

Separable two-dimensional discrete Hartley transform

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Bracewell has proposed the discrete Hartley transform (DHT) as a substitute for the discrete Fourier transform (DFT), particularly as a means of convolution [J. Opt. Soc. Am. **73**, 1832 (1983)]. Here we show that the most natural extension of the DHT to two dimensions fails to be separable in the two dimensions and is therefore inefficient. We consider an alternative separable form and derive a corresponding convolution theorem. We also argue that the DHT is unlikely to provide faster convolution than the DFT.

INTRODUCTION

In a series of recent papers Bracewell has drawn attention to the Hartley transform as a substitute for the much more widely known Fourier transform.¹⁻³ He has introduced a discrete Hartley transform (DHT) as an analog to the discrete Fourier transform (DFT) and has developed a fast algorithm (FHT) to compete with the fast Fourier transform. Sorensen *et al.*⁴ have developed a complete set of algorithms for computation of the DHT, analogous to those available for computation of the DFT. Buneman⁵ has shown how DFT algorithms may be converted to DHT algorithms through indexing changes.

The purpose of this paper is to point out certain problems encountered in the extension of the DHT from one dimension to two dimensions.⁶ There are two ways in which this extension can be made. Each has its virtues, but only one is practical. This choice requires a new convolution theorem, which we provide. However, even with this theorem, the two-dimensional (2D) DHT appears not to provide a computational advantage over the DFT.

THE DISCRETE FOURIER TRANSFORM

Consider a discrete sequence $p(x)$, $x = 0, \dots, N-1$. The forward DFT of $p(x)$ is given by

$$P_F(u) = \sum_{x=0}^{N-1} p(x) \exp(-i2\pi ux/N), \quad (1)$$

and the inverse DFT by

$$p(x) = \frac{1}{N} \sum_{u=0}^{N-1} P_F(u) \exp(i2\pi ux/N). \quad (2)$$

In our notation we depart slightly from Bracewell, who places the scalar $1/N$ in the forward transform in both the DFT and the DHT. This differs from general practice⁷⁻⁹ and is less efficient if forward transforms are more common than reverse transforms. We therefore place the scalar in the inverse transform in our definitions of the DFT, the DHT, and the separable discrete Hartley transform (SDHT; defined below).

SYMMETRY IN THE DISCRETE HARTLEY TRANSFORM OF A REAL SEQUENCE

Although the DHT is defined for both real and complex sequences, its practical value arises from the way in which it takes advantage of the symmetry in the DFT of a real sequence. Therefore in what follows we consider only real input sequences. A real sequence, defined by N real numbers, has a DFT that contains N complex entries, or $2N$ real numbers. However, only N of the $2N$ real transform values are independent because the transform is *Hermitian*; that is, its real part is even and its imaginary part is odd. The Hartley transform is one way of taking advantage of this redundancy. Note that the sum (or difference) of an odd and an even sequence can easily be broken apart again into its odd and even parts. Thus, because real and imaginary parts of the DFT are even and odd, respectively, we can add them together with the assurance that they can be broken apart to recover the full complex DFT when desired. We therefore define the one-dimensional (1D) DHT of a real sequence $p(x)$ as the real part minus the imaginary part of the DFT:

$$P_H(u) = \text{Re } P_F(u) - \text{Im } P_F(u). \quad (3)$$

On expanding the right-hand side by means of Eq. (1),

$$P_H(u) = \sum_{x=0}^{N-1} p(x) [\cos 2\pi ux/N + \sin 2\pi ux/N], \quad (4)$$

$$P_H(u) = \sum_{x=0}^{N-1} p(x) \text{cas}(2\pi ux/N). \quad (5)$$

The sequence cas , defined by

$$\text{cas}(2\pi ux/N) = \cos(2\pi ux/N) + \sin(2\pi ux/N), \quad (6)$$

therefore serves as the basis sequence for the DHT. Note that the DHT may be expressed as the sum of its even and odd parts

$$P_H(u) = P_{HE}(u) + P_{HO}(u), \quad (7)$$

where

$$P_{HE}(u) = \frac{1}{2}[P_H(u) + P_H(-u)], \quad P_{HO}(u) = \frac{1}{2}[P_H(u) - P_H(-u)]. \tag{8}$$

Throughout the text we interpret all sequence arguments modulo the length of the sequence, so, for example, if $P(u)$ is a sequence of length N then $P(-1) = P(N - 1)$.

