jet trajectory equations conducted by the Bureau of Reclamation has shown that a widely cited equation is flawed and produces jet trajectories that are much too flat for this application.

Jet Trajectory Equations

Fig. 1(a) shows the flow situation. The reservoir is surcharged to produce an overtopping head H_{ovtop}. A jet with velocity v_0 is produced and springs free from the dam crest. If we define the downstream edge of the crest as the origin of an x-y coordinate system, the equation of motion in a plane is

$$y = x \tan \theta_0 - \frac{gx^2}{2v_0^2 \cos^2 \theta_0} \quad (1)$$

where x and y=coordinates of the bottom edge of the jet, θ_0=initial angle of the jet from horizontal (zero in the figure, positive if the jet issues upward, and negative if the jet is initially

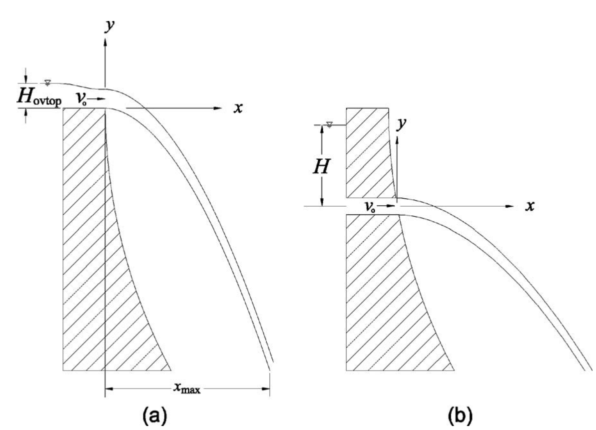

Fig. 1. Free jets (a) overtopping a dam; (b) issuing from an orifice through a dam

[1]Hydraulic Engineer, U.S. Dept. of the Interior, Hydraulic Investigations and Laboratory Services Group, Bureau of Reclamation, Denver, CO. E-mail: twahl@do.usbr.gov

[2]Hydraulic Engineer, U.S. Dept. of the Interior, Hydraulic Investigations and Laboratory Services Group, Bureau of Reclamation, Denver, CO. E-mail: kfrizell@do.usbr.gov

[3]Civil Engineer, U.S. Dept. of the Interior, Waterways and Concrete Dams Group, Bureau of Reclamation, Denver, CO. E-mail: bcohen@do.usbr.gov

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inclined downward), v_0=velocity of the jet as it leaves the dam crest, and g=acceleration due to gravity. When the jet issues horizontally, the equation is greatly simplified, becoming

$$y = -\frac{gx^2}{2v_0^2} \tag{2}$$

The derivation of these elementary equations is presented in many physics texts (Halliday and Resnick 1981), and these equations in a slightly modified form are given also by Chow (1959). They describe the motion of a projectile unaffected by wind resistance. In reality, projectiles or free jets will always travel an x distance somewhat shorter than that computed by these equations, due to wind resistance and jet breakup.

Through algebraic manipulation, the trajectory equation can be restated in terms of the velocity head $h_v = v_0^2/(2g)$

$$y = x \tan \theta_0 - \frac{x^2}{4h_v \cos^2 \theta_0} \tag{3}$$

One could also compute the trajectory of the top surface of the jet by simply adding the initial jet thickness t_0, assuming that the velocity and angle of orientation are the same as for the bottom edge of the jet

$$y = t_0 + x \tan \theta_0 - \frac{x^2}{4h_v \cos^2 \theta_0} \tag{4}$$

Davis et al. (1999) presented similar equations for computing the top surface of the nappe at a free overfall. They related the initial velocity of the jet to the velocity at a section upstream from the brink. Rouse (1943) related the initial jet thickness to the Froude number at an upstream section, and showed that the vertical jet thickness is nearly constant as the jet falls.

