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Sensitive determination of nonlinear properties of Berea sandstone at low strains

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Abstract. Resonant response functions of a sandstone bar, measured at strain magnitudes between 10^{-8} and 10^{-6} at several temperatures, are used to extract Young's modulus and loss tangent as functions of frequency and driving force. Expressing the distorted shape of nonlinear resonances in terms of shifts in the resonant frequency and Q enables correlation analysis of these functions against components of the strain response function. Resonance shifts are found to be strongly correlated with somewhat complicated functions of strain, but only weakly with frequency or phase of the response. Decomposing the data along contours of constant strain magnitude yields the quantitative scaling of resonance shifts with strain, which shows a superposition of both conventional and necessarily hysteretic nonlinear sources. No statistically significant temperature dependence is found in the coefficients of the fits.

Introduction

Rocks are known to be much more nonlinear than the crystal grains they comprise. The physical state conditions on which this nonlinearity depends, however, are not yet well understood. Young's mode resonance experiments on highly nonlinear materials, like sandstone, can yield extremely precise measurements of shifts in material properties (shifts of Young's modulus result in shifts of resonance frequency; shifts in loss tangent result in shifts of resonant quality factor Q.) The high resolution of a set of such measurements, presented here, places constraints on both models and mechanisms of nonlinearity to unprecedented low strains.

The classical Landau theory of nonlinear elasticity [*Landau and Lifschitz*, 1959] predicts a modulus shift proportional to the square of dynamical strain magnitude [*McCall and Guyer*, 1996], and harmonic generation proportional to strain at linear and quadratic orders. Reversible by construction, this theory predicts no net energy loss, but the cascade of energy from lower to higher harmonics results in a nonlinear attenuation of the fundamental resembling a shift in $1/Q$.

A description of non-conservative nonlinearity originating in hysteresis [*McCall and Guyer*, 1996, 1994] has recently been added to the classical Landau theory. The hysteretic mechanism is predicted (in its simplest formulation) to lead to shifts of both resonant frequency and Q proportional to the *first* power of strain magnitude. A subset of the data analyzed in this paper (at a single temperature) has already been argued [*Guyer et al.*, 1999] to show this scaling for strains from 3×10^{-8} to 10^{-6}.

It is important that neither Landau theory nor its hysteretic extension implies a particular microphysical origin for nonlinearity. Indeed, more structured distributions than the simplest one in the hysteretic many-element state space can reproduce all power-law scalings of the Landau theory, from either conservative or lossy elements. Thus, recovery of a Landau-scaling component does not alone imply reversibility, and only the *need* for hysteretic elements can be demonstrated from a scaling component linear in strain.

A concern in applying the mesoscopic theory of hysteresis, at strains nearing 10^{-8}, is the difficulty of activating obvious candidate hysteretic elements such as interlocking grain asperities, when total bar-end displacements are at most tens to hundreds of nanometers. It is thus necessary to consider the possibility that hysteresis is due to physico-chemical, and not mechanical, stick-slip processes [*Thompson and Robbins*, 1990]. If so, these might be thermally rather than acoustically activated. To test this premise, resonance measurements were made at temperatures from 35°C to 65°C . While the temperature range is only ten percent of absolute, it represents half of the practical range accessible in rocks, between irreversible damage thresholds at freezing (0°C), and the baking of interstitial clays (≈70°C).

It was once believed that 10^{-6} was a lower bound on the strain magnitudes at which nonlinear effects could be seen [*e.g., Gordon and Davis*, 1968]. However, this is not a true onset limit. Preliminary analysis has already demonstrated a transition strain, around 10^{-7}, between a lower range dominated by strain-linear scaling, and a range dominated by quadratic scaling at higher strains [*TenCate et al.*, 1998]. The analysis presented here explains the transition, showing that *both* linear and quadratic frequency shifts are present at all strains between 3×10^{-8} and 10^{-6}, with roughly constant coefficients. It also provides refined tests of whether the material property shifts are sensitive only to strain magnitude, to other strain components, or possibly to frequency or phase.

