For National Mathematics Advisory Panel

The task of the National Mathematics Advisory Panel is a noble endeavor. As civil rights activist, Dr. Robert Moses is saying: Algebra is the next civil right and that "the most urgent social issue affecting poor people and people of color is economic access. In today's world, economic access and full citizenship depend crucially on math and science literacy." [1]

For many students, learning Algebra is crucial to obtaining their first or second choice for a career or college major.

I will share some thoughts -- thoughts which are known to many of you. I second the opening remark by Norman C. Francis, president of Xavier University of Louisiana, on the importance of programs which **Stress Analytical Reasoning (SAR).**

When you define Algebra, please include Algebraic word problems. Outside of mathematics class, virtually all mathematics problems are word problems, also known as "problems in context".

Please specifically include non-trivial, multi-step Algebraic word problems, which Stress Analytical Reasoning (SAR). The popular Math Reform curriculum emphasizes wordy "real world problems", usually with little math content, that is, problems, which I call "Avoid Analytical Reasoning" (AAR). The items on MD's Algebra test exemplify AAR, for example read Problem 31 in Appendix C.

Solving Arithmetic and Algebraic word problems, which stress analytical reasoning (SAR), develops basic life skills of reasoning and analysis. It is valuable everywhere, both in quantitative settings and in daily life.

Most of the problems in high school physics courses and many in high school chemistry courses, as well as many in college freshmen nutrition and geography courses could be considered Arithmetic or Algebraic word problems, albeit ones that require science knowledge. Students fluent in Arithmetic and Algebraic word problems will have the solid math foundation needed for high school chemistry and physics courses and also for college freshmen nutrition and geography courses. This reasoning also provides a solid background for solving Calculus word problems. In this technological age, it should become the rule, rather than an exception, that high school students study both chemistry and physics.[2]

Instruction for problems such as the next Problem 1 is missing in American high schools and colleges. Nine out of ten of my students read the next Problem 1, and then translate it into *wrong*

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[1] Dionne, Jr., E.J. (6 March 2001). Into the Math Mix. *The Washington Post*, pg. A.23. As leader of the Mississippi Voter Rights Project, Dr. Moses was one of the ten most important civil rights activists in the1960s. Moses has a Harvard Ph.D. in the philosophy of mathematics.

[2] This means Math-based chemistry and physics courses, not *descriptive* chemistry and physics courses.

equations. [3] My students were mostly college seniors majoring in engineering and mathematics education (about to become high school mathematics teachers). Most engineering and mathematics education students did not take my course. Which is scarier -- an engineer or an Algebra teacher writing down wrong equations?

Problem 1. (A typical Russian high school Algebra problem[4]) Suppose that it takes Tom and Dick 2 hours to do a certain job, it takes Tom and Harry 3 hours to do the same job and it takes Dick and Harry 4 hours to do the same job. How long would it take Tom, Dick and Harry to do the same job if all 3 men worked together? (Assume that each works at his standard speed.)

The crucial four pillars for a solid foundation for the study of Algebraic word problems includes the analytical reasoning provided by:
1. Fluency in **Arithmetic word problems**, especially, non-trivial, multi-step Arithmetic word problems, which Stress Analytical Reasoning (SAR).
2. Fluency in **measurement** including measurement problems which SAR.
3. Arithmetic-based science lessons, especially those which SAR. This will provide "**real world**" problems as well as "**real world**" opportunities for **estimation** and use of measurement.
4. Instruction in reading comprehension and following directions, specifically for Arithmetic and Algebraic word problems.

Details on these four pillars for the study of Algebraic word problems follow:

1. Fluency in **Arithmetic word problems**, especially, non-trivial, multi-step Arithmetic word problems, which Stress Analytical Reasoning (SAR). Arithmetic word problems, which SAR, should be an overarching theme of the K-7 Math curriculum, and should be a Focal Point in each grade.

Problem 2. (A two-step problem) The price of a loaf of bread is two dollars. The price of a large jar of milk is two dollars. Johnny buys one loaf of bread and one large jar of milk. He gives the cashier a five-dollar bill. What is the change?

Two-step problems (like Problem 1) are NOT mentioned until Grade 6 in the NCTM's 2006 "Curriculum Focal Points for Mathematics in Prekindergarten" [http://www.nctm.org/focalpoints/bygrade.asp]

Problem 3. It is a fact that fat has 9 calories per gram and protein has 4 calories per gram. If a piece of meat consists of 100 grams of protein and 10 grams of fat, how many calories does it have altogether? (Answer: 490 calories; not a trick question)

College students receive instruction on how to do Problem 3 in an elementary nutrition course on my campus, (with selective admissions); it is the flagship campus of the University of Maryland. This should *not* be necessary

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[3] A correct way, to translate and analyze this algebraic word problem into correct equations, together with instruction on the needed mathematical reasoning appears, in my report: "Algebraic Word Problems ", on my website: http://www.math.umd.edu/~jnd/Algebraic_word_problems.pdf

[4] I found this problem in an article by Andre Toom in the American Educator.

Praxis II Middle School math content exam Sample Question 7. (A *simple* ratio problem) (Abbreviated) Suppose that 40% (by weight) of a county's trash is paper and 8% is plastic. "If approximately 60 tons of the trash consists of paper, approximately how many tons of the trash consists of plastics?"

The Praxis II "Middle School math" content exam (#0069) is used by some states to certify that a teacher is NCLB "highly qualified". Two of the sample multiple choice questions (at ftp://ftp.ets.org/pub/tandl/0069.pdf) are "ratio" questions (#7 and 10); both are simple ratio questions. None of the sample questions is an "advanced" ratio question like the following problem:

Problem 4. (An *advanced* ratio problem) Suppose that 40% (by weight) of a county's trash is paper and 8% is plastic. If approximately 72 tons of the trash consists of paper and plastic, approximately how many tons of the trash consists of plastics?

Note that one has to stop a minute to think and analyze Problem 1, in contrast to the straight-forward ratio calculation needed for the Praxis II Sample Question 7.

Arithmetic word problem instruction should be an incremental, evolving process, which carefully extends previous knowledge and often lays a solid foundation for conceptually more difficult `mathematical and scientific` word problems in the future. Instruction for the list of problems in Appendix B will do this, thereby developing a methodology for Arithmetic word problems, which SAR.

2. Measurement. Fluency in measurement including solving measurement problems which SAR.

Acquiring facility with measurement in middle school, will result in students being comfortable with the measurement aspects of high school chemistry class. This will free up the time used to teach measurement in chemistry class. Also it will enable more students to choose the more math-based high school chemistry (and physics) classes, instead of the simpler descriptive chemistry and physics class.

Problem 5. How many cubic feet are in a cubic yard?

This is Question #2 in Math Professor Betsy Darken's pretest, given to her college students in a specialized math course for future elementary and middle school teachers at the University of Tennessee at Chattanooga. Only one student in her class of thirty obtained the correct answer on the pretest. These students (mostly) had earned a grade of C or above in an entry level non-specialized college mathematics course. ["How Little Our Future Teachers Know" was the subtitle of Professor Betsy Darken, presentation at the 2005 annual meeting of the Mathematics Association of America (MAA), the professional society of college math professors, which emphasizes math instruction.]

"When [a] biology teacher had to teach [a chemistry class] at Howard High School [in suburban Howard County, MD], how to change centimeters to meters, he just told them to move the decimal two places -- rather than illustrating the concept. ... 'Forty-five minutes later, only three of them got it.' ". [5]

Not so difficult, 236 centimeters make 2.36 meters, just like 236 cents make \$2.36. Changing centimeters to meters does occur in Biology lab work.

Students should learn the formulas: {distance} = {constant speed} x {time} and d = s x t, (its formulation in Algebraic notation) and {average speed} = {distance} / {time} [when traveling in a straight line, with no backtracking]. Students should be using these formulas to solve word problems, which SAR (by Grade 6).

Problem 6. (**SAT medium level**) "How many <u>minutes</u> are required for a car to go 10 miles at a constant speed of 60 miles per hour?" (Item#5 of Section 7 of the May 2000 SAT Math test.)

The SAT rated this as a medium level problem, that is, medium level for Grade 12 students, (who plan to attend a four year college). It should be an easy problem for Grade 6 students.

3. Arithmetic-based science lessons in elementary school and middle school, especially those which Stress Analytical Reasoning (SAR). They will provide useful motivation for and practice with Arithmetic calculations and for measurement. They are the ideal place for students to practice **estimation**. Arithmetic-based science lessons naturally generate "**real world**" problems, and do so *in context*; as part of a coherent syllabus. In my opinion, this is far better than having science-based "real world" problems "parachute" into Mathematics class. Also a good place to use Algebraic notation, like d = s x t [for {distance} = {average speed} x {time}]. There is more in Appendix A, Arithmetic-based Science and Algebra-based Science.

For this middle school science teachers will need fluency in Arithmetic word problems, which SAR, and in measurement. Ideally, there would be a certification for a teacher, who would teach *both* middle school Mathematics and science (in contrast to the common general K-8 certification or a 7-12 Math certification).

