

Green's function for a two-dimensional exponentially graded elastic medium

BY YOUN-SHA CHAN¹†, L. J. GRAY¹, T. KAPLAN¹
AND GLAUCIO H. PAULINO²

¹*Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA (chany@ornl.gov)*

²*Newmark Laboratory, Department of Civil and Environmental Engineering, University of Illinois, 205 North Mathews Avenue, Urbana, IL 61801, USA*

Received 26 March 2003; accepted 9 August 2003; published online 16 March 2004

The free-space Green function for a two-dimensional exponentially graded elastic medium is derived. The shear modulus μ is assumed to be an exponential function of the Cartesian coordinates (x, y) , i.e. $\mu \equiv \mu(x, y) = \mu_0 e^{2(\beta_1 x + \beta_2 y)}$, where μ_0 , β_1 , and β_2 are material constants, and the Poisson ratio is assumed constant. The Green function is shown to consist of a singular part, involving modified Bessel functions, and a non-singular term. The non-singular component is expressed in terms of one-dimensional Fourier-type integrals that can be computed by the fast Fourier transform.

Keywords: functionally graded materials; Green's function; boundary-element methods

1. Introduction

The goal in this paper is to obtain the Green function for a class of functionally graded materials (FGMs) in two dimensions. Specifically, it is assumed that the Poisson ratio ν of the medium is constant and that the Lamé moduli λ and μ of the material are exponentially graded,

$$\mu(\mathbf{x}) = \mu(x, y) = \mu_0 e^{2(\beta_1 x + \beta_2 y)} = \mu_0 e^{2\boldsymbol{\beta} \cdot \mathbf{x}}, \quad \lambda(\mathbf{x}) = \lambda_0 e^{2\boldsymbol{\beta} \cdot \mathbf{x}}, \quad (1.1)$$

where μ_0 , λ_0 , and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ are material constants. Martin *et al.* (2002) employed a Fourier transform method to solve this problem in three dimensions and found that the Green function, \mathbf{G}_3 , is the sum (Martin *et al.* 2002, eqn (2.12))

$$\mathbf{G}_3(\mathbf{x}; \mathbf{x}') = e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')} [\mathbf{G}_3^0(\mathbf{x}; \mathbf{x}') + \mathbf{G}_3^g(\mathbf{x}; \mathbf{x}')], \quad (1.2)$$

where \mathbf{G}_3^0 is the well-known three-dimensional (3D) Kelvin solution (Mukherjee 1982; Mura 1987) for a homogeneous solid, and the *grading term* \mathbf{G}_3^g is a bounded well-behaved function of the distance between the field point \mathbf{x} and the source point \mathbf{x}' ,

† Present address: Department of Mathematics, University of Texas–Pan American, 1201 W. University Drive, Edinburg, TX 78541, USA.

$|\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$. It might therefore be reasonably expected that the Green function in two dimensions would be of the form

$$\mathbf{G}_2(\mathbf{x}; \mathbf{x}') = e^{-\beta \cdot (\mathbf{x} + \mathbf{x}')} [\mathbf{G}_2^0(\mathbf{x}; \mathbf{x}') + \mathbf{G}_2^g(\mathbf{x}; \mathbf{x}')], \quad (1.3)$$

where \mathbf{G}_2^0 is the two-dimensional (2D) Kelvin's solution, and \mathbf{G}_2^g is a corresponding grading term. However, if \mathbf{G}_2 is written in this form, \mathbf{G}_2^g is *not* bounded (see § 5*d*). It turns out that in order to correctly split off the singularity, \mathbf{G}_2 should be decomposed as the sum

$$\mathbf{G}_2(\mathbf{x}; \mathbf{x}') = e^{-\beta \cdot (\mathbf{x} + \mathbf{x}')} [\mathbf{G}_2^s(|\beta||\mathbf{r}|) + \mathbf{G}_2^{\text{ns}}(\mathbf{x}; \mathbf{x}')], \quad (1.4)$$