Analysis of resonance data

The experiment, described in [*TenCate and Shankland*, 1997] and [*Guyer et al.*, 1999], is to force the end of a thin cylindrical bar of Berea sandstone with a sequence of constant-magnitude, CW tones (i.e., an incremental sweep) passing through the fundamental longitudinal (Young's-mode) resonance. Such sweeps are taken at several different driving forces. The bar is 350

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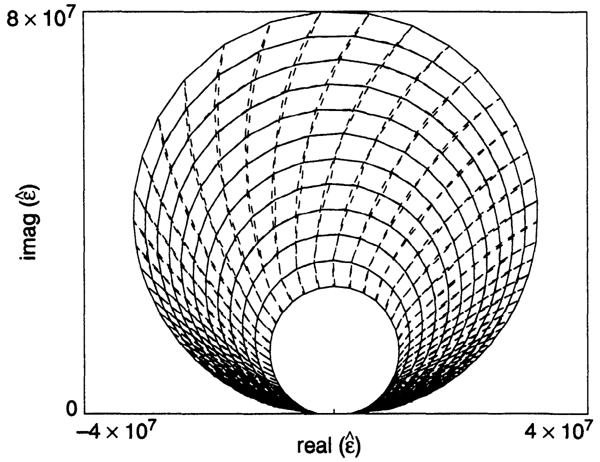

Figure 1. Complex strains (solid lines), at twelve drive voltages. Max($|\varepsilon|$) ranges from $2.5 \times 10^{-7} - 8 \times 10^{-7}$. Small deviations of solid lines from circles determine shifts in Q. Spiraling radials (dashed lines) show, at fixed frequencies, how phase through resonance advances with increasing voltage.

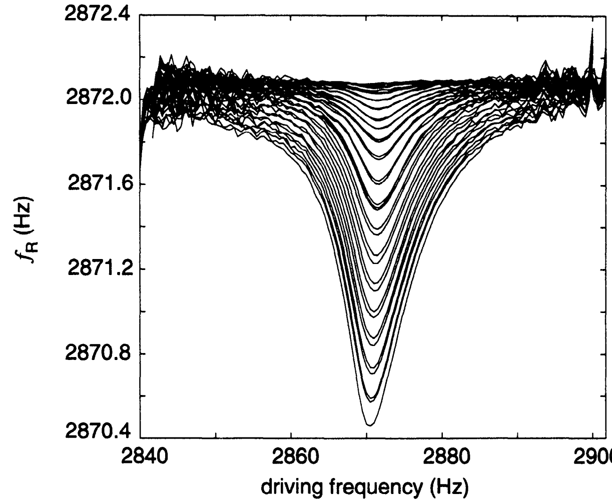

Figure 2. Instantaneous real resonance frequency ω_R, obtained by subtracting linear fits from $\hat{\varepsilon}^{-1}$. Corresponding strain maxima range from $6 \times 10^{-8} - 8 \times 10^{-7}$. Drive voltages were stepped up to maximum and then back down to check for repeatability. The shift at maximum strain is $\approx 0.1\%$.

mm long and approximately 24 mm in diameter. Force is delivered by a PZT transducer epoxied between one end of the bar and a brass backload, and checked to be proportional to the AC driving voltage V with a (highly linear) Plexiglas control sample of similar dimensions and acoustic properties. Acceleration is measured with a B&K 4374 accelerometer bonded to the opposite end of the bar, and converted to displacement using the known driving frequency. An EG&G lock-in amplifier is used to measure both in-phase and quadrature components during a sweep.

The complex strain response function

Previous analyses [*Johnson et al.*, 1996] of resonance nonlinearity have involved only the magnitude of the strain response function, which shows peak-shift and peak-bending reminiscent of a Duffing oscillator. In this analysis the components of the measured bar-end displacement will be expressed in terms of a *complex* strain response function ε, formed from the measured in-phase and quadrature components. Its two degrees of freedom can be inverted for effective resonance position and quality at *each* frequency in a sweep.