4. "**Reading Instruction for Arithmetic Word Problems**: If Johnny can't read well and follow directions, then he can't do math". This is the title of Appendix C, which discusses reading together with its accruements, that is, paraphrasing a word problem, accurately and precisely, into mathematical expressions, formulas or equations, as well as the reading of tables, charts and graphs. I would be surprised if just reading in Mathematics class, even with teacher feedback is sufficient; I think that reading instruction specific for Mathematics and for Word Problems is needed.

Great precision in both the reading and the writing of Mathematics, is a hallmark of the disciplines of Mathematics. I suggest that training of teachers in the precise reading and writing of Mathematics should occur in a Math content course taught by a professor of mathematics. For

[5] "Right Teacher, Wrong Class", Washington Post, February 15, 1999

an example of imprecise writing on a state exam read Problems 12 and 2 in Appendix C. The College of Education on my campus has a course in reading instruction in the content areas; this course satisfies a state certification requirement. But, it is a general course on reading instruction; it contains no instruction on reading instruction specifically for Mathematics.

Other considerations:

Recognition that different problems have the same problem pattern is a crucial part of solving word problems. This is far more important than the popular pattern-recognition problems.

It may be useful to have **grade by grade** descriptions of appropriate Arithmetic Word Problems. The list I suggested to the NCTM Focal Points team is in Appendix B. Several members of your panel have more expertise than me and could improve on my list.

Appendix A. Arithmetic-based Science and Algebra-based Science.

Lessons in Arithmetic-based Science and Algebra-based Science are very helpful in reinforcing students' understanding and reasoning about word problems and Arithmetic calculations, including measurement and estimations.

But Arithmetic and Algebra-based Science mostly belong in science classes, rather than in mathematics classes. In science class, science topics can be taught in context, as part of a focused coherent syllabus, with the appropriate scientific background. Be wary of having isolated science-based "real world" problems "parachute" into Mathematics class.

Arithmetic and Algebra-based Science instruction needs to be both mathematically and scientifically rigorous. In Math class, it is easy to misapply scientific methods (e.g. by treating dependent data as if it were independent), resulting in flawed conclusions. Be wary of having students plug into science formulas they may not understand and then to trust the answers.

Knowledge of science is often important, if not crucial, to an understanding of the science and its implications. Arithmetic and Algebra-based Science instruction in Math class sometimes reflects a real misunderstanding of the science. For example, students may be asked to assume that a problem is linear when there is no indication that the problem is indeed linear, or to model science without the proper scientific background, and then to draw inappropriate conclusions.

Problem 7. Physics tells us that weights of objects on the moon are proportional to their weights on Earth. Suppose an 180 lb man weighs 30 lb on the moon. What will a 60 lb boy weigh on the moon?

Remark. Of course, elementary school children should have been taught that the weight of an object is *fixed*. A bag of apples or a child has the same weight, no matter on which scale [on Earth] the weightings occur. The parenthetical phrase "[on Earth]" is (of course) omitted from instruction. It must be a complete *mystery* to students as to why a weight on the moon should be any *different* than the weight on Earth. This is why, Problem 7 should be presented in a science lesson, not in a math lesson. It should be presented only after a long discussion as to why weights on the moon are a fraction of weights on Earth and then why it is always the *same* fraction. Either is a sophisticated topic for middle school students.

As stated, Problem 7 is a straight-forward and correct proportion problem (albeit an inappropriate one). In contrast, here is a similar, but *impossible* problem:

Problem 8. (**Impossible**) Suppose an 180 lb man weighs 30 lb on the moon. What will a 60 lb boy weigh on the moon?

Remark. There is no way for a student to know about this proportionality of weights of objects on the moon and on Earth. Nevertheless, this type of problem occurs. There is the expectation that students will make the completely *unwarranted* assumption that the weights of objects on the moon are proportional to their weights on Earth. This is training students to make completely *unwarranted* and sometimes completely *incorrect* assumptions of proportional in many situations.

The next problem tempts students in a similar manner.

Problem 9. A ball dropped from a tall building, falls 16 feet in the first second. How far does it fall in two seconds? (Ignore friction)

Warning. It is tempting to assume that the distance the ball falls is a linear function of time. What else can one do given the limited information in the problem? But, multiplying 2 x 16 ft. = 32 ft. will produce a *wrong answer*. While the falling distance is a function of time, it is *not a linear function*.

Problems 9 and 7 should be postpone to high school physics course, wherein they can be taught with appropriate physics background and rigor.

Problem 10: (**Wrong physics**) The distance, s that a "freely-falling" object falls in time t is given by the formula: $s = 16t^2$. Using this formula, estimate how long it would take you to fall 10,000 feet. ("Freely-falling" means ignoring the effects of air resistance.)

As a math problem this is trivial, just plug in s = 10,000 feet, solve for t. But air resistance is significant; so ignoring air resistance will produce an answer that is physically wrong.

Work. The useful physical concept of work, defined as {force} x {distance} and its Algebraic notation, W = f x d, is a good topic for middle school science.

Problem 11 (a). Jack and Jill lift a W lb. weight vertically h ft.. Jack lifts with a force of x lb. and Jill lifts with a force of y lb., where x+y = W. Prove that the work required to lift the weight is equal to the sum of the work Jack did plus the work Jill did.
(b) Prove that the work required to lift a W lb. weight vertically, (x+y) ft. is equal to the sum of the work required to first lift the weight vertically x ft. plus the work required to then lift the weight vertically y ft.

Problem 11 provides examples of the general formula:
{Total Work required} = Sum of the various work required for the “parts” .
This formula is justified by the Distributive Rule of Algebra.

Appendix B. **A Grade by Grade Description of Appropriate Arithmetic Word Problems**

Slightly modified excerpts from Jerome Dancis's (November 30, 2005) email to the NCTM Focal Points Team:

I would like to see these three "Big" Mathematical Ideas listed explicitly:

1. Arithmetic. Fluency in Arithmetic is crucial background for Algebraic calculations, which in turn is crucial for not being relegated to remedial math in college. This will require half the lessons each year.
2. Reading instruction for Math textbooks and for Arithmetic Word Problems.
3. Arithmetic Word Problems, especially two-step problems and those, which require more than just reading and recall of isolated technical terms.

Some details below:

Grade 2. Students solve two-step problems involving addition AND subtraction (with small integers).

Problem 1. The price of a loaf of bread is two dollars. The price of a large jar of milk is two dollars. Johnny buys one loaf of bread and one large jar of milk. He gives the cashier a five-dollar bill. What is the change?

Grade 3. Students solve one-step problems involving units.

Problem 2. If one T-shirt cost 2 dollars, how much do 3 T- shirts cost?

Problem 3. If three T-shirts cost 9 dollars, how much does one T- shirt cost?

Grade 4. Students solve two-step problems involving units. Students solve simple problems involving ratios.

Problem 4. Three T-shirts cost 15 dollars. How much do 5 T- shirts cost?

Problem 5. A recipe for 5 cup cakes uses 2 cups of flour, 1 cup of milk and 2 eggs. What would you need for 10 cupcakes?

Problem 5B. (From a Grade 3 Russian math textbook.[6]) 2 televisions and 4 radios were purchased for a resort building. 756 rubles were paid for everything. The price of a television is 270 rubles. How much does a radio cost?

Grade 5. Students solve problems, which require detailed following of directions.

[6] From http://www.cbmsweb.org/NationalSummit/Plenary_Speakers/ma.htm Page 4

Problem 6. It costs 90 cents for The Striped Toothpaste Company to make, package and ship a tube of toothpaste. The company also has "overhead costs" of $3000 per month. The company sells (at wholesale) cartons of toothpaste at the price of $2.50 per tube. This month, the company sold 5000 tubes of toothpaste. What is this month's profit?

Problem 7. (**A "Forward" Geese Problem**) A flock of 100 geese on a pond were being observed continuously.
At 1:00 P.M., 1/5 of the geese flew away.
At 2:00 P.M., 1/8 of the geese that remained flew away.
At 3:00 P.M., 3 times as many geese as had flown away at 1:00 P.M. flew away, leaving 28 geese on the pond.
At no other time did any geese arrive or fly away or die.
(a) How many geese remain at 1:00 PM?
(b) How many geese remain at 2:00 PM?
(c) How many geese remain at 3:00 PM?

Grade 6

Problem 8. (An *advanced* ratio problem) Suppose that 40% (by weight) of a county's trash is paper and 8% is plastic. If approximately 72 tons of the trash consists of paper and plastic, approximately how many tons of the trash consists of plastics?

Problem 8B. We flew from Denver to Boston at an average speed of 500 MPH; we returned from Boston to Denver at an average speed of 400 MPH. (The distance from Denver to Boston is 2000 miles) What was our average speed for this round-trip?

<u>Students solve Catch-up and Overtake problems and problems which require more detailed following of directions.</u>

Problem 9. (**Catch-up and Overtake**) As the clock strikes noon, Jogger J is 2500 yards and Walker W is 4000 yards down the road (from here). Jogger J jogs at the constant pace of 10 yard/sec. Walker W walks at the constant pace of 5 yard/sec. How long will it take Jogger J to *catch up* to Walker B?