where \mathbf{G}_2^s and \mathbf{G}_2^{ns} stand for the *singular* and *non-singular* parts, respectively,

$$\beta = |\beta| = \sqrt{\beta_1^2 + \beta_2^2},$$

and $|\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$. As both singular and non-singular parts contain the grading parameter β , naming either one of them as a *grading term* is no longer appropriate. The singular part, \mathbf{G}_2^s , contains the modified Bessel functions $K_0(|\beta||\mathbf{r}|)$ and $K_1(|\beta||\mathbf{r}|)$, and the appearance of Bessel functions is consistent with the Green function for the 2D heat equation found by Gray *et al.* (2003) for graded materials.

As the 2D situation is of interest herein, the subscript in equation (1.4) is dropped and it will be shown that

$$\mathbf{G}(\mathbf{x}; \mathbf{x}') = e^{-\beta \cdot (\mathbf{x} + \mathbf{x}')} [K_0(|\beta||\mathbf{r}|)\mathbf{C}_0 + K_1(|\beta||\mathbf{r}|)\mathbf{C}_1 + \mathbf{G}^{\text{ns}}(\mathbf{x}; \mathbf{x}')], \quad (1.5)$$

where ($\kappa = 3 - 4\nu$ for plane strain),

$$\mathbf{C}_0 = \frac{\kappa}{2\pi\mu_0(\kappa + 1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C}_1 = \frac{D(\beta, \mathbf{x}, \mathbf{x}')}{4\pi\mu_0(\kappa + 1)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (1.6)$$

and

$$D(\beta, \mathbf{x}, \mathbf{x}') = |\beta| \frac{(x_1 - x'_1)^2 - (x_2 - x'_2)^2}{|\mathbf{r}|} - \frac{\beta_1^2 - \beta_2^2}{|\beta|} |\mathbf{r}|.$$

It is worth noting that while $K_0(|\beta||\mathbf{r}|)$ shares with the 2D Kelvin solution the necessary logarithmic singularity as $|\mathbf{r}| \rightarrow 0$, it also dies off exponentially as $|\mathbf{r}| \rightarrow \infty$. A scalar analogue is the *graded* Laplace equation (Gray *et al.* 2003)

$$\left(\nabla^2 + \beta_1 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial y} \right) \phi = 0, \quad (1.7)$$

and the corresponding Green function

$$\Phi(\mathbf{x}; \mathbf{x}') = \frac{e^{\beta \cdot (\mathbf{x} - \mathbf{x}')}}{2\pi} K_0(|\beta||\mathbf{r}|) \quad (1.8)$$

also decays exponentially at infinity. The exponential decay of the modified Bessel function K_0 at infinity (see Appendix B) will make the Fourier analysis straightforward.

The paper is organized as follows. The motivation for investigating this problem and a discussion of related work is given in § 2. The governing partial differential equations for the exponentially graded FGM and the definition of the Green function are presented in § 3. A Fourier transform method is used in § 4 to translate the partial

differential equations into a system of algebraic equations. The solution of the Green function for a normal (diagonal) term is obtained in § 5; the shear (off-diagonal) term is considered in § 6. In addition, an analysis of the singularity of the derived Green function is provided in § 5; in both § 5 and § 6, numerical evaluation of the non-singular terms using the fast Fourier transform algorithm is considered. A summary of the formulae for all components of the Green function is presented in § 7 and the last section contains some concluding remarks. Two appendixes, providing useful double Fourier transform formulae and asymptotics of the modified Bessel functions, supplement the paper.