For an oscillatory driving voltage $V(t) \equiv V\mathrm{Re}\left(e^{i\omega t}\right)$, the displacement $\delta l_{\mathrm{bar}}(t)$ will be defined as

$$\delta l_{\mathrm{bar}}/l_{\mathrm{bar}} \equiv \mathrm{Re}\left(\varepsilon e^{i\omega t}\right) \tag{1}$$

where l_{bar} is the static bar length. The complex strain then has (approximately) the form of a simple pole [*Skudrzyk*, 1981]:

$$\varepsilon = R_{\mathrm{other}} \frac{V}{V_{\mathrm{ref}}} \left[\frac{\omega_{\mathrm{ref}}}{\omega - (\omega_R + i\omega_I)} \right] \quad . \tag{2}$$

where the collection of terms $\omega_{\mathrm{ref}}/\omega_I V_{\mathrm{ref}}$ is a calibration factor. R_{other} is a relative response function of the form 1 + (small terms), representing the complicated structure associated with other resonances of the bar. In processing these data, the average of many curves at the lowest strains measured is used to deduce R_{other}, which is then divided from all higher-strain curves before analysis for nonlinearities begins. The resulting (complex) conditioned response $(\varepsilon/R_{\mathrm{other}}) \equiv \hat{\varepsilon}$, for a set of data at a single temperature, is plotted in Fig. 1.

Frequency-dependent resonance shifts

The deviations from circularity in Fig. 1 are nearly invisible; $\hat{\varepsilon}^{-1} = (V_{\mathrm{ref}}/V\omega_{\mathrm{ref}})\left(\omega - (\omega_R + i\omega_I)\right)$, the inverse, is (visually) indistinguishable from a straight line. However, subtraction of a linear term, fit from the low- and high- frequency asymptotes, extracts the variable $\omega_R + i\omega_I$ as a residual. ω_R is shown in Fig. 2; ω_I (shown in Fig. 3), is proportional to $1/Q$.

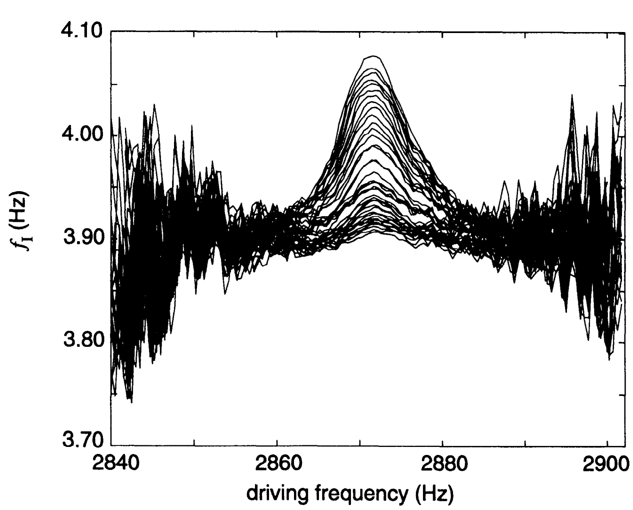

Figure 3. Instantaneous imaginary resonance frequency ω_I, for same curves as in Fig. 2. $\omega_R/2\omega_I = Q$ of the resonator.

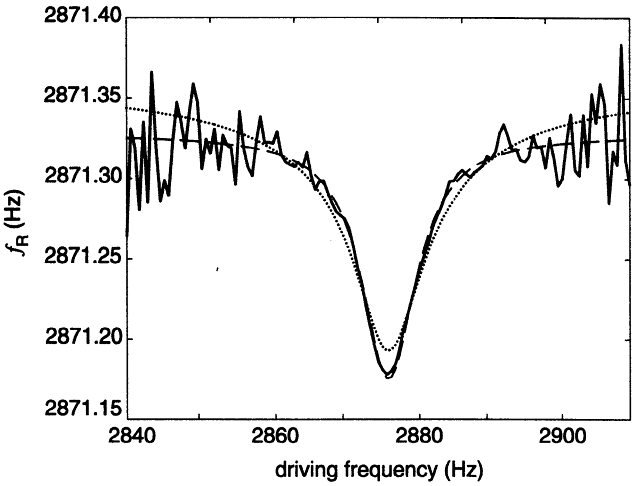

Figure 4. Fits of imaginary (dashed) and absolute (dotted) components of strain to the frequency shift for a sweep with Max($|\varepsilon|$) $\sim 2 \times 10^{-7}$. Baseline and scale factor of template function are fit to data in a least-squares sense.