Problem 10. It costs one dollar for The Striped Toothpaste Company to make, package and ship a tube of toothpaste. The company also has "overhead costs" (machinery or rent or whatever) of $3000. The Striped Toothpaste Company sells (at wholesale) cartons of toothpaste at the price of three dollars per tube. How many tubes of toothpaste does the company need to sell to cover/balance-out the fixed costs?

There are arithmetic solutions to the last two problems in my report, "Supposedly Difficult Arithmetic Word Problems".

Grades 7 and 8. More detailed following of directions and involved Arithmetic problems using units.

Problem 11. (The Geese Problem [7]) "A flock of geese on a pond were being observed continuously.
At 1:00 P.M., 1/5 of the geese flew away.
At 2:00 P.M., 1/8 of the geese that remained flew away.
At 3:00 P.M., 3 times as many geese as had flown away at 1:00 P.M. flew away, leaving 28 geese on the pond.
At no other time did any geese arrive or fly away or die. How many geese were in the original flock?"

There is an arithmetic solution to this Geese Problem in Appendix C. This solution is useful background for an algebraic solution. This is why it is very useful that students first practice with involved arithmetic solutions.

The Geese Problem 11 is quite similar to the next problem:

Problem 11B. **(From a 5th grade Singapore math textbook [8])** "Mrs. Chen made some tarts. She sold 3/5 of them in the morning and 1/4 of the remainder in the afternoon. If she sold 200 more tarts in the morning than in the afternoon, how many tarts did she make?"

Problem 12. (Work) Suppose that it takes Sally 3 hours to mow a lawn, and it takes Tom 4 hours to mow the same lawn; Tom's mower is less powerful than Sally's. *Without* using algebra (x or other variables) determine how long it would take Sally and Tom to mow the lawn if they worked together (using both lawn mowers)? (Assume that each works at his/her standard speed and they never get in each others way.)

There is an arithmetic solution to the last problem in my report, "Supposedly Difficult Arithmetic Word Problems" (http://www.math.umd.edu/~jnd/Difficult_Word_Problems.html).

At what grades (if any) do you see these problems being included in a hypothetical textbook, written by a commercial publisher, following the Focal Points?

[7] This was Question #25 of Section 4 of the May 2000 Math SAT.

[8] From http://www.cbmsweb.org/NationalSummit/Plenary_Speakers/ma.htm

Appendix C. (From http://www.math.umd.edu/~jnd/subhome/Reading_Instruction.htm)

Reading Instruction for Arithmetic Word Problems:

If Johnny can't read and follow directions, then he can't do math.[9]

By Jerome Dancis [10]

1. Introduction and Summary.

Comprehending (Arithmetic) word problems correctly and then translating them into organized mathematical expressions and equations, is a crucial part of doing math and science. Although all state math standards repeatedly mention problem solving, there is also a critical need for instruction in the reading and comprehension of word problems and for instruction in following directions, with increasing levels of sophistication in higher grades. Reading includes its accruements, that is, paraphrasing a word problem, accurately and precisely, into mathematical expressions, formulas or equations, as well as the reading of tables, charts and graphs.

Students are further handicapped, when textbooks do *not* clearly explain the mathematics, worse when the book presents mathematics in an unnecessarily complicated and confusing manner as well as topics presented in unnecessarily difficult ways. For another report.

The problem below, demonstrate the difficulty students are having with comprehension and following directions of appropriately written problems. Problem 1 requires precise reading and the ability to follow directions, together with Grade 5 arithmetic. But it was an SAT Math test problem; one that the SAT rated at the highest level of difficulty. Also it stumped almost all the students in my Univ. of Maryland pre-calculus class, last summer; the math by itself did not cause difficulty. Five out of 25 students in an *honors* precalculus classes solved Problem 1 correctly.[11]

Being able to do this type of problem is important for doing organized logical analysis in science and economics. Greatly increasing the percentage of Grade 5 students who can do this "following-the-directions" problem would be a worthy goal/challenge. And, it would enable middle school science teachers to considerably raise the level of arithmetic-based science lessons. Until this occurs, greatly increasing the percentage of high school students who can do this "following-the-directions" problem will be a worthy goal/challenge. And, it would enable high school science teachers to considerably raise the level of algebra-based science lessons.

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[9] This article is a major expansion of my short article, "When It Comes To Math, Words Count", Washington Post, September 8, 2002; on the web at http://www.washingtonpost.com/wp-dyn/articles/A49346-2002Sep7.html or may be accessed from my website, http://www.math.umd.edu/~jnd.

[10] Mathematics Dept., University of Maryland, College Park, MD, 20742-4015.

[11] At Montgomery Blair High School, Montgomery County, MD

Ideally, instruction in following directions should be shared by both arithmetic-based science lessons and math lessons.

Problem 1. (The Geese Problem [12]**)** (SAT Level 5) "A flock of geese on a pond were being observed continuously.
At 1:00 P.M., 1/5 of the geese flew away.
At 2:00 P.M., 1/8 of the geese that remained flew away.
At 3:00 P.M., 3 times as many geese as had flown away at 1:00 P.M. flew away, leaving 28 geese on the pond.
At no other time did any geese arrive or fly away or die. How many geese were in the original flock?"

Arithmetic Solution. At 1:00 P.M., 1/5 = 20% of the [original flock of] geese flew away; leaving 80% of the original flock of geese.
At 2:00 P.M., 1/8 of 80% = 10% [of the original flock of geese] flew away.
At 3:00 P.M., 3 times 20% = 60% [of the original flock of geese] flew away; leaving 28 geese.
A total of 20% + 10% + 60% = 90% of the [original flock of] geese have flown away; leaving 100% - 90% = 10% of the original flock of geese.
Thus the remaining 28 geese are 10% of the original flock of geese. The original flock is represented by 100% = 10 times 10%.
Hence the original number of geese was: 10 x 28 = 280.

Allocating time – beginning in elementary school – for substantive instruction and drill on precise reading and on following lists of directions, should considerably increase the number of Grade 5 students able to do this problem. My pre-calculus students were only able to do Problem l after I provided them with training. Students should build up to Problem 1 by doing simplified versions of it (like Problems 15 and 16 later) in previous grades. This may be unnecessary for proficient readers, but there are large numbers of non-proficient readers.

The Geese Problem 1 is quite similar to Problem 17 (below) from a 5th grade Singapore math textbook. So training students, from Singapore textbooks in elementary school, should enable them to do some difficult Math SAT problems like Problem 1.

Comment. Yes, the Geese Problem 1 is usually considered a problem in Algebra. But, the algebraic solution can be written largely by having the word processor replace each "%" sign by "x/100" in the Arithmetic Solution. [13] As such the arithmetic solution is useful background for

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[12] It was Question #25 of Section 4 of the May 2000 Math SAT; the SAT rated it as Level #5 on its scale of 1 to 5.

[13] **Algebraic Solution.** Let x be the number of geese in the original flock.
At 1:00 P.M., (1/5)x = 20x/100 geese flew away; Leaving 80x/100 geese.
At 2:00 P.M., 1/8 of 80x/100 = 10x/100 geese flew away.
At 3:00 P.M., 3 times 20x/100 = 60x/100 geese flew away; leaving 28 geese.
A total of 20x/100 + 10x/100 + 60x/100 = 90x/100 geese have flown away; leaving 10x/100 geese.
Thus the remaining 28 geese are 10x/100 geese. That is 28 = 10x/100 = x/10.
Hence, x = 10 (28) = 280.
Notice that these steps mimic the steps of the Arithmetic Solution.

the algebraic solution. This is why it is very useful that students first practice with involved arithmetic solutions.

Comment. Two college professors, who were teaching college mathematics courses to future elementary school teachers, told me that many of these students could not do the Geese problem. One noted that the Geese problem was a four-step problem and that such involved problems are beyond many of her students. I expect that these future elementary school teachers could be trained to do the Geese problem; after all 5th grade Singapore students learn how to do it. But, such training is not included in the college mathematics training of future elementary school teachers, partly because of the absurdly low amount of time devoted to the mathematics training of future teachers. [14] Patricia Kenschaft discusses the low level of mathematics knowledge of many elementary school teachers in her recent article[15].

The next problem suggests the extent of the reading-for-math difficulties of Grade 9. Problem 2 requires a modicum of reading and interpretation; it stymied more than 5 of 8 (65%) Grade 9 students, when it was field tested in Maryland.

Problem 2. (2000 sample MD High School Assessment Algebra test[16], Item #48, on the web at http://www.mdk12.org/mspp/high_school/look_like/algebra/v48.html:)
"The table below shows how a typical household spends money on utilities.

Utility	**Percentage of Total Utility Costs**
Lighting	6
Refrigeration	9
Water heating	14
Appliances	27
Heating and cooling	44.

A typical household spent $1,400 on utilities last year. If there are no significant changes in their utility usage for this year, how much should they budget for heating and cooling their home this year?