For convenient application to the situation of flow overtopping a dam, practitioners prefer an equation that expresses the jet trajectory as a function of the reservoir head. In the case of flow issuing from an orifice, this can be easily obtained. Fig. 1(b) shows this case, in which the initial velocity, neglecting losses in the conduit leading to the outlet, is

$$v_0 = \sqrt{2gH} \tag{5}$$

where H=total head on the centerline of the opening (assuming the velocity in the reservoir to be negligible), and the velocity head is

$$h_v = \frac{v_0^2}{2g} = \frac{2gH}{2g} = H \tag{6}$$

All of the potential energy of the reservoir above the orifice centerline is converted to velocity head. In reality, the velocities at the top and bottom of the jet may be slightly different due to the difference in head across the height of the orifice, but this detail is never considered. Inserting H in place of h_v in Eq. (3), we obtain

$$y = x \tan \theta_0 - \frac{x^2}{4H \cos^2 \theta_0} \tag{7}$$

This equation is presented in the Bureau of Reclamation's *Design of Small Dams* (3rd Ed., p. 376) (Bureau of Reclamation 1987) to compute the trajectory of flow issuing through an orifice (or beneath a gate) at an angle θ_0. (It should be noted that a different sign convention is used in *Design of Small Dams*, but the equation is otherwise the same.)

Returning to the case of flow overtopping a dam, a similar jet trajectory equation is presented in *Design of Small Dams* (pp. 385, 387) for use in designing open-channel spillway chutes and deflector buckets

$$y = x \tan \theta_0 - \frac{x^2}{4K(d + h_v)\cos^2 \theta_0} \tag{8}$$

where d=depth of flow, and h_v=velocity head, as defined previously. This equation is described as the trajectory of a free jet. The equation is modified with the factor K, a constant suggested to have values less than or equal to 1.0 (0.9 and 0.75 are commonly suggested) when computing a real jet trajectory. When designing a spillway chute profile, a value of K=1.5 is suggested to ensure that the floor of the spillway will fully support the jet to reduce the possibility of negative pressures and cavitation on the spillway surface. Eq. (8) is also cited in Reclamation's *Design of Gravity Dams* (Bureau of Reclamation 1976) and *Design of Arch Dams* (Bureau of Reclamation 1977), and in a recent treatise on the subject of scour prediction (Annandale 2006, pp. 146–151), where it is suggested for use in computing the trajectory of free jets overtopping dams.

Eq. (8) may not raise suspicion at first glance, because it appears to be a simple modification of Eq. (7), replacing the total head acting on the orifice with the total head of the open channel flow. In the dam overtopping case, this is approximately equal to the overtopping head if losses are neglected. However, comparing Eqs. (8) and (3), which was developed directly from the projectile motion equation, we see that they are not equivalent, even when K=1. Eq. (3) contains only the velocity head in the denominator of the second term, but Eq. (8) contains an additional depth term, which causes it to compute a flatter trajectory than Eq. (3). Eq. (8) would be correct if the entire overtopping head were being converted to velocity head, but this does not occur because until the flow springs free from the crest, a nearly hydrostatic pressure profile exists in the flow, and part of the energy is in the form of pressure head. As a first approximation, the flow overtopping the dam should be near critical depth and velocity, with the depth being about two-thirds of overtopping head and the velocity head being the remaining one third. Further confusion is created because most citations of this equation (including *Design of Small Dams*) do not clearly define the terms d and h_v in context, leading some to conclude that the d value used should be the total overtopping head, with h_v then added to it. This causes an even flatter trajectory to be computed.

The development of Eq. (8) is not well documented. The equation and accompanying discussion of its use appeared in the 1st edition of *Design of Small Dams* (Bureau of Reclamation 1960) and has appeared essentially unchanged to this date in subsequent editions and in the other publications already mentioned, including many foreign-language translations. It has always been presented without derivation or citation of its origin, and no occurrence of it has been found in literature prior to that time. One can surmise that Eq. (8) was developed by simple inspection, starting with Eq. (7) for the trajectory of an inclined jet issuing from an orifice and replacing the total head term H with the total head of an open channel flow $d+h_v$ and adding the K coefficient to account for other factors, such as jet breakup and wind resistance. Unfortunately, since a rigorous derivation process was not followed, the fact was missed that the velocity head was the crucial quantity, not the total head. None of the publications presenting this equation make a specific argument for the inclusion of the depth term. One other Bureau of Reclamation publication addresses this topic. Engineering Monograph No. 25—*Hydraulic Design of Stilling Basins and Energy Dissipators* (Peterka 1958)