It has been proposed that frequency shifts are purely a function of the strain magnitude. This may be tested by correlating the curves above with components of the (complex) strain response function. Fig. 4 shows fits of the real frequency shift at $|\varepsilon| \sim 2 \times 10^{-7}$ to $|\varepsilon|$ and $\mathrm{Im}\,(\varepsilon)$, corresponding to the absolute and out-of-phase components of strain, respectively. The preference for out-of-phase component at this strain is clear, and differs from the minimal hysteretic result [*McCall and Guyer*, 1994].

At higher strains ($\sim 8 \times 10^{-7}$), a similar comparison shows a transition toward dependence only on the magnitude of strain. The interplay of these two dependencies leads to frequency shifts which are a somewhat complicated function of strain. There appears to be no asymmetry at any strain level, which would indicate an intrinsic dependence on frequency or phase of the resonant system.

Comparisons with earlier work

Constant strain sectioning. Based on an assumption that frequency shift depends only on magnitude of strain, an alternative method of data reduction was developed in [*Guyer et al.*, 1999]. Resonant response was extracted along contours of (ω, V) with constant strain magnitude. Applying this method to the data of Fig. 2, and subtracting the mean of the lowest-strain curve as a linear baseline, the residual frequency shifts on contours between $|\varepsilon| \approx 6 \times 10^{-8}$ and $|\varepsilon| \approx 4 \times 10^{-7}$ are shown in Fig. 5. The significant non-flat structure along each contour shows that $|\varepsilon|$ is *not* the unique variable determining strain, as expected from Fig. 4.

Shift of material properties. It is clear from Fig. 5, however, that the fine differences between strain components are small corrections to the fits to either $|\varepsilon|$ or $\mathrm{Im}\,(\varepsilon)$, so the mean values of contours in Fig. 5 determine the leading scaling of frequency shift with dynamic strain magnitude. This is expressed as a fractional shift in Young's modulus, $\delta E/E$, in Fig. 6. Similar analysis of the loss tangent produces curves of $1/Q$ increasing with larger strain, and proportional to the curve of Fig. 6 by a factor of roughly $-1/9$.

All temperature data from 35°C to 65°C are plotted in Fig. 6. The statistical scatter, between sets of overlapping strain range at a single temperature, is larger than any systematic differences that can be discerned between temperatures.

A second-order polynomial fit of the modulus shift to strain magnitude, averaged over all sets, produces the main result of this note:

$$\frac{\delta E}{E} \approx -1.8 \times 10^{-4} \left[\frac{|\varepsilon|}{2.8 \times 10^{-7}} + \frac{1}{2} \left(\frac{|\varepsilon|}{2.8 \times 10^{-7}} \right)^2 \right] \quad (3)$$

An equivalent expression for frequency shift,

$$\frac{\delta f}{f_0} = -328\,|\varepsilon| - 5.8 \times 10^8 |\varepsilon|^2, \quad (4)$$

may be compared with inferred sound speed shift $\delta c/c_0$ from pulse-mode measurements of harmonic generation [*TenCate et al.*, 1996]. The modulus (hence sound speed) shifts in that work were associated with the magnitudes of second and third harmonic generation. The second-harmonic coefficient, $\beta \approx 400$, leads to no resonant frequency shift in dynamic averages [*McCall and Guyer*, 1994]. It does, however, generate harmonics with magnitude linear in the strain and should lead to apparent shift in the quality of the fundamental with the same scaling as the linear coefficient in Eq. (4). It is interesting, therefore, that the two coefficients also have very close values.

The third-harmonic coefficient, $(3/2)\,\delta \approx 3 \times 10^8$, leads to both third harmonic generation and shift of the fundamental resonance, proportional to the square of dynamic strain magnitude. It is not possible to compare literally the coefficient from harmonic generation (a short-length, first-order perturbation experiment), with that for resonance shift, which results from effectively high-order iteration of the perturbative results, as carried out in [*VanDenAbeele*, 1996]. However, in this range of parameters, δ is dominant and the order of magnitude of the parameters in both experiments