[Multiple Choice] F $196 G $378 H $616 J $784 "

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14 Mathematics Education Professor, Betsy Darken, of the University of Tennessee at Chattanooga, discusses this in her report: "How Little Our Future Teachers Know".

15 "Racial Equity Requires Teaching Elementary School Teachers More Mathematics" by Patricia Clark Kenschaft in the Notices of the AMS (American Mathematics Society), February 2005, Volume 52, Number 2 at http://www.ams.org/notices/200502/fea-kenschaft.pdf
From her final paragraph:
"One lesson our current elementary school teachers convey powerfully is that math is too difficult to understand. Because knowledge of mathematics correlates strongly with economic and political achievement, the mathematical education of all elementary school teachers is the paramount equity issue. As Will Rogers said long ago, 'You can't teach what you don't know any more than you can come back from where you ain't been.' "

16 Item #48, on the web at http://www.mdk12.org/mspp/high_school/look_like/algebra/v48.html; for the 63%, click on "view field test results", then add the percentage of students, who skipped this problem to the percentage, who obtained a wrong answer.

Comments. Yes, the wording is awkward, and it needs to be read twice and one needs to ignore what is not relevant. Also, students need to assume (without justification) that there are no significant changes in the price of heating oil for this year.

To do this problem, students need to concentrate on what is crucial while cutting through the camouflage. They need to reformulate the problem, accurately and precisely, in concise, coherent form. The end product should be a translation of this wordy problem into the really simple statement: "Find 44% of $1400".

The arithmetic level of Problem 2, is much lower than the reading comprehension level, since students had *calculators* to calculate 44% of $1400. [17] So it is reasonable to suspect that reading (and its accoutrements) was a major reason for 5 of 8 (65%) Grade 9 students not solving this problem correctly. That about 70% of students start Grade 9 *reading below grade level* cannot help but contribute to this.[18] This calls out for reading instruction, directed at deciphering, then analyzing, then reformulating and paraphrasing such problems, at least for the 5 of 8 Grade 9 students, who did not solve this problem. (Of course, many Grade 9 students are analyzing this problem correctly, without any special instruction.)

For additional comments on Problem 2, see Problems 12 and 13.

A word problem, that requires several times as much reading as Problem 2, is Item #39 on the 2001 MD High School Assessment Algebra exam (on the web at http://www.mdk12.org/mspp/high_school/look_like/2001/algebra/v39.html). But, after the camouflage is removed, the item is merely asking the student to count how many of the ten numbers listed, consist of the digits from 1 to 8, only, (No 0 or 9). This item was showcased by the Maryland State Department of Education.

The reading level on the Maryland Algebra exam is higher than the math level. This produces false negatives.

Similarly, mathematics educator, Liping Ma, has said[19]: "To compose a mathematical expression" [which is a mathematical translation of a word problem] has potential significant pedagogical power in mathematics teaching and learning. It is wildly used in math education in other countries. The "[composing of a] math expression" is only one of quite a few important concepts I have noticed missing in current math education in this country."

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17 Alternatively, observe that 44% is a little less than 50% = 1/2; then pick out Answer H as the only choice which is a little less than 1/2 of $1400.

18 "A dismaying Carnegie Foundation report entitled 'Reading Next' shows that the states have set the reading achievement bar very low so students can be moved from grade to grade - even though about 70 percent enter the first year of high school reading below grade level." Quote from a New York Times editorial March 1, 2005.

19 During her presentation, "Arithmetic in American Mathematics Education: An Abandoned Arena?" during the Opening Session ("Challenges in the Mathematical Education of Teachers: Why is the Preparation of Mathematics Teachers so Difficult?") of the National Summit on Mathematical Education of Teachers, on the web at http://www.cbmsweb.org/NationalSummit/Plenary_Speakers/ma.htm

Herb Gross, math instructor at a community college, has written: "The ability to paraphrase is one of the greatest aids I have when I solve problems. I find that most of my students are very weak in this regard. As important as mathematics is, it is a distant second to the need for good reading comprehension. The shortage of training in translating word problems into mathematical expressions is why we teachers so often hear students summarize a course by saying 'I could do everything except the word problems'. Sadly, in the textbook of life, there are only word problems."

Of course, being able to paraphrase is crucial in all reading, not just math and not just in school. Specific instruction, on how to paraphrase accurately and precisely, in concise, coherent form, should be a pedagogical strand in all subjects in all grades (in my opinion).

Relatedly, Dr. Mel Levine[20], directs[21] teachers toward "placing a strong emphasis on paraphrasing" for students with short-term memory problems. Also students "[with weak saliency determination] need opportunities to develop skills at summarizing, finding main ideas in paragraphs [and] paraphrasing what a teacher has said … ". Paraphrasing a math word problem is in the same spirit. I am in favor of this for all students. Yes, it is much more important for students, with learning difficulties.

Comment. Ideally, reading-for-math instruction should be *shared* by science, social studies, English and mathematics lessons. The reading instruction for Problem 2 belongs in English lessons. The reading of tables and charts should occur naturally in science and social studies lessons; therefore instruction, in the reading of tables and charts, should occur *mainly* in science and social studies lessons.

Virgina Anderson[22] provides extensive training to her college biology students in the reading and drawing of tables, graphs and charts.[23] The general need for such instruction in college is indicated by a federal study conducted by the National Center for Education Statistics. " … far fewer [Americans] are leaving higher education with the skills needed to comprehend routine data, such as reading a table about the relationship between blood pressure and physical activity, … 'What's disturbing is that the assessment is not designed to test your understanding of Proust, but to test your ability to read labels,' [Mark S. Schneider, commissioner of education statistics] added." [24]

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[20] He is Professor of Pediatrics and director of the Clinical Center for the Study of [Child] Development at the University of North Carolina.

[21] in one of his books, <u>Educational Care A System for Understanding and Helping Children with Learning Differences at Home and in School</u>, Second edition, Page 74 Item #3 and to a lesser extent Page 40 Item #3.

[22] Professor of Biological Sciences, at Towson State University, MD

[23] " 'As you can see …' But Students Don't: Helping Students Read and Interpret Graphic Information", her Oct. 31, 2005 presentation at Univ. of MD, College Park

24 "Literacy of College Graduates Is on Decline Survey's Finding of a Drop in Reading Proficiency Is Inexplicable, Experts Say", Washington Post, December 25, 2005; A12

Relatedly, a report Reading Next [25] notes that: "Some 70 percent of older readers [between fourth and twelfth grade] require some form of remediation. Very few of these older struggling readers need help to read the words on a page; their most common problem is that they are *not* able to *comprehend* what they read." This report strongly recommends literacy (reading and writing) programs for the bulk of middle and high school students; a crucial element of such a program would be:
"Effective [literacy] instructional principles embedded in *content* [for example math class], including language arts teachers using *content*-area texts and *content-area teachers* providing instruction and practice in reading and writing skills *specific to their subject area*". (Emphasis added.)

Relatedly, a report published by the National Association of Secondary School [26] Principals (NASSP) states (http://www.principals.org/s_nassp/sec.asp?CID=858&DID=52759):
"Historically, direct literacy instruction has been supported up to the third grade. However, there is a glaring need for it to continue so students can not only read narrative text, but also learn specific strategies to *derive meaning* from expository and descriptive text. When literacy instruction stops early, how can middle and high school students learn the strategies to read increasingly difficult text and to *comprehend* more abstract ideas? If a *regular* student continues to need direct instruction to read and *comprehend* the text found in secondary *textbooks*, consider the tremendous need for instruction and intervention that struggling readers must require. And sadly, if students two to three grade levels behind their peers do not receive intensive literacy instruction, the results can be devastating because the struggling reader will not experience success within the content areas. Therefore, it becomes even more critical that *secondary content area teachers* better understand and teach specific literacy strategies to help students read and extract meaning from the written material used to teach the course content. Conclusions from the RAND Reading Study Group [2002] clearly support the need for continued literacy instruction at the middle and high school levels … * Secondary students in the United States are scoring lower than students in other comparable nations. This is especially evident as secondary students deal *with understanding discipline-specific content text*." (Emphasis added.)

This NASSP report quotes a 1999 position statement by the International Reading Association, which argued for " * Highly skilled teachers who model and explicitly teach reading comprehension and study strategies across the content areas".
I have allocated class time to reading instruction for the somewhat complicated sentences and paragraphs, which come up in my college math courses. [27].

Arithmetic-based science. The next Problem 3 is a very practical "real-world" problem. The skills that it draws on can be used in many situations. Problems of this sort should be presented

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[25] "Reading Next: A Vision for Action and Research in Middle and High School Literacy" at http://www.all4ed.org/publications/ReadingNext/ExecutiveSummary.html

[26] "Secondary Schools" means "middle schools, (junior high schools) and high schools".

[27] Most of the seniors (in engineering), entering my (second semester) linear algebra course, cannot make sense of the statement of the following theorem, even though they learned all the terms in a previous course. Theorem. Linear combinations of solutions, to a homogeneous linear equation, are more solutions to the same equation. I teach my students how to translate this theorem, into mathematical formulas.

as soon as students can be taught the concept of the word "per". Problems like this should be an important part of arithmetic-based science lessons in middle school.