2. Motivation and brief literature review

Many applications (e.g. coatings) involve dissimilar materials jointed at an interface. However, it is well known (Erdogan 1995) that stress concentration near the interface may result. FGMs are of interest in the materials community primarily because a continuous change in the material composition may avoid these local stress concentrations. Moreover, by controlling the gradation, the material performance can possibly be tailored and optimized to fulfil particular service requirements. FGMs have been investigated for many applications: thermal barrier coatings for aerospace applications, graded refractive index materials in optical devices, and biomaterials for dental and other implants. Good introductions to the general field of FGMs are found in the review articles by Hirai (1996) and Paulino *et al.* (2003), and the books by Suresh & Mortensen (1998) and Miyamoto *et al.* (1999). Erdogan (1995) provides a good review of fracture mechanics in FGMs, and Eischen (1987) discusses the crack-tip fields in FGMs.

As the study of FGMs is relatively new, it is not surprising that the literature on computational fracture analysis in these materials is not extensive, especially in regards to the boundary-element method (BEM). Using a singular integral-equation method, Konda & Erdogan (1994) have solved the mixed-mode plane elasticity crack problem. Kim & Paulino (2002*a*) have proposed graded elements for modelling bulk FGMs by the finite-element method (FEM) and have also employed this idea to evaluate mixed-mode stress intensity factors and T-stress in FGMs (Kim & Paulino 2002*b*, 2003). Although singular integral-equation methods can capture the crack-tip singularity for crack problems and provide accurate numerical results (Erdogan 1995), the extension to general boundary-value problems is very limited. For the FEM, the task of re-meshing for problems involving moving boundaries such as crack propagation is, in general, substantial. Thus, a specific motivation of the present work is to develop the 2D Green function for exponentially graded materials, which will allow boundary-integral fracture-analysis simulations using boundary-only meshing and discretization.

The boundary-integral approach can have advantages in treating FGMs, and especially fracture problems. The required mesh is for a lower-order dimensional surface, and the stress singularity at the crack tips can be easily captured in the $\sqrt{|r|}$ behaviour of the displacements on the crack surfaces (Cruse 1988). The Green function is essential for formulating boundary-integral equations, and the ability to do 2D simulations is important. Two-dimensional analyses are commonly used in engineering practice and they are often a good starting point for many practical applications. Previous works on Green's functions for non-homogeneous materials can be found

in Gray *et al.* (2003), Martin *et al.* (2002) and Sutradhar *et al.* (2002, 2003). The books by Banerjee (1994) and Bonnet (1995) give access to the extensive references on boundary-integral-equation methods.

3. Green's function equations

In classical linear elasticity, if the Lamé moduli λ and μ are functions of $\mathbf{x} = (x, y, z)$, the equilibrium equations (in the absence of body forces) are

$$\mu(\mathbf{x})\nabla^2\mathbf{u} + [\lambda(\mathbf{x}) + \mu(\mathbf{x})]\nabla\nabla\cdot\mathbf{u} + (\nabla\mathbf{u} + \nabla\mathbf{u}^T)\nabla\mu(\mathbf{x}) + (\nabla\cdot\mathbf{u})\nabla\lambda(\mathbf{x}) = \mathbf{0}, \quad (3.1)$$

where \mathbf{u} is the displacement vector, ∇ , $\nabla\cdot$, and ∇^2 are the gradient, divergence, and Laplacian operators, respectively, and $\nabla\mathbf{u}^T$ is the transpose of $\nabla\mathbf{u}$. If a 2D plane problem is considered and the Lamé moduli μ and λ are assumed to be exponential functions of (x, y) , as in equation (1.1), then (3.1) can be written as the following system of partial differential equations (Konda & Erdogan 1994):

$$\mathcal{L} \begin{bmatrix} u \\ v \end{bmatrix} = (\mathcal{L}^0 + \mathcal{L}^g) \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{0}, \quad (3.2)$$

where the linear differential operator \mathcal{L} has been split as a sum of the operator for homogeneous materials,

$$\mathcal{L}^0 = \frac{\mu_0}{\kappa - 1} \begin{bmatrix} (\kappa + 1)\partial_x^2 + (\kappa - 1)\partial_y^2 & 2\partial_x\partial_y \\ 2\partial_x\partial_y & (\kappa - 1)\partial_x^2 + (\kappa + 1)\partial_y^2 \end{bmatrix}, \quad (3.3)$$