Note that the DFT is easily reconstituted from the even and odd parts of the DHT:

$$P_F(u) = P_{HE}(u) - iP_{HO}(u). \tag{9}$$

The inverse transform is

$$p(x) = \frac{1}{N} \sum_{u=0}^{N-1} P_H(u) \text{cas}(2\pi ux/N). \tag{10}$$

The principal advantage of the DHT over the DFT, as pointed out by Bracewell, is that it transforms a real input into a real output. Furthermore, the convolution theorem can be expressed in terms of the DHT. While this expression involves two sequence products, rather than the single product of the DFT convolution theorem, the former are products of real quantities and will therefore require on the order of $1/4$ the number of real operations. Furthermore, if either of the inputs to the convolution is even, as is quite common in filtering applications, the result reduces to a single product of real sequences, requiring on the order of $1/4$ as many operations as straightforward convolution through the DFT.

THE DISCRETE HARTLEY TRANSFORM IN TWO DIMENSIONS

One of the most frequent applications of the DFT is in image processing, where it is frequently used to implement convolution. For the DHT to be of use in image processing it must be extensible to two (and more) dimensions, and these extensions must carry with them a simple form of the convolution theorem. In one of his papers, Bracewell has offered a definition of the 2D DHT.² But this definition does not agree with the general 1D definition that he proposes, as embodied in Eq. (9), or with the convolution theorem that he has offered (Ref. 2, p. 1834).

There are two natural candidates for the extension of the DHT to two dimensions. The basis functions of these two possibilities are

$$\text{cas}[2\pi(ux/N + vy/M)], \quad \text{cas}(2\pi ux/N)\text{cas}(2\pi vy/M). \tag{11}$$

In what follows, we will consider the advantages of each alternative.

First, which definition is consistent with the definition of the Hartley transform as the difference of even and odd parts of the DFT? Recall the observation that motivated the construction of the 1D DHT, namely, that the DFT of a real input is Hermitian. Let $p(x, y)$ be a real 2D sequence (image) with $x = 0, \dots, N - 1$ and $y = 0, \dots, M - 1$. In two dimensions, oddness and evenness are defined with respect to reflections about both axes:

$$\text{Even } \{p(x, y)\} = p_E(x, y) = \frac{1}{2}[p(x, y) + p(-x, -y)], \tag{12}$$

$$\text{Odd } \{p(x, y)\} = p_O(x, y) = \frac{1}{2}[p(x, y) - p(-x, -y)]. \tag{13}$$

Under these definitions, the real part of the DFT of a real

input is even, and the imaginary part is odd. We can again define the DHT to be the even part minus the odd part of the DFT:

$$P_H(u, v) = \text{Re } P_F(u, v) - \text{Im } P_F(u, v) \tag{14}$$

$$= P_{HE}(u, v) + P_{HO}(u, v). \tag{15}$$

In this case the DFT, as for one dimension, is given by

$$P_F(u, v) = P_{HE}(u, v) - iP_{HO}(u, v). \tag{16}$$

To discover the basis sequences of the 2D DHT we expand Eq. (15). Let $a = 2\pi ux/N$ and $b = 2\pi vy/M$. Then

$$P_H(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y) [\cos(a + b) + \sin(a + b)] \tag{17}$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y) \text{cas}(a + b). \tag{18}$$

Thus we see that a definition of the 2D DHT that preserves the symmetries of the 1D transform leads to the first of the two possible basis functions introduced above.

This definition, while it forms a natural extension of the 1D transform and leads to the convolution theorem offered by Bracewell,² has a serious drawback. Unlike the 2D DFT, it is not separable in u and v . Fast algorithms for the 2D DFT rely on separability, and a straightforward extension of the FHT to two dimensions would also require separability.

THE SEPARABLE TWO-DIMENSIONAL HARTLEY TRANSFORM

As a solution to this dilemma, we consider an alternative definition of the 2D DHT, which we call the separable discrete Hartley transform (SDHT), defined by

$$P_S(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y) \text{cas}(2\pi ux/N) \text{cas}(2\pi vy/M), \tag{19}$$

and the inverse transform

$$p(x, y) = \frac{1}{NM} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} P_S(u, v) \text{cas}(2\pi ux/N) \text{cas}(2\pi vy/M). \tag{20}$$

This is essentially the 2D DHT as given by Bracewell (only the position of the scale factor is different).² Note that both transforms are separable in x and y and can be executed by first applying a 1D DHT to each row and then a 1D DHT to each column (or vice versa). Since this transform no longer obeys Eq. (9), and does not lead to the convolution theorem derived therefrom, we must discover the relationship between the SDHT and the DFT and derive a convolution theorem for the SDHT.