Problem 3. It is a fact that fat has 9 calories per gram and protein has 4 calories per gram. If a piece of meat consists of 100 grams of protein and 10 grams of fat, how many calories does it have altogether? (Answer: 490 calories)

College students receive instruction on how to do Problem 3 in an elementary nutrition course at the Univ. of Maryland. This should not be necessary

Comment. *Little* mathematical reasoning is required for the word problems in this report. (Some mathematical calculations are required, but it's the reading and the following of directions that causes difficulties.) For contrast, I will now state three exceptions, Problem 4, which requires modest mathematical reasoning (about unit prices) and Problems 4-B and 5, which require serious mathematical reasoning.

Problem 4. Three white T-shirts cost 15 dollars. How much do 5 white T- shirts cost?

Problem 4-B. (Work) Suppose that it takes Sally 3 hours to mow a lawn, and it takes Tom 4 hours to mow the same lawn; Tom's mower is less powerful than Sally's. *Without* using algebra (x or other variables) determine how long it would take Sally and Tom to mow the lawn if they worked together (using both lawn mowers)? (Assume that each works at his/her standard pace and they never get in each other's way.)

Comment. Detailed instruction, on how to translate and analyze this and other Arithmetic word problems, together with instruction on the needed mathematical reasoning, appears in my report: "Supposedly Difficult Arithmetic Word Problems", accessible from my website: http://www.math.umd.edu/~jnd/Difficult_Word_Problems.html

Nine out of ten of my students read the next Problem 5, and then translate it into *wrong* equations, when I first assign it for homework (without instruction). My students are mostly college seniors majoring in engineering and mathematics education (about to become high school mathematics teachers). After a little training, most of the students are able to solve similar problems under timed test conditions. Most engineering and mathematics education students do not take my course. What is scarier, an engineer or an Algebra teacher writing down wrong equations?

Problem 5. (A typical Russian high school Algebra problem[28]) Suppose that it takes Tom and Dick 2 hours to do a certain job, it takes Tom and Harry 3 hours to do the same job and it takes Dick and Harry 4 hours to do the same job. How long would it take Tom, Dick and Harry to do the same job if all 3 men worked together? (Assume that each works at his standard speed.)

Comment. A correct way, to translate and analyze this algebraic word problem into useful equations, together with instruction on the needed mathematical reasoning appears, in my report: "Algebraic Word Problems ", on my website:
http://www.math.umd.edu/~jnd/Algebraic_word_problems.pdf

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[28] I found this problem in an article by Andre Toom in the American Educator.

2. English.

Grammar. Understanding grammar is crucial for correct understanding of word problems.

Passive Voice. Pupils need to understand the passive voice of verbs; and not just for math. This is crucial for understanding how "3 divided by 6" differs grammatically, and hence mathematically, from the common colloquial expression: "3 divides 6". (Note that: 3 pies may be divided by 6 girls and 3 girls may divide 6 pies; but 3 girls will not be divided by 6 pies.) Also how "6 divided by one-half" differs from "6 divided in half". The latter is short for "6 divided into two equal halves" which is 3, unlike "6 divided by one-half", which is 12.

Comment. Mathematics educator, Betsy Darken, described[29] the too little math knowledge, of the college students in her specialized Mathematics course for future elementary and middle school teachers at the University of Tennessee at Chattanooga: "Only 32% percent of the pretest students realized that $M \div 1/2$ is not equivalent to the statement, 'If M is divided into two equal parts, how much is in each part?' There was little improvement on this question on the posttest, with the percent correct rising to only 45%. Apparently this is a misunderstanding that is difficult to correct." These future teachers are not going to be able to explain how "6 divided by one-half" differs from "6 divided into halves" to their future students. Another reason to increase the reading and mathematics training of future teachers.

Quantitative vocabulary. Instruction in plain English quantitative vocabulary is important, both for Arithmetic word problems and outside of math lessons. Teachers' choice as to whether this occurs in a math lesson or a reading lesson.

For example: "Than" in the many ways it appears, including "more than", "less than", "fewer than", "older than",
also "at least" versus "more than" versus "not more than",
and "each" and "per", and "percent" versus "percentile".

As Andre Toom wrote [30] "In addition to those limited formalisms of pure mathematics, which are available at the K-12 level, word problems bring a plethora of images, such as coins, buttons, matches and nuts, time and age, work and rate, distance and speed, length, width, perimeter and area, fields, boxes, barrels, balls and planets, price, percentage, interest and discount, volume, mass and mixture, ships and current, planes and wind, pumps and pools etc. etc. It is an invaluable experience for children to discern those formal characteristics of these images, which should be taken into account to solve the problem."

"What is at least equally important, in my opinion, is that solving word problems, children have to comprehend and translate into mathematics a multitude of verbs, adverbs and syntactic words

[29] During her presentation, subtitled: "How Little Our Future Teachers Know —and How Much They Can Learn" at the Joint Mathematics Meetings in Atlanta, Georgia January 6, 2005.

[30] "Between Childhood and Math" in the "Humanistic Mathematics Network J.

indicating actions and *relations* between objects, such as put, give, take, bring, fill, drain, move, meet, overtake, more, less, later, earlier, before, after, from, to, between, against, away etc. Although I say "children", I actually mean a wide range of ages, including college undergraduates, for whom all this may be quite a challenge. "

Sophisticated math words and phrases are: "each", "per" and "pro rate" and the "*one-way*" significance of implication words and phrases, such as "because" and "then" (as in "A then B") and "because" and "implies" and "If … then" versus "only if". Relatedly, "If and only if". Also the "quantifiers": "there exists" versus "for all".

Important that students be taught that, in *proper* English, two negative words or prefixes cancel. This is useful outside of math, for figuring out that the word "antidisestablishment" means "in favor of retaining the establishment". But two negative words are commonly used to emphasize the negative in colloquial and slang phrases, such as in "A No, No!" and in "I do not know nothing!". Also in *proper* Spanish, two negative words are commonly used to emphasize the negative. Even in standard English, when does the answer "No" mean to the "negative" question: "Didn't you know that?" It usually means: "(No;) I did not know"; but double-negative-is-a-positive sticklers will say it means "(Not so;) I did know".

Set Theory vocabulary is descriptive, as such, it is useful and not just in math: "union", "intersection", "complement", "and", "neither", "or", and "nor". Also, "one" versus "a" ("at least one"). Venn Diagrams are useful; Boolean Algebra is not crucial.

English quantitative vocabulary (important for reading outside of math than for math) are: "much" versus "many", "list" versus "set", "payee" versus "payer" and "the" versus "a". Also: "biannually" and "biweekly" versus "semiannually" and "semiweekly" also, "bicycle" and "semicircle".

Converses. (Understanding the "*one-way*" significance of implication words.) Ezra Shahn wrote[31]: "In descriptions of many biological phenomena … 'understanding' means mastery of a sequence such as A then B then C then D … . It was as though in reading or hearing 'then' the student was understanding 'and'. … [But] the sequential relationship is more restrictive, hence more precise and it is this distinction that many students apparently fail to grasp." Shahn also wrote: " … it seems that students often misread conjunctions [including the implication words 'because' and 'then' (as in 'A then B')] so that they mean 'and'. "

Arnold Arons[32] wrote: "… essentially the same problem frequently arises in connection with "if … then" statements of reasoning." Arons elaborated: "Crucial to understanding scientific reasoning and explanation [in beginning physics classes] as opposed to recall of isolated technical terms, resides in the use of [implication words] words such as 'then' and 'because'. A perceptive description of the difficulties exhibited by many students [with the words 'then' and 'because'] is given by Shahn".

[31] Ezra Shahn, "On Science Literacy," in Educational Philosophy and Theory. Journal of the Philosophy of Education Society of Australia, (1988).

[32] Arnold Arons's book, A guide to introductory physics teaching, Chapter I Underpinnings, Section 1.16 Language

Understanding the "one-way" significance of the implication word "then" (even for just "A then B") and the word "and" is crucial in mathematics for understanding the difference between a statement and its converse.

Students should learn the difference between a statement and its converse and should not expect one to imply the other. Students should know that a contrapositive is equivalent to the original statement. This is useful both inside and outside of math. It is not necessary that the formal terms "converse" and "contrapositive" be used in elementary or middle school.

Caveats: Sophisticated language and math jargon, where it is unnecessary and cumbersome will *interfere* with students doing the math. When the reading level is higher than the math level on a math exam, *false negatives* will occur; that is, students, who understand the required math (both calculations and problem solving procedures), but flunk the exam due to reading comprehension difficulties.

Math trivia. Words that might be taught but not tested: A "googol" is the super large number, ten to the one hundred power, 10^{100} or 10^100 (A one followed by 100 zeros.) The unimaginable large number, a "googolplex" is ten to the googol power, 10^{googol}. (A one followed by a googol zeros.) A billion in England is a million million; that is, the number 10^12 is called a billion in England and is called a trillion in America. A "fortnight" is two weeks (fourteen nights). "Half again as much" means "1.5 times as much".