and the operator for the grading part,

$$\mathcal{L}^g = \frac{2\mu_0}{\kappa - 1} \begin{bmatrix} \beta_1(\kappa + 1)\partial_x + \beta_2(\kappa - 1)\partial_y & \beta_2(\kappa - 1)\partial_x + \beta_1(3 - \kappa)\partial_y \\ \beta_2(3 - \kappa)\partial_x + \beta_1(\kappa - 1)\partial_y & \beta_1(\kappa - 1)\partial_x + \beta_2(\kappa + 1)\partial_y \end{bmatrix}, \quad (3.4)$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$. From equation (1.1), we have assumed that the ratio

$$\frac{\lambda}{\mu} = \frac{\lambda_0}{\mu_0} = \frac{3 - \kappa}{\kappa - 1}$$

is constant. Moreover, $\kappa = 3 - 4\nu$ if plane strain is considered, and $\kappa = (3 - \nu)/(1 + \nu)$ for a plane stress problem. A constant Poisson ratio, ν , is widely invoked in the FGM literature, and appears to be physically reasonable (Erdogan 1995). In particular, for a crack problem, ν may not have significant effect on the stress intensity factor (Delale & Erdogan 1983; Konda & Erdogan 1994). Finally, if β_1 and β_2 are set to 0, then the system of partial differential equations (3.2) becomes the standard Navier–Cauchy equations for homogeneous elastic materials.

The free-space Green function is obtained by solving the above partial differential equations in the plane $\mathbf{x} = (x, y)$ under a concentrated point force, at $\mathbf{x}' = (x', y')$. Let

$$\mathbf{G} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix},$$

where u_α and v_α , $\alpha = 1, 2$, denote the first and second displacement components, respectively, at the point \mathbf{x} due to a force in the α -direction at point \mathbf{x}' . The Green

function components u_α and v_α satisfy

$$\mathcal{L} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = -e^{-2\beta \cdot \mathbf{x}} \begin{bmatrix} \delta(\mathbf{x} - \mathbf{x}') \\ 0 \end{bmatrix}, \quad \mathcal{L} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = -e^{-2\beta \cdot \mathbf{x}} \begin{bmatrix} 0 \\ \delta(\mathbf{x} - \mathbf{x}') \end{bmatrix}, \quad (3.5)$$

where $\delta(\mathbf{x})$ denotes the 2D Dirac delta, and we have moved the common factor $e^{2\beta \cdot \mathbf{x}}$ from the left-hand side of the equals sign to the right. Clearly, in case of homogeneous materials, by setting $\beta_1 = \beta_2 = 0$, the equations (3.5) reduce to

$$\mathcal{L}^0 \begin{bmatrix} u_1^0 \\ v_1^0 \end{bmatrix} = - \begin{bmatrix} \delta(\mathbf{x} - \mathbf{x}') \\ 0 \end{bmatrix}, \quad \mathcal{L}^0 \begin{bmatrix} u_2^0 \\ v_2^0 \end{bmatrix} = - \begin{bmatrix} 0 \\ \delta(\mathbf{x} - \mathbf{x}') \end{bmatrix}, \quad (3.6)$$

and the corresponding Green's function components u_α^0 and v_α^0 ($\alpha = 1, 2$) are Kelvin's solution.

