Nonlinear propagation of pulsed ultrasonic waves in stressed granular media

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Abstract

Elastic waves propagating in granular media provide a unique probe of both its structure and nonlinear dynamic properties. Here we focus our attention on the second-harmonic generation of a compressional wave train travelling through a glass bead packing under various effective applied loads. Amplitudes of the filtered second harmonic are measured as a function of the input amplitude and distance of propagation. The experimental results are in a good agreement with the theoretical prediction based on the Hertz model for granular elasticity. Also we examine the dependence of the nonlinear parameters on the applied loading.

Introduction

Finite-amplitude sound propagation in dissipative continuous media is a very rich topic [1]. The fundamental achievements were made first for fluids [2, 3, 4] and later for solids [5, 6]. Various analytical models and approximation methods (e.g., perturbation analysis) are developed in terms of the strain field, to describe the nonlinear acoustic effects such as cumulative wave distortion and the concept of the parametric array. More recently there is a great deal of interest in the nonlinear acoustics in earth materials. Consolidated aggregates such as rocks are shown to exhibit enormously strong hysteretic nonlinearities as compared with ordinary solid materials [7, 8]. Similar nonlinear acoustic effects have also been found in non cohesive granular materials [9] by travelling wave [10] and resonance experiments [11]. In this work, we report the observation of the generation of the second-harmonic in a stressed granular medium. Our experimental results will be compared with theoretical calculations based on Hertzian nonlinear elasticity developed in granular materials. The effect of energy loss on the nonlinear propagation will be emphasised.

1 Theoretical Analysis

In material (Lagrangian) coordinates, the 1D equation of motion in a continous medium can be written as [1]:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial a} \tag{1}$$

where u is the particle displacement, σ the stress and a the particle coordinate.

In the case of a solid, the equation of state is characterized by an elastic modulus K_0 through a stress-strain relation:

$$\sigma = K_0 \epsilon (1 + \beta\epsilon + ...) = K_0 \frac{\partial u}{\partial a}\left[1 + \beta \frac{\partial u}{\partial a} + ...\right] \tag{2}$$

where β is the first order classical nonlinear parameter and $c_0 = (K_0/\rho_0)^{1/2}$ the sound velocity in the unperturbated medium. Here, we'll neglect higher-order nonlinear terms. For a granular medium, the stress-strain relation derived from Hertz's contact theory, gives $\beta = 1/4\epsilon_0$ with ϵ_0 the static strain due to the external load [8, 10, 11].

To take into account the internal dissipation of energy, one usually adds the viscous stress $\eta \partial^3 u/\partial t \partial a^2$ to the equation of motion Eq.(1), where $\nu = \eta/\rho_0$ is an effective kinetic viscosity. So, the equation of motion becomes:

$$\frac{\partial^2 u}{\partial t^2} - \nu \frac{\partial^3 u}{\partial t \partial a^2} - c_0^2 \frac{\partial^2 u}{\partial a^2} = c_0^2 \beta \frac{\partial}{\partial a}\left(\frac{\partial u}{\partial a}\right)^2 \tag{3}$$

This equation has the characteristic form of a second-order approximation problem. The left side of this equation coincides with the linear equation, but the right side represent a quadratic correction. The analysis of wave propagation on the basis of Eq.(3) is carried out in the case of finite but moderate amplitude by means of a perturbation analysis [12, 13] assuming that the particle displacement can be written as $u = u_{1\omega} + u_{2\omega}$. One can obtain the solution with the boundary thecondition ($u_{2\omega}(a = 0) = 0$), i.e., the second-harmonic is not present at the source:

$$u_{1\omega} = A e^{-\alpha a} cos(ka - \omega t) \tag{4a}$$

$$u_{2\omega} = \frac{A^2 \beta \omega^2}{8 \alpha c_0^2}(e^{-2\alpha a} - e^{-4\alpha a}) cos2(ka - \omega t) \tag{4b}$$

where A is the amplitude of the displacement of the source along the direction of propagation of the wave and $\alpha = \nu\omega^2/2c_0^3$ the coefficient of attenuation. A priori, the second-harmonic first increases, reaches a maximum and then decreases.

2 Experiments

Our dry granular media are composed of polydisperse glass beads of diameter $d \approx 0.6 - 0.8mm$, confined in a cylinder closed by two fitted pistons. We then apply a normal load P of the order of a few hundred kPa. Lengths of the cell are less than its diameter, ranging from $25mm$ to $65mm$. A large source transducer of diameter $30mm$ and an identical detecting transducer are placed on the axis, respectively, at the top and bottom of the cell in direct contact with glass beads. A ten-cycle tone burst excitations centered at $50kHz$ is applied to the longitudinal source transducer (Fig.1). At low frequency ($\lambda >> d$) as in this work, coherent wave propagation is ballistic.