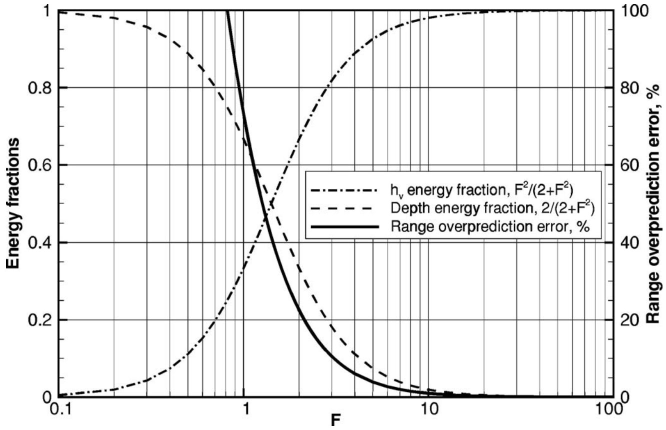

Fig. 2. Jet trajectory error as a function of the Froude number

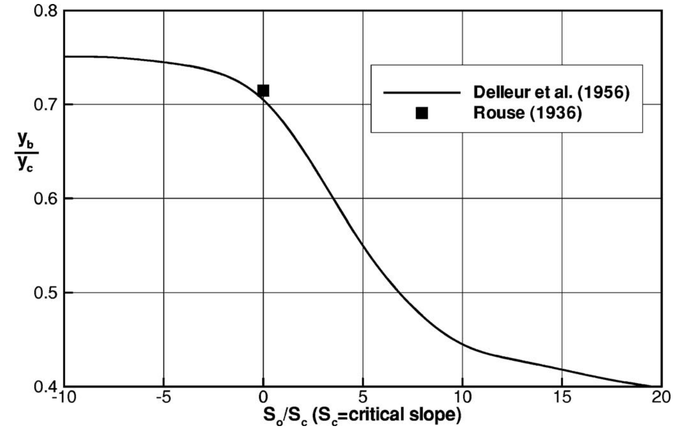

Fig. 3. Ratio of brink depth to critical depth for sloped (Delleur et al. 1956) and horizontal (Rouse 1936) free overfalls

presents equations for computing the throw distance of tunnel spillway flip buckets. The equations are based on the velocity at the bucket entrance and are thus correct. Eq. (8) does not appear in the monograph.

Trajectory Calculation Errors

The error introduced into a trajectory calculation by using Eq. (8) rather than Eq. (3) depends upon the initial Froude number of the jet. Wahl (2001) showed that the fraction of the specific energy associated with the flow depth is $2/(2+\mathsf{F}^2)$, where F=Froude number $[\mathsf{F}=v_o/(gD)^{0.5}]$, with D being the hydraulic depth. Similarly, the fraction of the specific energy associated with the velocity head is $\mathsf{F}^2/(2+\mathsf{F}^2)$. This information can be used in this case to show that the ratio of the erroneous $d+h_v$ to the correct h_v is $(2+\mathsf{F}^2)/\mathsf{F}^2$. The range of the jet at any point on the trajectory x is proportional to the square root of the head term in the denominator of the trajectory equation, so the ratio of the range computed by Eq. (8) and that computed by Eq. (3) is proportional to $(2+\mathsf{F}^2)^{0.5}/\mathsf{F}$ and the percentage error in range prediction is $[(2+\mathsf{F}^2)^{0.5}/\mathsf{F}-1]\times(100\%)$. Fig. 2 shows the variation of these ratios and the range prediction error.

The errors shown in Fig. 2 are dramatic for low Froude numbers and still significant for Froude numbers as high as 10. Given the context within which Eq. (8) is presented in *Design of Small Dams*, it was probably developed initially from analysis of flows having large Froude numbers, where the error is relatively small. It would have been difficult to detect the small error in experimental results because of the greater effects of aeration, jet spread, and air resistance, and the difficulty of precisely measuring the trajectory of a jet during an experiment. The factor K was used to account for these effects and also unknowingly adjusted for the error caused by the erroneous equation. The use of K values as low as 0.75 may have arisen from an attempt to obtain agreement with measured trajectories of flows with lower Froude numbers.