4. Fourier transform

The Green function equations in (3.5) will be solved by using the method of Fourier transforms (Sneddon 1972). The Fourier transform is defined by

$$\mathcal{F}(f)(\xi_1, \xi_2) = \hat{f}(\boldsymbol{\xi}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i(x\xi_1 + y\xi_2)} dx dy, \quad (4.1)$$

and the inverse Fourier theorem by

$$\mathcal{F}^{-1}(\hat{f})(x, y) = f(\mathbf{x}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi_1, \xi_2) e^{-i(x\xi_1 + y\xi_2)} d\xi_1 d\xi_2. \quad (4.2)$$

Taking the Fourier transform of equation (3.5), one obtains

$$\mu_0 \left(\begin{bmatrix} q_{11} & q_{12} & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 \\ 0 & 0 & q_{33} & q_{34} \\ 0 & 0 & q_{43} & q_{44} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix} \right) \begin{bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \hat{u}_2 \\ \hat{v}_2 \end{bmatrix} = e^{-2\beta \cdot \mathbf{x}'} \begin{bmatrix} e^{i\boldsymbol{\xi} \cdot \mathbf{x}'} \\ 0 \\ 0 \\ e^{i\boldsymbol{\xi} \cdot \mathbf{x}'} \end{bmatrix}, \quad (4.3)$$

where

$$q_{11} = q_{33} = \left(\frac{\kappa + 1}{\kappa - 1} \right) \xi_1^2 + \xi_2^2, \quad q_{22} = q_{44} = \xi_1^2 + \left(\frac{\kappa + 1}{\kappa - 1} \right) \xi_2^2,$$

$$q_{12} = q_{21} = q_{34} = q_{43} = \frac{2\xi_1 \xi_2}{\kappa - 1};$$

$$b_{11} = b_{33} = 2i \left[\beta_1 \left(\frac{\kappa + 1}{\kappa - 1} \right) \xi_1 + \beta_2 \xi_2 \right], \quad b_{12} = b_{34} = 2i \left[\beta_2 \xi_1 + \beta_1 \left(\frac{3 - \kappa}{\kappa - 1} \right) \xi_2 \right],$$

$$b_{21} = b_{43} = 2i \left[\beta_2 \left(\frac{3 - \kappa}{\kappa - 1} \right) \xi_1 + \beta_1 \xi_2 \right], \quad b_{22} = b_{44} = 2i \left[\beta_1 \xi_1 + \beta_2 \left(\frac{\kappa + 1}{\kappa - 1} \right) \xi_2 \right].$$

The equations for (u_1, v_1) are of course independent of those for (u_2, v_2) , and thus the 4×4 matrices in equation (4.3) have a diagonal block structure; instead of solving a 4×4 linear system, it suffices to consider the 2×2 subsystem.

Denoting

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} q_{33} & q_{34} \\ q_{43} & q_{44} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix}, \quad \left. \vphantom{Q} \right\} \quad (4.4)$$

$$S = Q + B,$$

we observe that matrix Q is symmetric, while matrices B and S are not symmetric. Thus, it is not clear at this point if the symmetry

$$v_1(\mathbf{x}; \mathbf{x}') = u_2(\mathbf{x}'; \mathbf{x}), \quad (4.5)$$

which holds for \mathbf{G}^0 , still holds for the graded Green function \mathbf{G} . Martin *et al.* (2002) have proposed a neat way to verify the symmetry by rewriting the right-hand side of equation (3.5) as

$$e^{-2\boldsymbol{\beta} \cdot \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}') = e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')} \delta(\mathbf{x} - \mathbf{x}'), \quad (4.6)$$

so that the non-singular term of \mathbf{G} is symmetric after decomposition as in equation (1.3). The proof of equation (4.5) for an anisotropic inhomogeneous elastic medium can be found in Ben-Menahem & Singh (1981); the re-formulation (4.6) is easily justified by the integral rules for the Dirac delta function. The symmetry is important in the numerical implementation of the symmetric Galerkin approximation in boundary-element methods.