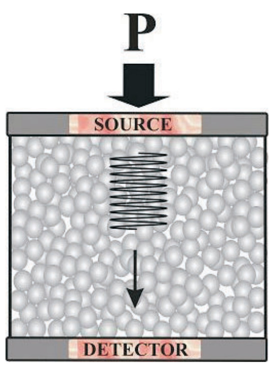

Figure 1: Coherent propagation of the longitudinal waves through a glass bead packing under stress P.

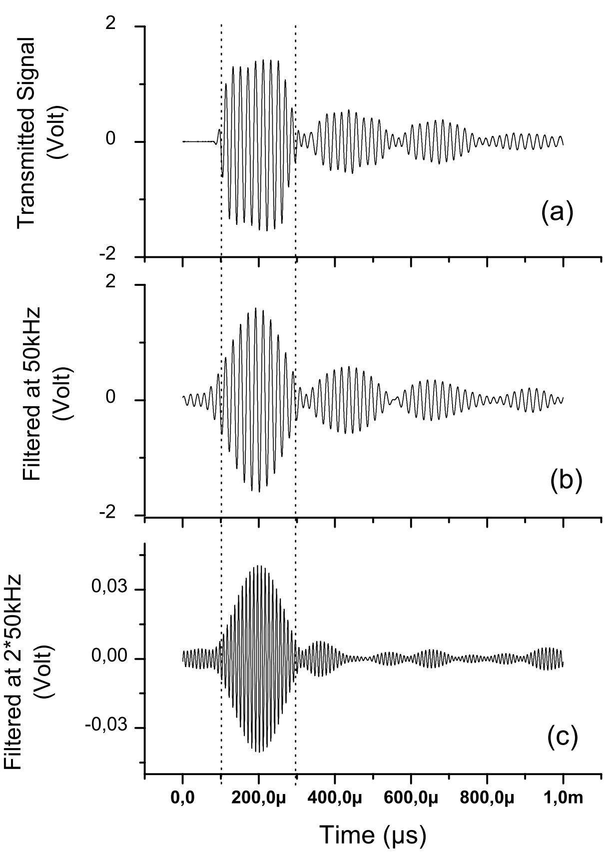

Figure 2: Transmitted signal through a glass bead packing under stress $P = 500kPa$. (a) Transmitted signal, (b) Signal filtered at $50kHz$, (c) Signal filtered at $2*50kHz = 100kHz$.

In Fig.(2.a), we show the arrival of the longitudinal pulse followed by its echos from the bottom and top boundaries in a bead packing under $P = 500kPa$. To study the generation of the second harmonic, we filtered the transmitted ultrasonic signal through the material by a temporal filter centred at $50kHz$ and $2*50kHz$, respectively (Figs.2.b & 2.c).The fundamental and second-harmonic amplitudes are then measured as a function of the amplitude of the input source voltage V_{input} (Fig.3). As the amplitude of vibration at the source is proportional to V_{input}, Fig.(3) clearly illustrates a linear dependence of the fundamental and quadratic dependence of the second harmonic as a function of the input amplitude. As predicted by Eq.(4), these results confirm the classic nonlinear behaviour of the granular materials in the present experiment. Note that the absence of the second-harmonic at lowest amplitude in Fig.(3) is due to a poor signal/noise ratio.

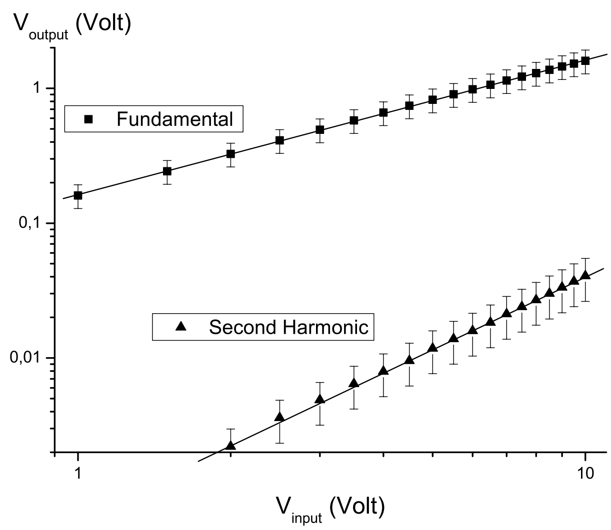

Figure 3: Amplitude of the fundamental (■) and second-harmonic (▲) are detected versus input voltage. Solid lines correspond to the theoretical prediction of linear and quadratic dependences.