Modeling Jets Overtopping Dam Crests

Considering the jet arising from overtopping of a dam with a horizontal crest, the flow should pass through critical depth on the crest and spring free from the crest as a slightly supercritical flow. Rouse (1936) concluded from experimental work that the flow depth at the brink of a free overfall is $y_b=0.715y_c$, and this result has since been confirmed with only slight variation by many investigators. Henderson (1966) presents an excellent summary of the early work in this area, including results from Delleur et al. (1956) for the case of an overfall that is sloped (Fig. 3). More recently, Rajaratnam et al. (1976) examined the effects of roughness on the brink depth and Davis et al. (1998) explored the combined effects of roughness and slope. One should take note that much of the work on free overfalls is targeted at the use of free overfalls for discharge measurement in canals and assumes a long crest (i.e., a canal) leading up to the overfall; when the crest is narrower than about $3H_{\text{ovtop}}$, the situation tends toward the case of a sharp-crested weir, and the brink depth will begin to approach the critical depth.

In practice, to determine the brink depth and velocity, we would compute the discharge Q using a weir equation $Q=\text{CLH}^{1.5}$ where C=discharge coefficient of the crest, L=crest length, and H=overtopping head. We could then compute the corresponding critical depth for that discharge $y_c=(q^2/g)^{1/3}$ where q=discharge per unit of crest length. The brink depth could then be determined using the relationships given by Rouse (1936) or Delleur et al. (1956), and the brink velocity can be determined from the continuity equation. It should be noted that because critical flow produces the minimum specific energy for a given discharge, if one were to compute the specific energy at the brink, where flow is supercritical, it will be greater than the specific energy at the critical section. This seems contradictory, but because the pressure distribution at the brink is no longer hydrostatic, the specific energy at the brink is no longer the simple sum of $y_b+v_b^2/(2g)$.

Let us now perform a general analysis of the flow regime overtopping a dam with a horizontal crest and evaluate the errors caused by using the incorrect trajectory equation. We will assume no losses, so that the critical depth is two-thirds of the overtopping depth $y_c=(2/3)H_{\text{ovtop}}$. The critical velocity head is thus the remaining $(1/3)H_{\text{ovtop}}$ and the critical velocity is $v_c=(2gH_{\text{ovtop}}/3)^{0.5}$. The discharge is $Q=y_cv_cL$. If the flow over this dam were being computed with a traditional weir equation $Q=\text{CLH}^{3/2}$, the discharge coefficient would be $C=(2/3)^{1.5}(g)^{0.5}$. Applying Rouse's relation for the brink depth yields

$$y_b = 0.715y_c = 0.715(2/3)H_{\text{ovtop}} = 0.477H_{\text{ovtop}} \tag{9}$$

The continuity equation at the critical section and the brink requires $v_cy_c=v_by_b$, so the brink velocity is $v_b=(1/0.715)v_c$

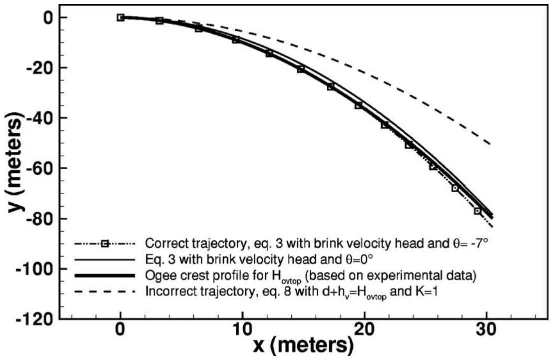

Fig. 4. Comparison of computed jet trajectories and ogee crest profile representing experimental data

$=1.399v_c$. The brink velocity expressed in terms of the overtopping head is

$$v_b = 1.399v_c = 1.399(2gH_{\text{ovtop}}/3)^{0.5} = 0.808(2gH_{\text{ovtop}})^{0.5} \quad (10)$$

The velocity head at the brink is thus $v_b^2/(2g)=0.652H_{\text{ovtop}}$, and the Froude number at the brink is $\mathsf{F}=v_b/(gy_b)^{0.5}=1.65$. From Fig. 2, the resulting overprediction of the range of the jet with Eq. (8) is about 32%. For comparison, if the depth at the brink were exactly critical depth ($\mathsf{F}=1$), then Fig. 2 shows that Eq. (8) would overestimate the range of the jet trajectory by about 70%. One other consideration is the effect of streamline curvature at the brink. In truth, only the bottom streamline of the jet actually parallels the crest; the middepth streamline is deflected about 7 deg downward at the brink [see Henderson (1966)], further shortening the real trajectory, but only by a small amount, as shown in the next section.