Substituting (4.6) into equations in (3.5), and also recalling that it suffices to handle only the 2×2 subsystem, we obtain

$$(\mathcal{L}_{\text{II}} + \mathcal{L}_{\text{I}}) \begin{bmatrix} u_1^{\text{g}} \\ v_1^{\text{g}} \end{bmatrix} = -\mathcal{L}_{\text{I}} \begin{bmatrix} u_1^0 \\ v_1^0 \end{bmatrix}, \quad (\mathcal{L}_{\text{II}} + \mathcal{L}_{\text{I}}) \begin{bmatrix} u_2^{\text{g}} \\ v_2^{\text{g}} \end{bmatrix} = -\mathcal{L}_{\text{I}} \begin{bmatrix} u_2^0 \\ v_2^0 \end{bmatrix}, \quad (4.7)$$

where the second-order linear differential operator is

$$\mathcal{L}_{\text{II}} = \frac{1}{\kappa - 1} \begin{bmatrix} (\kappa + 1)\partial_x^2 + (\kappa - 1)\partial_y^2 & 2\partial_x\partial_y \\ 2\partial_x\partial_y & (\kappa - 1)\partial_x^2 + (\kappa + 1)\partial_y^2 \end{bmatrix}, \quad (4.8)$$

and the first-order operator is

$$\mathcal{L}_{\text{I}} = \frac{2}{\kappa - 1} \begin{bmatrix} -\frac{1}{2}[\beta_1^2(\kappa + 1) + \beta_2^2(\kappa - 1)] & (\kappa - 2)(\beta_2\partial_x - \beta_1\partial_y) - \beta_1\beta_2 \\ (\kappa - 2)(\beta_1\partial_y - \beta_2\partial_x) - \beta_1\beta_2 & -\frac{1}{2}[\beta_1^2(\kappa - 1) + \beta_2^2(\kappa + 1)] \end{bmatrix}. \quad (4.9)$$

From this point on, only the details of deriving u_1^{g} and v_1^{g} will be given, as finding u_2^{g} and v_2^{g} follows exactly the same route. Taking the Fourier transform of equation (4.7), we get

$$(\hat{\mathcal{L}}_{\text{II}} + \hat{\mathcal{L}}_{\text{I}}) \begin{bmatrix} \hat{u}_1^{\text{g}} \\ \hat{v}_1^{\text{g}} \end{bmatrix} = -\hat{\mathcal{L}}_{\text{I}} \begin{bmatrix} \hat{u}_1^0 \\ \hat{v}_1^0 \end{bmatrix}, \quad (4.10)$$

where

$$\hat{\mathcal{L}}_{\text{II}} = Q,$$

and

$$\hat{\mathcal{L}}_{\text{I}} = \begin{bmatrix} \beta_1^2 \frac{\kappa + 1}{\kappa - 1} + \beta_2^2 & \frac{2[\text{i}(\kappa - 2)(\beta_2\xi_1 - \beta_1\xi_2) + \beta_1\beta_2]}{\kappa - 1} \\ \frac{2[\text{i}(\kappa - 2)(\beta_1\xi_2 - \beta_2\xi_1) + \beta_1\beta_2]}{\kappa - 1} & \beta_1^2 + \beta_2^2 \frac{\kappa + 1}{\kappa - 1} \end{bmatrix}.$$

Equation (4.10) combined with

$$\begin{bmatrix} \hat{u}_1^0 \\ \hat{v}_1^0 \end{bmatrix} = Q^{-1} \begin{bmatrix} e^{i\xi \cdot \mathbf{x}'} \\ 0 \end{bmatrix}$$

leads to the following system of algebraic equations

$$\begin{bmatrix} \hat{u}_1^g \\ \hat{v}_1^g \end{bmatrix} = \{-(Q + \hat{\mathcal{L}}_1)^{-1} \hat{\mathcal{L}}_1 Q^{-1}\} \begin{bmatrix} e^{i\xi \cdot \mathbf{x}'} \\ 0 \end{bmatrix}, \quad (4.11)$$

which is the same equation as that derived for the 3D case (Martin *et al.* 2002, eqn (3.2)).