3 Discussion

Nonlinear behaviour of sound propagation in the granular material can also be demonstrated by the attenuation measurement of the second harmonic. In Fig.(4), we present the fundamental wave amplitude ($50kHz$) and the generated second-harmonic ($100kHz$) V_{output} as a function of the propagation distance; we then readily extract the coefficient of attenuation α from the slope of these curves. The attenuation coefficient of the second-harmonic is twice as large as that of the fundamental: $\alpha_{2*50kHz} \approx 2 * \alpha_{50kHz}$ (Table1). For the range of distance with $\alpha a \sim 1$ as the case in our experiments, the theoretical predictions from Eqs.(4) can be simplified by:

$$u_{1\omega} = Ae^{-\alpha a} cos(ka - \omega t) \tag{5a}$$

$$u_{2\omega} = \frac{A^2 \beta \omega^2}{8 \alpha c_0^2} e^{-2\alpha a} cos(2ka - 2\omega t) \tag{5b}$$

which are in a good agreement with the experimental results. For comparison in Fig.(4), we show the fundamental wave attenuation of an input signal centered at $100kHz$. It clearly appears that $\alpha_{100kHz} \neq \alpha_{2*50kHz}$. This finding confirms the nonlinear behaviours of attenuation observed with the second harmonic $2*50kHz$ in our granular medium.

Frequency	α (mm^{-1})	$1/\alpha$ (mm)
$50kHz$	$0,022$	$45,5$
$2*50kHz$	$0,045$	$28,5$
$100kHz$	$0,035$	$22,2$

Table 1: Experimental values of the coefficient and length of attenuation for $50kHz$, $2*50kHz$ and $100kHz$.

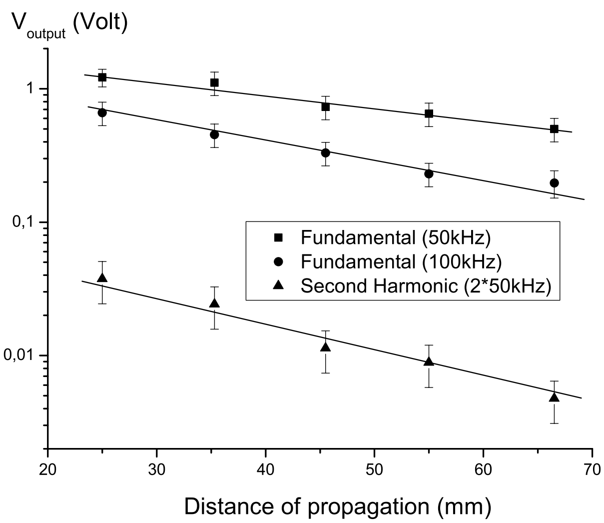

Figure 4: Amplitudes of the fundamental and the second harmonics measured as a function of distance of propagation. Solid lines correspond to the calculations from Eqs.(5) for a fundamental $50kHz$ (■), $100kHz$ (●) and the second-harmonic at $2*50kHz$ (▲).

Finally, we examine the stress-dependence of the nonlinear parameter β. To do so, we apply the ratio $u_{2\omega}/u_{1\omega}^2$, as a function of the applied pressure from Eqs.(5). According to Hertz's contact theory, c_0 scales as pressure $P^{1/6}$ [14] and accordingly the attenuation α scales as $P^{-1/2}$. If the nonlinear parameter β is $1/4\epsilon_0$, we can expect the scaling behaviour:

$$\left|\frac{u_{2\omega}}{u_{1\omega}^2}\right| = \frac{\beta\omega^2}{8\alpha c_0^2} \propto P^{-1/2} \tag{6}$$

To verify this behaviour, we have measured the ratio of the second-harmonic amplitude to the square of the fundamental one for external load P ranging from $50kPa$ to $500kPa$. As illustrated in Fig.(5), the experiment and the theory are in a reasonable good agreement.

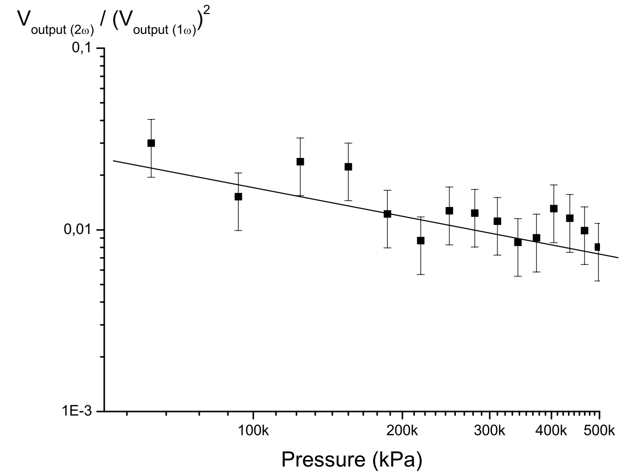

Figure 5: Stress dependance of the second-harmonic. Experimental results (■) and theoretical prediction for scaling behaviour.

Conclusion

We have experimentally observed the spectral evolution of a travelling longitudinal wave train as a function of propagation distance in strong loss granular materials. The second-harmonic amplitude was shown to exhibit a quadratic dependence on the fundamental wave amplitude. A theoretical analysis of the nonlinear elasticity is given which includes the energy loss in granular materials. This heuristic model based on the Hertz contact theory describes very satisfactorily the experimental results in terms of a classical nonlinear acoustics and also predicts the stress-dependence of the nonlinear parameters (e.g., β) in granular materials.

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