Thermal–Stochastic Properties of Hysteretic Elastic System: The Quasi–Static Limit

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Abstract. It has recently become clear that a large category of materials exhibit an identical, nonlinear elastic signature [1]. These materials (rocks, sand, cement, concrete and ceramics) have been termed nonlinear mesoscopic elastic (*NME*) materials, in contrast to the well-known nonlinear atomic elastic (*AE*) materials, where the traditional theory of elasticity can be used to describe their elastic properties. In this work we describe a model in the quasi–static limit of slow dynamics, one of the most intriguing nonlinear phenomena of these *NME* materials.

INTRODUCTION

There is a class of materials whose elastic properties cannot be explained by classical Landau elasticity theory [2]. Hysteresis and point memory in quasi–static measurements of stress as a function of strain, a non-classical nonlinear elasticity in dynamic measurements and relaxation effects termed slow dynamics on time scale of 10^2 – 10^4 sec are the principal features of this class of materials. These materials are in contrast to *AE* materials. The difference between these two categories of materials is both in the manifestation and the origin of their nonlinear responses. Elasticity of *AE* arises from atomic-level forces between atoms and molecules and they can be described by the traditional theory of elasticity. In contrast, the *NME* materials contain soft features contained in a hard matrix termed the ”bond system” where the elastic nonlinearity arises.

Recent dynamical stress–stain measurements [3] draw attention to the presence of broad time scales in the elastic response of rocks and other materials. These experiments are complementary to [4] *creep* that we call *quasi–static slow dynamics*. In an experiment applying a constant force one observes a logarithmic recovery in time.

The purpose of this paper is to study the effects of the temperature on these systems numerically and analytically and to reproduce the slow dynamical response seen in the *NME* materials.

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THE MODEL

A phenomenological model termed the *DMG* (Dynamical McCall Guyer model) has been developed that describes most of the nonlinear features seen in dynamical experiments [5]. The model proposed in this paper is a generalization of the *DMG* model to slow dynamics where the response of the system to a fluctuating thermal environment has been included in order to study the approach of the system to equilibrium. In the *DMG* model a rock is represented as a chain of N particles (rigid units) connected by hysteretic elastic elements (bond system) that describe the mesoscopic nonlinear elastic properties. Two quantities are used in the model, the displacement $u_i(t)$, describing the displacement for the $i-th$ particle and $\eta_i(t;u_i,u_{i+1},u_{i-1})$ the state variable associated with $i-th$ elastic units describing the nonlinear behavior of the system [5].

In this work we added a stochastic force to this system. The thermal noise is a perturbation of the system assumed to drive the slow dynamics. Therefore the thermal fluctuations that drive the displacement $u(t)$ must be small. In this limit of low temperature the displacement can be neglected in the dynamics of η_i, and are therefore uncoupled from the $u(t)$ equation. The equations for $u_i(t)$ and $\eta_i(t)$ under a force F become simpler as a result. The thermal and mechanical equilibrium length of the chain under the forcing F is $L(t;F) = u_N - u_1 = NF - b\sum_{i=1}^{N-1}\langle\eta_i\rangle(t)$, [5] where $\langle\eta_i\rangle(t)$ is the ensemble average. Its behavior can be studied by analyzing separately the $\langle\eta_i\rangle(t)$ of each elastic element in the chain. To introduce the temperature in this system we have added stochastic noise,the Langevine force $f_s(t;T)$, with amplitude proportional to the temperature T and inversely proportional to the damping term τ_η [6]:

$$\ddot{\eta}_i + \frac{\dot{\eta}_i}{\tau_\eta} = -\alpha_i + \beta_i\eta_i - \eta_i^3 + f_s(t;T) \tag{1}$$

where

$$\alpha_i = \frac{tanh[k\cdot(f_{c,i}+F)] + tanh[k\cdot(f_{o,i}+F)]}{2} \tag{2}$$

$$\beta_i = \frac{tanh[k\cdot(f_{c,i}+F)] - tanh[k\cdot(f_{o,i}+F)]}{2} \tag{3}$$

The parameter b and f_o, f_c determine the nonlinear behavior of the unit [5] and are different for each elastic unit in the chain. The state variable is driven by a potential $W(\eta) = \alpha\cdot\eta - 1/2\beta\cdot\eta^2 + 1/4\eta^4$, whose shape depends on f_o, f_c so it will be different for each elastic unit.

In this work we attempt to reproduce observations of placing a small strain ($\sim 10^{-6}$) on a sample removing it and then follow the creep evolution of the material. This translates in the model to applying an external force F_{ext} to the chain and analyzing the the time evolution of the chain's length once the external force is removed to study its approach to equilibrium.