RELATION BETWEEN SEPARABLE TWO-DIMENSIONAL HARTLEY TRANSFORM AND DISCRETE FOURIER TRANSFORM

In the SDHT we have gained separability but have lost the simple relationships between the DHT and the DFT, name-

ly, DHT = Re DFT - Im DFT and DFT = Even DHT - *i* Odd DHT. However, as we show below, a similarly simple relation exists for the SDHT and the DFT. We begin by expanding the cas terms in the definition of the SDHT:

$$P_S(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)(\cos a + \sin a)(\cos b + \sin b) \quad (21)$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)(\cos a \cos b + \sin a \sin b + \cos a \sin b + \sin a \cos b) \quad (22)$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)[\cos(a - b) + \sin(a + b)] \quad (23)$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)\{\text{Re}[\cos(a - b) - i \sin(a - b)] - \text{Im}[\cos(a + b) - i \sin(a + b)]\} \quad (24)$$

$$= \text{Re } P_F(u, -v) - \text{Im } P_F(u, v). \quad (25)$$

Thus the SDHT is equal to the real part of the DFT reflected about the *u* axis, minus the imaginary part of the DFT. As might be imagined, it does not matter whether the reflection is about the *u* or the *v* axis.

Going in the other direction, we can expand the expression for the DFT, again using the abbreviations *a* and *b*,

$$P_F(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)\exp[-i(a + b)] \quad (26)$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)[(\cos a \cos b - \sin a \sin b) - i(\sin a \cos b + \sin b \cos a)] \quad (27)$$

$$= \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} p(x, y)\frac{1}{2}(\text{cas } a \text{ cas } - b + \text{cas } - a \text{ cas } b) - \frac{i}{2}(\text{cas } a \text{ cas } b - \text{cas } - a \text{ cas } - b) \quad (28)$$

$$= \frac{1}{2}[P_S(u, -v) + P_S(-u, v)] - \frac{i}{2}[P_S(u, v) - P_S(-u, -v)] \quad (29)$$

$$= P_{SE}(u, -v) - iP_{SO}(u, v). \quad (30)$$

SUMMARY OF RELATIONSHIPS

For future reference we summarize the relationships among the DFT, DHT, and SDHT:

$$P_S(u, v) = P_{HO}(u, v) + P_{HE}(u, -v), \quad (31)$$

$$P_S(u, v) = \text{Re } P_F(u, -v) - \text{Im } P_F(u, v), \quad (32)$$

$$P_H(u, v) = P_{SO}(u, v) + P_{SE}(u, -v), \quad (33)$$

$$P_H(u, v) = \text{Re } P_F(u, v) - \text{Im } P_F(u, v), \quad (34)$$

$$P_F(u, v) = P_{SE}(u, -v) - iP_{SO}(u, v), \quad (35)$$

$$P_F(u, v) = P_{HE}(u, v) - iP_{HO}(u, v). \quad (36)$$

SEPARABLE TWO-DIMENSIONAL HARTLEY TRANSFORM CONVOLUTION THEOREM

To be useful, the SDHT must yield a simple convolution theorem. We can derive this theorem from the preceding relations between DFT and SDHT. Let *r*(*x*, *y*) be the result of convolving the two real sequences *p*(*x*, *y*) and *q*(*x*, *y*):

$$r(x, y) = p(x, y) * q(x, y) = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} p(i, j)q(x - i, y - j). \quad (37)$$

From the convolution theorem for the DFT we have

$$R_F(u, v) = P_F(u, v)Q_F(u, v). \quad (38)$$

From Eq. (32) we can convert this to an expression for the SDHT:

$$R_S(u, v) = \text{Re } R_F(u, -v) - \text{Im } R_F(u, v) \quad (39)$$

$$= \text{Re}[P_F(u, -v)Q_F(u, -v)] - \text{Im}[P_F(u, v)Q_F(u, v)]. \quad (40)$$