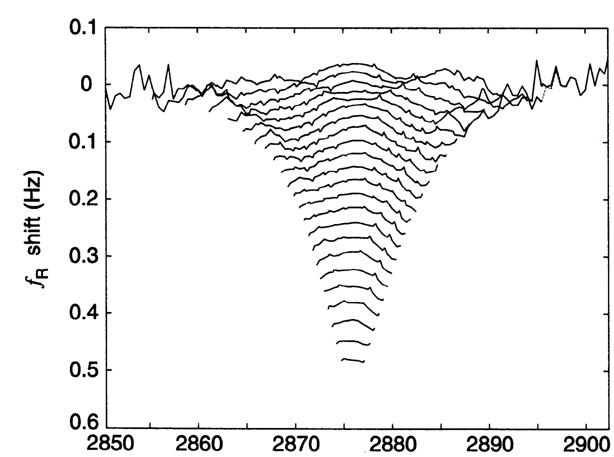

Figure 5. Real part of the residual resonance frequency, cut along contours of from $|\varepsilon| \approx 6 \times 10^{-8}$ to $|\varepsilon| \approx 4 \times 10^{-7}$.

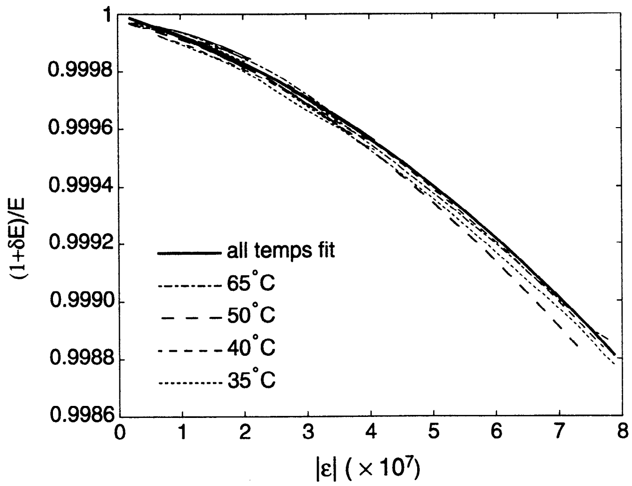

Figure 6. Relative shift in Young's modulus along contours of constant strain magnitude, measured at temperatures from 35°C to 65°C.

should be the same. Again, the experimental results show reasonable agreement for similar samples (within a factor of 5), for the two very different measurement methods.

The strain-linear coefficient for small dynamic strains in Eq. (4) is also in rough agreement with the linear coefficient ($\sim$ 1000), attributed to hysteresis in large-strain quasi-static experiments [*Guyer et al.*, 1999]. Finally, the scaling of Eq. (3) exchanges dominance from linear to quadratic contributions at $|\varepsilon| \sim 0.28 \times 10^{-6}$, in rough agreement (factor of 3 too large) with the analysis of [*TenCate et al.*, 1998]. It is this onset of quadratic scaling that coincides with the traditionally expected lower bound for observation of nonlinear effects.

Conclusions

Nonlinear shifts of Young's modulus and loss tangent have been observed in Berea sandstone at strains as low as 3×10^{-8}. Both strain-linear and strain-quadratic components to the frequency shift are found, with roughly constant coefficients, across this range. No statistically significant temperature dependence is observed in the coefficients, over a temperature range representing half that accessible between lower and upper damage thresholds. A thermal activation mechanism for hysteresis, expected to scale as $\exp(-E_{\text{char}}/kT)$, where E_{char} is a characteristic activation energy, k is Boltzmann's constant and T is temperature, should not maintain a constant rate over this temperature range. This result seems to indicate that all instantaneous nonlinear mechanisms are mechanically, and not thermally, activated.

While the quadratic component of observed scaling is consistent with classical nonlinearity, the linear component requires some form of hysteretic nonlinear elements. More complex models than the minimal case of uniform hysteresis are possible, so it is interesting that the relations of modulus and loss-tangent shifts are different from the minimal predictions. The imaginary frequency shift, like the real shift, has a component quadratic in strain, and the linear component of the imaginary resonance shift is proportional to the real component by a factor of roughly $-1/9$, where the model of [*McCall and Guyer*, 1994] predicts a proportionality of $-1/2$. Whether more information about underlying mechanism can be gleaned from these differences remains a topic for future work.

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