RESULTS

The time evolution of the average, $\langle \eta_i \rangle(t)$, can be analytically determined and written in terms of probabilities:

$$\langle \eta_i \rangle(t) = \int d\eta_i \int d\eta_i' \eta_i P(\eta_i t \,|\, \eta_i' 0; F = 0, f_o, f_c) P_0(\eta_i'; F_{ext}, f_{o,i}, f_{c,i}) \quad (4)$$

where $P_0(\eta_i'; F_{ext}, f_o, f_c)$ is the thermal equilibrium distribution for η_i in the first part of the simulation where a F_{ext} is applied to the chain. $P(\eta_i t \,|\, \eta_i' 0; F = 0, f_o, f_c)$, conditional probability, is the probability to have a value of η_i in $d\eta_i$ at time t if at time $t' = 0$, when the external force is removed, η is equal to η_i'. Both of these distributions can be analytically calculated by solving the Fokker–Planck equations [6]. In solving the Fokker-Planck equation for P and P_0 solutions, we have calculated the solution for $\langle \eta_i \rangle(t)$ in the limit of $t \to \infty$ and low temperature as [6],

$$\langle \eta_i \rangle(t) = \eta_0(\tau_\eta, T)\, exp\{-\frac{t}{\tau_i(T, f_{o,i}, f_{c,i})}\}. \quad (5)$$

Eq.5 represents the time evolution of the state variable, in response to a step change in the applied external forced, in terms of a spectrum of relaxation times (τ_n). The relaxation time τ_i that characterizes the approach to equilibrium is a function (with fixed temperature and damping) of the potential $W(\eta; f_{o,i}, f_{c,i})$ that is different for each unit in the chain, as one can see in Fig.1. If the potential is symmetric (Fig.1a where $f_o = -f_c = -0.7$) and the temperature is low compared to the barrier potential, the unit reaches equilibrium slowly. The unit reaches equilibrium quickly and the relaxation time is therefore small if the potential is asymmetric as shown in Fig.1b. The time evolution of the η averages has been calculated as an ensemble average over 5000. The beginning values for each ensemble have been chosen randomly from the probability distribution $P_0(\eta'; F_{ext}, f_o, f_c)$ of η before removing the external force.

The state variable for the chain of 600 elements was calculated as the sum of all state variables of each unit. The f_o, f_c for each unit were chosen using a uniform distribution in PM space [5]. Each $\langle \eta_i \rangle$ was fitted with an exponential and the spectrum of relaxation times was calculated. Fig.1c shows a set of relaxation times that characterizes the return to $F = 0$ to equilibrium as broadly distributed. The state variable for the chain $\langle \eta \rangle(t) = L(F = 0)$ shows a logarithmic recovery (quasi–static slow dynamics) in Fig.1d.

CONCLUSIONS

A generalization of the DMG model where a fluctuating thermal environment is included has been introduced and studied numerically. The numerical results of this new model SDMG (Stochastic thermal Dynamical McCall and Guyer) have reproduced observations and the behavior expected from the theory. In particular the results here presented show that the slow dynamics of the hysteretic elastic systems can be described including the thermal forces in the DMG model in the quasi–static limit.

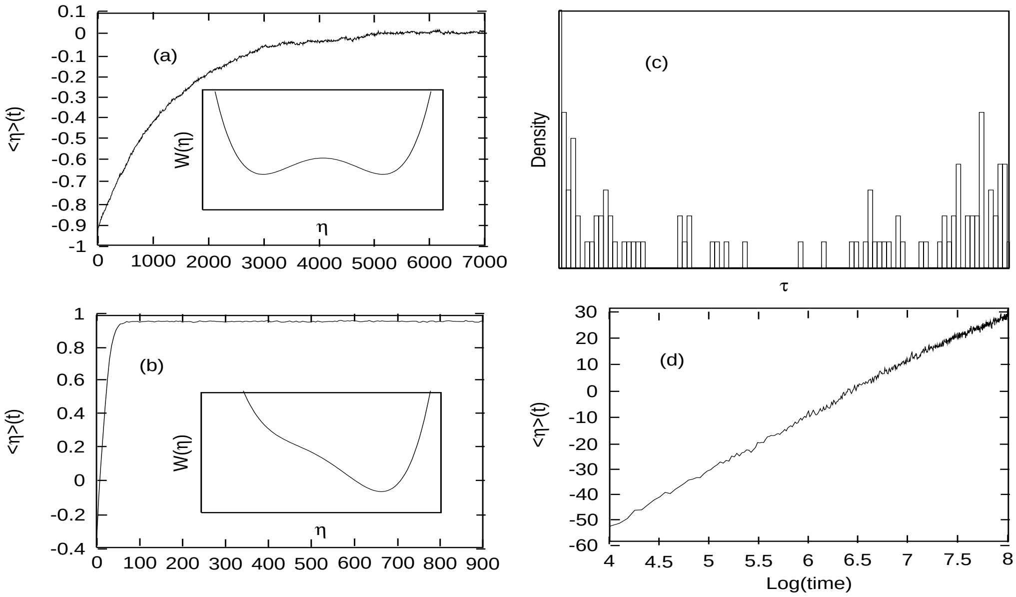

FIGURE 1. a) $f_o = -f_c = -0.7$: the potential $W(\eta)$ is symmetric and the $\langle\eta\rangle(t)$ once removed external force reaches equilibrium quickly; b) asymmetric potential ($f_o = -0.9$ and $f_c = -0.01$) where equilibrium is reached very fast. c) Spectrum of relaxation times for a chain of 600 elements ; d) the ensemble average of state variable for the chain in semi logarithmic scale.

A description of the dynamic response in terms of slow dynamics is the natural next step in this work, step that is the subject of current research started in collaboration with S.Habib and K. Heitmann at Los Alamos.

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