Following directions. Science and Math classes provide a good opportunity for instruction. Learning the standard algorithms of arithmetic calculations, is training in following directions. In addition, students should be able to do problems, which require following new directions, ones, which have not been repeatedly practiced or explained by the teacher. For example, doing the Geese Problem 1.

The "Key-Word" method. A popular avoid-the-reading method of instruction for Arithmetic Word problems is the "Key-Word" method, in which student are given lists of "addition" and "subtraction" words and phrases.

The "Key-Word" method says
- (i) Do not read the problem (or at most speed read it).
- (ii) Zoom in on the key word or phrase.
- (iii) Do the operation that the key word triggers.

When the Key word is "total" add up all the numbers in sight. When the Key word is "average", then average all the numbers in sight. But, the "Key-Word" method often produces *wrong* answers.

Problem 6. What must we add to 4 to obtain 9 as the sum?

Wrong Answer. The Key Word method directs students to zoom in on the two key "addition" words "add" and "sum", then add $4 + 9 = 13$.

The "Key-Word" method is very easy to teach. It does not place struggling readers at any disadvantage. This pseudo-altruistically levels the playing field for struggling readers. But, by not reading problems, students miss out on an important opportunity to practice reading for understanding. This is especially detrimental for weak students.

The key word "less" suggests subtraction, but subtracting will yield a wrong answer in Part (a) of the next problem.

Problem 6-B. Jack has five less dollars than Jill.
(a) If Jack has seven dollars, how many dollars does Jill have?
(b) If Jill has seven dollars, how many dollars does Jack have?

Instruction. Since Jack has five less dollars than Jill, one needs to count up 5 dollars from Jacks number of dollars to obtain Jill's number of dollars; and one needs to count down 5 dollars from Jill's number of dollars to obtain Jack's number of dollars.

There are similar problems written with other comparative words; for example:

Problem 6-C. Jack is five years younger than Jill.
(a) When Jack is seven years old, how old is Jill?
(b) When Jill is seven years old, how old is Jack?

Instruction. Since Jack is five years younger than Jill, one needs to count up 5 years from Jack's age to obtain Jill's age and one needs to count down 5 years from Jill's age to obtain Jack's age.

In the two paragraphs labeled instruction just above, I purposely used the phrases, "count up" and "count down" instead of the formal words: "addition" and "subtraction". These are simpler, more descriptive words for Grade 1 students. Students can transition to the formal words: "addition" and "subtraction" after becoming comfortable with "counting up" and "counting down". To me the phrase, "count up" is more descriptive than the phrase, "count on", which I saw in a current textbook.

Influenced by the Math Reform movement, students are now expected to read word problems. But some textbooks still provide lists of words that suggest "addition" or "subtraction". This is similar to the Key-Word method.

Some college students regularly look for key-words -- to their detriment. They will point to a paragraph of explanation and then trivialize it as "If I see the [key] word ..., then I do such and such". This over-simplification often ignores one of the hypotheses, which leads to wrong answers. The "Key-Word" bad habit is hard to break.

The avoidance of reading in math problems, in the 1980s and 1990s, is exemplified by the following very "functional" Problem 3, which was *not* permitted on the State of Maryland Functional Mathematics Test; it *exceeded* the test specifications; which specified that a change problem must state the total sale price. In fact, no problem requiring both addition and subtraction was permitted on the State of Maryland Functional Mathematics Test. Passing this exam was a high school graduation requirement until about 2004.

Problem 6-D. Sally buys a loaf of bread for two dollars and a gallon of milk, also for two dollars; she gives the clerk a five dollar bill. What is the change? (No taxes or special sales prices.)

Mathematical English versus Ordinary English. Sometimes ordinary English words take on special meanings in mathematics. For example: "There is a man in the next room" means (in mathematics) that "There is at least one man in the next room"; it includes the possibilities that there are three men in the next room or that there are a man and a woman in the next room.

Also, the word "or" in mathematics means "and/or". For example: In mathematics, the statement: "The Secretary of State was a woman or a black" includes the possibility of the Secretary of State being a black woman.

Mathematics	Ordinary English
There is	There is at least one
or	And/or
line segment between points P and Q	line between points P and Q
line through points P and Q	line through points P and Q, extended to infinity in both directions

Indirect definitions. In mathematics, some definitions are *indirect* and convoluted[33]. Indirect definitions require more instruction. The square of a number is directly defined as the number multiplied by itself; in symbols $A^2 = AxA$. So $3^2 = 3x3 = 9$. No problem. In contrast, the square root of a number is defined *indirectly* as another number, which when multiplied by itself returns the original number; in words and symbols: Given a number A, another number B is a square root of A, if $B^2 = A$. To find $\sqrt{9}$, one checks the multiplication table and finds that $3x3 = 9$, so 3 is a square root of 9; but so is -3, (since $(-3)(-3) = 9$). The only easy way to find $\sqrt{7}$ is to use a calculator.[34]

3. Precision.

Precision for Averages. When calculating averages, one must be precise about just which numbers are being averaged.

Problem 7. We flew from Denver to Boston at an average speed of 500 MPH; we returned from Boston to Denver at an average speed of 400 MPH. (The distance from Denver to Boston is 2000 miles) What was our average speed for this round-trip?

Wrong Answer. A student trained in the "Key-Word" method is likely to see the word "average" and then simply average 500 and 400, which is 450 MPH, an incorrect answer.

Comment. A crucial directive is to *use the defining equation,* that is, explicitly use the equation or formula provided by the definition of a mathematical or scientific term; the defining equation for average speed is:

$$\text{Average speed} = \{\text{Total distance}\} / \{\text{Total time}\}$$

Solution. Time to go: 2000/5 = 4 hours. Time to return: 2000/4 = 5 hours.
Total travel time: 9 hours. Now we use the defining equation:

$$\text{Average speed} = \{\text{Total distance}\} / \{\text{Total time}\} = 4000 \text{ miles} / 9 \text{ hours} = 444.4 \text{ MPH}.$$

In both math and in math-based science lessons, students should be provided with training in following this direction: "Use defining equations".

Percents[35]. Even more careful reading is often required when dealing with percents and fractions. The reason is that one must be clear as to what the "base" is, that is, the percent or

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[33] In Algebra II, $\log_{10} 5$ is defined as the number c, such that $10^c = 5$. Having a discussion on the meaning of $\log_{10} 5$ is counterproductive and likely to add to the confusion. Students should become comfortable just using the equations, namely $\log_{10} 5 = c$ if and only if $10^c = 5$.

[34] An involved way to calculate square roots used to be taught in Algebra I classes. Fortunately, no longer.

[35] I thank Professor Betsy Darken, University of Tennessee at Chattanooga, for alerting me to this difficulty.

fractions is being taken of precisely which number. Sometimes, this is tricky, especially when the base is not specified explicitly. Again, using the definition directly is useful.

Definitions. "Number N is 30% larger than (or more than or greater than) Number M" means that "Number N is 30% [of M] larger than Number M", that is:

$N = M + 30\%M$; NOT $N = M + 30\%N$ and NOT $N = M + 30\%$.

"Number N is 30% less than Number M" means that "Number N is 30% [of M] less than Number M", that is:

$N = M - 30\%M$; NOT $N = M - 30\%N$ and NOT $N = M - 30\%$.

Literal English versus common English: "Number N is 30% larger than Number M" *literally* means "$N = M + 30\%$"; but in common English usage it means "$N = M + 30\%M$".

Example. Thus, 125 is 25% more than 100, but 100 is 20% less than 125.

Problem 7-B. What is 4% larger than 49 and what is 4% larger than 49 million?
Answers. $49 + (.04) \times 49 = 50.96 \approx 51$.
49 million + (.04) x 49 million = 50,960,000 ≈ 51 million.

Here are a few interesting examples.

Definitions. (Retail Store) A store's "markup", on an item, is its sale's price minus the cost (purchase price), as a percentage of the <u>cost</u>. A store's "[gross] margin", on an item, is the sales price minus the cost, as a percentage of the <u>price</u>. A store's "net margin" is the store's total sales minus the total costs (to purchase its inventory), and minus all the other of the store's expenses (labor, rent, etc.), as a percentage of the store's total sales.

Problem 8. A store buys food for \$100 and sells it for \$125, what is its markup and what is its gross margin in percents?

Answer. Its markup is 25% and its margin is 20% (= 25 / 125).

Remark. This is mainly a *reading* problem, knowing the definitions of markup and margin. A *math* problem would be: A store's markup is 50%, what is its [gross] margin? (After some algebraic calculations, Answer 33 1/3%.)

That a 20% decrease does not exactly cancel a 20% increase is a consequence of the "changing" base, as occurs in the next problem.

Problem 9 . In May, the price of a gadget is \$100. In June, the store raises the price by 20%. In July, the store has a 20% off sale; what is the price in July?

Answer. The price in June was \$120. The 20% off sale is off the last price of \$120, which is \$24. So the price in June is \$120 – 24 = \$96.