The next step is to substitute Eq. (35) for each DFT:

$$= \text{Re}[\{P_{SE}(u, v) - iP_{SO}(u, -v)\}[\{Q_{SE}(u, v) - iQ_{SO}(u, -v)\}]] - \text{Im}[\{P_{SE}(u, -v) - iP_{SO}(u, v)\}[\{Q_{SE}(u, -v) - iQ_{SO}(u, v)\}]]. \quad (41)$$

Thus we arrive at a convolution theorem

$$R_S(u, v) = P_{SE}(u, v)Q_{SE}(u, v) - P_{SO}(u, -v)Q_{SO}(u, -v) + P_{SE}(u, -v)Q_{SO}(u, v) + P_{SO}(u, v)Q_{SE}(u, -v). \quad (42)$$

SYMMETRICAL INPUTS

Bracewell notes that if one of the inputs to a convolution is even, then the DHT of the convolution reduces to the product of the DHT's. Note that if *p*(*x*, *y*) is real and even, then *P_F*(*u*, *v*) is real and even, and consequently, from Eq. (25),

$$P_S(u, v) = P_F(u, -v). \quad (43)$$

Since *P_F*(*u*, *v*) is even, so is *P_F*(*u*, -*v*), and therefore so are *P_S*(*u*, *v*) and *P_S*(*u*, -*v*). Accordingly, the SDHT of the convolution reduces to

$$R_S(u, v) = P_{SE}(u, v)Q_{SE}(u, v) + P_{SE}(u, -v)Q_{SO}(u, v). \quad (44)$$

Thus only two products need to be computed.

COMPLEXITY

As noted above, the SDHT is one method of exploiting the symmetry of the DFT of a real sequence. Other well-known methods allow one to compute the DFT of a real sequence in

essentially half the number of operations required for a complex sequence.^{8,10} Let us call this sort of transform a real DFT (RDFT). Furthermore, the result of a RDFT on a real sequence of length N is $N/2$ distinct complex values. Following Bracewell, let us assume that for a real input sequence of length N the number of real operations required by SDHT and RDFT are approximately equal. (Sorensen *et al.*⁴ recently showed that the 1D DHT in general requires slightly more real operations than the DFT.) Convolution by the RDFT will require $N/2$ complex multiplications, or approximately $2N$ real multiplications. Convolution by the SDHT will require $4N$ real multiplications. When one of the inputs is symmetrical, then, as noted above, only $2N$ real multiplications are required. In neither case, however, does the SDHT confer an advantage over the DFT.

SUMMARY

We have shown that the most natural extension of the DHT to two dimensions fails to be separable in the two dimensions. We have examined an alternative, separable form for the 2D DHT, originally proposed by Bracewell. We have determined its relation to the DFT and have derived a corresponding convolution theorem. Finally, we note that the DHT is unlikely to provide faster convolution than the DFT in the general case.

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REFERENCES

1. R. V. L. Hartley, "A more symmetrical Fourier analysis applied to transmission problems," *Proc. Inst. Radio Eng.*, **30**, 144-150 (1942).
2. R. N. Bracewell, "Discrete Hartley transform," *J. Opt. Soc. Am.*, **73**, 1832-1835 (1983).
3. R. N. Bracewell, "The fast Hartley transform," *Proc. IEEE*, **72**, 1010-1018 (1984).
4. H. V. Sorensen, D. L. Jones, C. S. Burrus, and M. T. Heideman, "On computing the discrete Hartley transform," *IEEE Trans. Acoustics Speech Signal Process.*, **AASP-33**, 1231-1238 (1985).
5. O. Buneman, "Conversion of FFT's to fast Hartley transforms," *SIAM J. Sci. Statist. Comput.*, **7**, (1986).
6. A. B. Watson and Allen Poirson, "A separable two-dimensional discrete Hartley transform," NASA Tech. Memo 88203 (NASA, Washington, D.C., 1985).
7. H. J. Nussbaumer, *Fast Fourier Transformation and Convolution Algorithms* (Springer-Verlag, New York, 1982).
8. R. C. Singleton, "Mixed radix fast Fourier transforms," *Programs for Digital Signal Processing*, (Institute of Electrical and Electronics Engineers, New York, 1979), pp. 1.4-1-1.4-18.
9. A. V. Oppenheim and R. W. Schaffer, *Digital Signal Processing* (Prentice-Hall, Englewood Cliffs, N.J., 1975).
10. G. D. Bergland, "A fast Fourier transform for real-valued series," *Commun. ACM*, **11**, 703-710 (1968).