Comparing Trajectory Equations

To demonstrate the differences between the correct and incorrect trajectory equations in a real-world case, trajectories were computed with Eqs. (3) and (8) for an example dam being analyzed recently by the Bureau of Reclamation. In addition, an ogee crest spillway profile was computed for the same overtopping head. The ogee crest shape closely matches a real nappe profile; equations describing the shape are based upon large bodies of experimental data compiled by the Bureau of Reclamation *Design of Small Dams*, pp. 365–367 (Bureau of Reclamation 1987) and U.S. Army Corps of Engineers (*Hydraulic Design Criteria*, Sheets 111-1 to 111-2/1) (U.S. Army Corps of Engineers 1987). Fig. 4 shows the ogee crest shape compared to the trajectories computed with Eqs. (3) and (8) for an overtopping head of 4.54 m. The Froude number at the brink is 1.65, as in the previous general analysis. As expected, the ogee crest profile closely matches Eq. (3), while the trajectory computed by Eq. (8) is much flatter. Including an approximate 7 deg downward deflection of the flow at the brink has only a small effect on the computed trajectories. Eq. (3) also produces results identical to a dimensionless equation provided by the U.S. Army Corps of Engineers (1964) for computing flip bucket throw distances

$$\frac{x}{h_v} = \sin 2\theta_0 + 2\cos\theta_0\sqrt{\sin^2\theta_0 + \frac{y}{h_v}} \quad (11)$$

Eq. (8) can be brought into reasonable agreement with Eq. (3) by utilizing a K factor significantly less than 1.0. A value of 0.75 has been suggested by Annandale (2006), but a value of $K=0.652$ produces the best agreement, since we found earlier that the velocity head at the brink is $0.652H_{\text{ovtop}}$ for a horizontal crest assuming no losses. The established practice of designing spillway chute profiles using Eq. (8) with $K=1.5$ is still acceptable, since the large K value ensures that the jet is fully supported. However, for correctness, the $(d+h_v)$ term in Eq. (8) should be replaced with just h_v. In spillway applications where the Froude number is large, the effect of this change is minor.

Conclusions

The computation of the trajectory for a free jet should always utilize either the initial velocity of the jet itself [Eq. (1)], or the initial velocity head $h_v=v_0^2/(2g)$ of the flow as it springs free into the atmosphere [Eqs. (3) and (11)]. For jets overtopping dams, the initial velocity can be estimated with reasonable accuracy using relationships first offered by Rouse (1936) for determining the brink depth of the flow.

The flawed Eq. (8) presented in several Bureau of Reclamation manuals and other literature should not be used to model a free jet. It predicts a flatter trajectory than is theoretically possible, which may lead analysts to underestimate the potential for scour near hydraulic structures or overestimate the extent of scour protection required. The distance traveled by the jet is overpredicted with Eq. (8) by 10% to 70% when the initial Froude number of the jet is between 4 and 1, respectively, as is the case for most flows overtopping dam crests. In practice, Eq. (8) has traditionally been adjusted with a K factor varying from 0.75 to 1.0. This has helped to reduce the errors, but the physical basis for different values of K has never been established. In reality, when the correct trajectory equations are used, the value of K should be nearly 1.0 (i.e., K should not be included) for analysis of flows overtopping dams. For modeling high velocity jets that might be subject to greater aeration and wind drag, modification of Eq. (3) with a K factor such as that included in Eq. (8) may be appropriate, but further research is needed to determine appropriate values for K.

Notation

The following symbols are used in this technical note:

C = discharge coefficient;
D = hydraulic depth, equal to flow area divided by top width;
d = flow depth;
F = Froude number;
g = gravitational acceleration;
H = total head;
H_{ovtop} = overtopping head;
h_v = velocity head;
K = coefficient in jet trajectory equation;
L = weir crest length;
Q = discharge;
q = discharge per unit crest length;
v_0 = initial jet velocity;
v_b = velocity at brink of free overfall;
v_c = critical velocity;
x = horizontal position of jet trajectory;
y = vertical position of jet trajectory;

y_b = flow depth at brink of free overfall;
y_c = critical depth; and
θ_0 = initial angle of inclination of jet.

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