Students should (be trained to) realize that the next problem is *ambiguous*, since there are two natural bases for the percent, leading to *two* correct answers.

Problem 10. Store A is selling a radio at 10 dollars. Store B is selling the same radio at 8 dollars. Express the difference in the prices as a percentage.

Answers. The difference in the prices is 2 dollars, which is 20% of Store A's price, which is also 25% of Store B's price. This may also be written as:
Store B's price is 20% less than Store A's price.
Store A's price is 25% higher than Store B's price.

Problem 10-B. (**Ambiguous**) In a presidential election, Candidate A received 51 million votes, Candidate B received 49 million votes and 100 million registered voters abstained by not voting for either. (No third party or write in votes; some abstainers did vote for senator that day.) By what percentage did Candidate A beat Candidate B?

Candidate A beat Candidate B by 2 million votes. But, depending on your perspective (or choice of "base"), Candidate A could have beaten Candidate B by 1% or 2% or 4%:

Answers. The 2 million votes out of the 100 million votes cast for the two candidates, is 2%.
The 2 million voters out of the 200 million registered voters is 1%.
In Problem 7-B, we saw that 51 million is about 4% more than 49 million.

Newspaper and TV normally call it a 2 point victory, by which they mean first subtract:
51% - 49% = 2%; then drop the percent sign.

Teachers should be trained to realize that the next problem is *ambiguous*, since there are two natural bases for the fraction, leading to *two* reasonable interpretations.

Problem 10-C. (**Ambiguous**) I have two identical pizza pies, each cut into five pieces of equal size. I eat one piece from each pie. What fraction have I eaten?

The *two* reasonable interpretations are:
(a) What fractional amount [of a pie] have I eaten?
(b) What fraction of the two pies have I eaten?

Yes, the wordings, of the two parts, are almost identical; training in precise reading is needed to understand the difference.

Answers. (a) Each piece is 1/5 of a pie. I ate 1/5 of one pie, and 1/5 of the other pie; so the amount I ate was 1/5 + 1/5 = 2/5 of a pie.
(b) I have eaten two of the ten identical pieces, so I have eaten 2/10 = 1/5 of the two pies.

Having a semantic discussion over an ambiguous, imprecise or unclear sentence (like "What fraction have I eaten?") is rarely useful, often counterproductive. One should simply note the ambiguity and then explicitly state the "base" (here one pie or two) for the fraction (or percent).

The next problem is a relatively standard middle school problem. But it stumped all the students, in Betsy Darken's college Mathematics course for future elementary and middle school teachers, on the pretest.[36]

Problem 11. An item's sale price is \$360 at a 33 1/3% off sale. Find its original price.

Darken noted that the major obstacle to solving this problem was a fundamental lack of understanding that the 33 1/3% discount is of *the original price NOT of the sale price*.

Common wrong Answer. The discount is 33 1/3% of \$360 = \$120. So its original price was \$360 + \$120 = \$480.

Solution. A synonym for 33 1/3% is 1/3.
The discount is 1/3 of the original price.
So the sale price (\$360) is the [remaining] 2/3 of the original price.
Dividing the last phrase by 2 yields: \$180 is 1/3 of the original price.
Hence (multiplying the last phrase by 3 yields:) \$540 was the original price.

Comment. It is *absurd* that the prerequisites, for college Mathematics courses for future elementary and middle school teachers, do *not* include being able to do relatively standard middle school mathematics problems. This *excessively* reduces the amount of elementary and middle school mathematics that can be taught to future teachers in college courses.

Here is a condensed version of a similar problem that high school teacher Gary Klauminzer gave to his students.

Problem 11B. On your lunch bill, a 5% tax was added to the cost of the food. After you add a 15% tip (applied to the cost of the food, not the tax), the final total is \$46.32. How much was the cost of the food.

Klauminzer reports that most students realized that 5% and 15% can be combined to make 20%. Then nine out of ten of his Algebra II students (and about a third of his advanced math/pre-calculus students) subtracted 20% of the final total. But it should be 20% of the bill for food and tax, not the final total, (which includes the tip).

Comments on Problem 2.

Problem 12. What are the deficiencies in the wording of Problem 2?

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[36] also in her presentation, subtitled: "How Little Our Future Teachers Know —and How Much They Can Learn" at the Joint Mathematics Meetings in Atlanta, Georgia January 6, 2005. A pictorial solution and an Algebraic solution appear in Darken's paper, albeit to the related question: "How much money do you save by buying the item on sale?"

As written, the correct answer for Problem 2 should be: *Not enough information is given*. The cost of heating and cooling depends on *both* the prices of heating oil and electricity and on utility usage, not just on the amount of heating oil and electricity used. In addition, the "typical household", which spent $1,400 on utilities, and the "typical household", for whom the table on utilities applies, may be two very *different* typical households. For each of these reasons, Problem 2, as stated, is unworkable. Students and writers of state assessments should be trained to spot imprecise wording of problems. Such training should help students to more precisely read non-mathematical material, also.

In the comment after Problem 2, we noted that 5 of 8 (65%) Grade 9 students did not solve that multiple-choice problem correctly. The next problem and its solution, shows how we obtained the number 65%.

Problem 13. (From http://www.mdk12.org/mspp/high_school/look_like/algebra/v48.html, click on "View field test results" on the right)
"**The correct answer** [to Problem 2] **is H.**
This item was field-tested in January and May of 2000. The number of students who were administered this item was 698. 22.20% percent of students omitted this item. The chart shows the percentage of student responses to each choice.

	Percentage of student who each chose each answer
F	16.80%
G	27.70%
H	44.40%
J	11.20%

Percentages were rounded up to the nearest .1%. Totals may not equal 100%."
Calculate the percentage of all the field-test students, who did not solve this problem correctly.

This problem requires careful reading and interpretation of the reading. We round the numbers in our calculation.

Solution. That "22% of students omitted this item", implies that only 100% - 22% = 78% of the students chose any answer to this item.
"That 45% of the students {who chose an answer} chose the *correct* answer, means that 45% of the 78% (of the students), chose the correct answer", hence
45% x 78% = .45 x .78 = .35 = 35% of the students chose the correct answer.
Hence, 100% - 35% = 65% of the field-tested students did not solve this problem correctly.

Newspapers. Precise reading is even more important when one is reading the newspaper.

Problem 13B. The newspaper reports that the earnings (or price or revenue) increased by 200% this year. Say this in simpler language, without using percents.

Solution. This year's earnings are three times last year's.

Reading includes realizing, sometimes after calculation, that the numbers are not reasonable. The next problem is an example. It and its answer are from The Public Editor [column]: "Numbed by the Numbers, When They Just Don't Add Up" in New York Times , January 23, 2005, http://www.nytimes.com/2005/01/23/weekinreview/23bott.html?ex=1107520994&ei=1&en=0424b2aabb40d565)

Problem 14. "In November [2004], when New York City Comptroller William Thompson released a study purporting to show that New Yorkers purchase more than $23 billion in counterfeit goods each year, The Times repeated the analysis as if it were credible." Is this story credible, why?
FYI, New York City has about 8 million people living in about 3 million households.

Answer. "Quick arithmetic would have demonstrated that $23 billion would work out to roughly $8,000 per city household, a number ludicrous on its face."

That statements using percentages can be quite misleading and that different bases for a percent can be significant is demonstrated by the following discussion, from the same NY Times Public Editor column.

The New York Times "third-party-paid" subscriptions are subscriptions paid for or subsidized by advertisers. A NY Times article noted that "the quantity of the paper's third-party-paid subscriptions on a given weekday is 79 percent higher than the comparable Sunday number. This sounds very ominous."

"It would sounds somewhat less ominous when you realize that these same third-party-paid subscriptions account for [a mere] 1.4 percent of Sunday circulation, and 2.5 percent of weekday circulation."

Comment. This "79% higher" phrase tempts readers to conclude that third-party-paid subscriptions are a significant part of weekday circulation. But, when third-party-paid subscriptions are only a tiny part, then being "79% higher" or even double is still tiny.

4. Other notes on the Geese Problem 1.

The statement of Problem 1 is like an outline of the solution; most of the sentences need to be comprehended and interpreted as a direction to be followed. The description of what occurred at 2 PM and at 3 PM does require precise and careful reading. (This arithmetic solution, also, demonstrates the usefulness of memorizing basic fraction-percent synonyms, like 1/5 = 20%.)

Of course, there should be instruction on simpler problems in earlier grades, which will serve as background for the Geese Problem. For example:

Problem 15. (A "Forward" Geese Problem) A flock of 100 geese on a pond were being observed continuously.

At 1:00 P.M., 1/5 of the geese flew away.
At 2:00 P.M., 1/8 of the geese that remained flew away.
At 3:00 P.M., 3 times as many geese as had flown away at 1:00 P.M. flew away, leaving 28 geese on the pond.
At no other time did any geese arrive or fly away or die.
(a) How many geese remain at 1:00 PM?
(b) How many geese remain at 2:00 PM?
(c) How many geese remain at 3:00 PM?

The successive parts for this Problem 15, might be assigned in successive grades[37].