It is worth pointing out that the matrix in the curly brackets in equation (4.11) is actually the difference between matrices $(Q + \hat{\mathcal{L}}_1)^{-1}$ and Q^{-1} , that is,

$$-(Q + \hat{\mathcal{L}}_1)^{-1} \hat{\mathcal{L}}_1 Q^{-1} = (Q + \hat{\mathcal{L}}_1)^{-1} - Q^{-1}. \quad (4.12)$$

Thus, equation (4.11) can be viewed as the general Green function equation for an exponentially graded medium from which the Kelvin solution is taken away. Although splitting off the singularity of the Green function for the 3D case according to equation (4.12) is appropriate, such is *not* the case for 2D Green function.

5. Green's function solution for u_1

The solution of u_1 will be obtained as

$$u_1(\mathbf{x}; \mathbf{x}') = \frac{e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')}}{4\pi\mu_0(\kappa + 1)} [u_1^s(\mathbf{x}; \mathbf{x}') + u_1^{ns}(\mathbf{x}; \mathbf{x}')], \quad (5.1)$$

where the singular part

$$\begin{aligned} u_1^s(\mathbf{x}; \mathbf{x}') &= 2\kappa K_0(|\boldsymbol{\beta}||\mathbf{r}|) \\ &+ \left[|\boldsymbol{\beta}| \frac{(x_1 - x_1')^2 - (x_2 - x_2')^2}{|\mathbf{r}|} - \kappa|\boldsymbol{\beta}||\mathbf{r}| + \frac{(\kappa - 1)\beta_1^2 + (\kappa + 1)\beta_2^2}{\sqrt{\beta_1^2 + \beta_2^2}} |\mathbf{r}| \right] K_1(|\boldsymbol{\beta}||\mathbf{r}|) \end{aligned}$$

contains the modified Bessel functions $K_0(x)$ and $K_1(x)$, and the non-singular part $u_1^{ns}(\mathbf{x}; \mathbf{x}')$ can be expressed as the linear combinations of single Fourier-type integrals (see § 5*b*). We shall give a detailed derivation of (5.1) and also discuss the numerical evaluation of $u_1^{ns}(\mathbf{x}; \mathbf{x}')$ (see § 5*c*).

(a) Splitting-off the modified Bessel function

By inverting the Fourier transform of (4.3), and after somewhat lengthy algebra, we obtain

$$\begin{aligned} u_1(\mathbf{x}; \mathbf{x}') &= \frac{e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')}}{4\pi^2\mu_0(\kappa + 1)} \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{\Delta} e^{-i\xi \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \end{aligned} \quad (5.2)$$

where

$$\Delta = (|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2 + \frac{4(3 - \kappa)}{\kappa + 1}(\beta_2 \xi_1 - \beta_1 \xi_2)^2, \quad |\boldsymbol{\xi}|^2 = \xi_1^2 + \xi_2^2, \quad |\boldsymbol{\beta}|^2 = \beta_1^2 + \beta_2^2.$$

Gray *et al.* (2003) suggest that instead of subtracting the Kelvin solution, one should split off a modified Bessel function, and this can be accomplished by decomposing the fraction of the integrand in equation (5.2):

$$\frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{\Delta} = U_1^s + U_1^{ns}, \quad (5.3)$$

where the singular term is

$$U_1^s = \frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2}, \quad (5.4)$$

and the remaining term (which will be seen to be non-singular) is

$$U_1^{ns} = \frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{\Delta} - \frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2}. \quad (5.5)$$

By the formulae provided in Appendix A, we obtain

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1^s(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2 \\ &= 2\kappa K_0(|\boldsymbol{\beta}||\mathbf{r}|) - \kappa|\boldsymbol{\beta}||\mathbf{r}|K_1(|\boldsymbol{\beta}||\mathbf{r}|) + |\boldsymbol{\beta}| \frac{(x_1 - x_1')^2 - (x_2 - x_2')^2}{|\mathbf{r}|} K_1(|\boldsymbol{\beta}||\mathbf{r}|) \\ & \quad + \frac{(\kappa - 1)\beta_1^2 + (\kappa + 1)\beta_2^2}{\sqrt{\beta_1^2 + \beta_2^2}} |\mathbf{r}| K_1(|\boldsymbol{\beta}||\mathbf{r}|), \end{aligned} \quad (5.6)$$

the term $u_1^s(\mathbf{x}; \mathbf{x}')$ in equation (5.1). Note that $K_0(|\boldsymbol{\beta}||\mathbf{r}|)$ has the desired logarithmic singularity as $|\mathbf{r}| \rightarrow 0$.