Problem 16. (A Short Geese Problem) A flock of geese on a pond were being observed continuously.
At 1:00 P.M., 1/5 of the geese flew away.
At 2:00 P.M., 1/8 of the geese that remained flew away, leaving 30 geese on the pond.
At no other time did any geese arrive or fly away or die.
How many were there at noon?

Here is a small variant on the Geese Problem:

Problem 17. **(From a 5th grade Singapore math textbook [38]**) "Mrs. Chen made some tarts. She sold 3/5 of them in the morning and 1/4 of the remainder in the afternoon. If she sold 200 more tarts in the morning than in the afternoon, how many tarts did she make?"

A solution[39] to Problem 15 is quite similar to the one for Problem 1. Singapore math textbooks train students to do such problems in a pictorial manner.

5. Other Problems.

Word order matters! The next problem demonstrates that "the sum of the squares" is different from "the square of the sum"; [40] it will be the English, which will determine the mathematics. Similarly, "the average of the squares" is different from "the square of the average".

Problem 19. For the set of numbers {1, 2 and 3},

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[37] I thank Mathematics professor Ralph A. Raimi for spurring me to include this sentence.
[38] From http://www.cbmsweb.org/NationalSummit/Plenary_Speakers/ma.htm
[39] **Arithmetic Solution**. In morning, 3/5 = 60% of tarts sold, leaving 40%.
In afternoon, 1⁄4 of remaining 40% sold = 10%.
That "She sold 200 more tarts in the morning than in the afternoon" translates into:
200 tarts = {morning sales} - {Afternoon sales} = (60% – 10%) of tarts = 50% of tarts = 1⁄2 of tarts.
Thus 200 tarts = half of the tarts. Hence total tarts were 400 tarts.
[40] Even though a sizable percentage of college students believe that the two are equal; they regularly mis-calculate $(a+b)^2$ as $a^2 + b^2$.

(a) Find the sum of the squares of these three numbers.
(b) Find the square of the sum of these three numbers.
(c) Find the average of the squares of these three numbers.
(d) Find the square of the average of these three numbers.

Each part requires following the directions, exactly as stated, so that the correct operation (squaring or summing or averaging) is done first.

Answers. (a) One first squares the numbers (obtaining 1, 4 and 9), then sums them, which results in 14.
(b) Here, one first sums the numbers, obtaining 6, then squares 6, which is 36.
(c) 14/5 and **(d)** 4.

A half-price sale suggests a 50% savings, but accurately reading the full description will show otherwise in the next problem:

Problem 20. The store sale sign reads: "Buy one tie, get a second tie at half price, when the second is of equal or lesser value (price)". John buys two ties, one priced at 4 dollars, the other at 6 dollars.
(a) How much is the sale price on these two ties?
(b) What percentage saving does John obtain on his purchase of the two ties?

Answer. (a) $6 + $4/2 = $8.
(b) The non-sale price is $10. The saving is $2. The percentage saving is $2/$10 = 20%.

The next four problems involve only addition and subtraction of small integers. They are appropriate for early grades; careful reading is required.
Both parts of the next problem start with the phase "How many", but different things are happening. This may trick some followers of the Key Word Method.

Problem 21. Mom gave Tom 6 toys on Monday and 4 toys on Tuesday.
(a) How many toys did Tom receive from Mom on Monday and Tuesday?
(b) How many more toys did Tom receive from Mom on Monday than on Tuesday?

The next problem contains the "subtraction" phrase "more than", but it is NOT a "subtraction" problem.

Problem 22. Mom gave Tom 6 toys on Monday and 4 toys on Tuesday. On Wednesday, Mom gave Tom 3 more toys than she gave him on Monday. How many toys did Tom receive from Mom on Wednesday?

The next problem is a "two-step" variant on the preceding problem, it requires more thought, when it is not preceded by Problem 22.

Problem 23. Mom gave Tom 6 toys on Monday and 4 toys on Tuesday. On Wednesday, Mom gave Tom 3 more toys than she gave him on Monday. How many toys did Tom receive from Mom on Monday and Wednesday?

The wording can be more sophisticated and more complicated, which will require appropriate reading instruction. For example:

Problem 24. Mom gave Tom 6 toys on Monday and 4 toys on Tuesday. On Wednesday, Mom gave Tom 3 more toys than she gave him on Monday. The number of toys Mom gave Tom on Thursday was the sum of the numbers of toys Mom gave Tom on Monday and Tuesday or it was twice the number she gave him on Wednesday, whichever is the lesser. How many toys did Tom receive from Mom on Thursday?

Problem 25. World War II started in 1939 and ended in 1945. How many years did the war last?

Comment. Based on *just* this data, the answer is between 5 and 7 years. It might have been the five years from Dec. 31, 1939 to Jan. 1, 1945, or it might have been the seven years from Jan. 1, 1939 to Dec. 31, 1945. One needs more specific information, like the months that the war started and ended, in order to get a more specific answer.

Problem 26. World War II started Sept. 1, 1939 and ended in August, 1945. How long did the war last?

Problem 27. Something is known to have occurred in Egypt around 2222 BC. How many centuries ago was that? (Round to the nearest century.)

Problem 28. (At least vs. more than) [41] Tom is about to throw a single fair die[42].
(a) What is the probability, that Tom's throw will be more than 5?
(b) What is the probability, that Tom's throw will be at least 5?
(c) What is the probability, that Tom's throw will *not* be more than 5?

Problem 29. (Pro-rated). I company rents temporary office space at $2000 for the first month and $1000 per month, thereafter, pro-rated for part of a month. The company uses this office space for three and a half months, what is the total rent? Answer. $4500.

Problem 30. (From a Grade 3 Russian math textbook.[43]) 2 televisions and 4 radios were purchased for a resort building. 756 rubles were paid for everything. The price of a television is 270 rubles. How much does a radio cost?

Solution. The first two sentences translate into: Cost of 2 TVs and 4 radios is 756 rubles.

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[41] This problem was inspired by an instructor teaching Statistics 100, on my college campus, who reported his students confusing parts of this problem. We have selective admissions.
[42] A "fair die" is one for which the probabilities of each side being thrown are equal (namely 1/6). "Loaded dice" are NOT fair dice.
[43] From http://www.cbmsweb.org/NationalSummit/Plenary_Speakers/ma.htm Page 4

Using the data from the third sentence:	2 x 270 + {Cost of 4 radios} = 756 (rubles)
Now, ready to do arithmetic:	{Cost of 4 radios} = (756 - 2 x 270) (rubles)
	{Cost of 1 radio} = (756 - 2 x 270) ÷ 4

Problem 31 requires a little reading, but it still stymied about 1 of 5 Grade 9 students, when it was field tested in Maryland.

Problem 31. (MD Algebra exam [44]) The United States Congress is composed of the Senate and the House of Representatives. The matrices [tables] below show the number of members in Congress from 1983 through 1989.

Senate	1983	1985	1987	1989
D	54	53	55	55
R	46	47	45	45
I	0	0	0	0

House	1983	1985	1987	1989
D	269	252	258	259
R	165	182	177	174
I	0	0	0	0

What was the total number of Democrats in Congress in 1985?

[Multiple Choice] F 229 G 235 H 305 J 534

The bulk of the Grade 9 students got this problem, when it was field tested in 2000. Most likely reading comprehension was the major reason that one out of five students missed this question, [45] since students can surely use a calculator to add 53 + 252. This should be of some concern.

Problem 32. We are about to fly from Denver to Boston, which are 2000 miles apart. Our plane's basic speed is 450 MPH. Today, there is a 50 MPH west wind (blowing in the direction from Denver to Boston). So, we expect to be flying from Denver to Boston at an average speed of 500 MPH, and we expect to be returning from Boston to Denver at an average speed of 400 MPH. Our plane burns 10 gallons of fuel each hour. The fuel tank is filled with 89 gallons. We are in a hurry, should we take on more fuel during the stop at Boston?

Comment. Students, using the Key Word method (as in the wrong solution to Problem 7), will calculate that 89 gallons is enough. Students, using the correct method of solution to Problem 7, will calculate that 90 gallons are needed.

Problem 33. A small company has two trucks, a small one, which gets 20 miles-per-gallon (mpg) and a large one, which gets 14 mpg. Last year, the small truck was driven 100,000

[44] 2000 sample MD High School Assessment Algebra test, Item #2, on the web at http://www.mdk12.org/mspp/high_school/look_like/algebra/v2.htmlc.

[45] Go to http://www.mdk12.org/mspp/high_school/look_like/algebra/v2.htmlc. Click on "View field test results" on the right

miles, while the large truck was driven 14,000 miles. Find the miles-per-gallon for the two trucks together, as used last year.

Wrong Answer. The Key Word method directs students to average the mpg's of the two trucks, which is: Average{14 and 20} = 17.

Solution. Correct reading of the phrase "the miles-per-gallon for the two trucks together", includes *directly* using the definition of "average miles-per-gallon", namely:

{Average mpg for the two trucks together} = {Total mileage} / {Total gas}

= 114,000 miles / 6,000 gal = 19 mpg.