(b) *Contour integral for the non-singular part*

The next step is to show that

$$u_1^{ns}(\mathbf{x}; \mathbf{x}') = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1^{ns}(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2 \quad (5.7)$$

is well behaved, and to obtain a better form for numerical computation. By using contour integration to integrate out one of the variables, ξ_1 , the double Fourier transform can be reduced to a single Fourier integral. For simplicity, it can be assumed that $\beta_2 = 0$; otherwise, by a simple change of variables (a rotation of $\arctan(\beta_2/\beta_1)$)

$$\xi_1 = \frac{\beta_1}{|\boldsymbol{\beta}|} s_1 - \frac{\beta_2}{|\boldsymbol{\beta}|} s_2, \quad \xi_2 = \frac{\beta_2}{|\boldsymbol{\beta}|} s_1 + \frac{\beta_1}{|\boldsymbol{\beta}|} s_2,$$

and thus the term $(\beta_2 \xi_1 - \beta_1 \xi_2)^2$ becomes $|\boldsymbol{\beta}|^2 s_2^2$, equivalent to letting $\beta_2 = 0$.

At $\beta_2 = 0$, the double Fourier integral in equation (5.7) can be written as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-\eta \beta_1^2 \xi_2^2 [(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2)]}{[(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi_2^2](\xi_1^2 + \xi_2^2 + \beta_1^2)^2} e^{-i\boldsymbol{\xi} \cdot \mathbf{r}} d\xi_1 d\xi_2, \quad (5.8)$$

where $\eta = 4(3 - \kappa)/(\kappa + 1)$, and we have abbreviated $(\mathbf{x} - \mathbf{x}')$ as $\mathbf{r} = (r_1, r_2)$. We choose to integrate with respect to ξ_1 first, and, as ξ_2 is fixed, the fraction in equation (5.8) has three poles located in the upper half-plane: they are two simple poles

$$p_1 = \frac{1}{2}\sqrt{2}(\mathcal{P} + i\mathcal{Q}), \quad p_2 = \frac{1}{2}\sqrt{2}(i\mathcal{Q} - \mathcal{P}),$$

and one pole of order 2,

$$p_3 = i\sqrt{\xi_2^2 + \beta_1^2},$$

with

$$\mathcal{P} = \sqrt{\sqrt{(\xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2} - \xi_2^2 - \beta_1^2}, \tag{5.9}$$

$$\mathcal{Q} = \sqrt{\sqrt{(\xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2} + \xi_2^2 + \beta_1^2}. \tag{5.10}$$

As a function of ξ_2 , \mathcal{P} has the following asymptotics as $|\xi_2| \rightarrow 0$:

$$\begin{aligned} \mathcal{P} &= \sqrt{\sqrt{(\xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2} - \xi_2^2 - \beta_1^2} \\ &\sim \frac{\sqrt{2\eta}}{2}|\xi_2| - \frac{\sqrt{2\eta}(4 + \eta)}{\beta_1^2}|\xi_2|^3 + O(|\xi_2|^5). \end{aligned} \tag{5.11}$$

The numerator in equation (5.8) is the linear combination of terms ξ_2^2 , $\xi_1^2\xi_2^2$ and ξ_2^4 , and the individual integrals are

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_2^2 e^{-i\boldsymbol{\xi} \cdot \mathbf{r}}}{[(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2](\xi_1^2 + \xi_2^2 + \beta_1^2)^2} d\xi_1 d\xi_2 \\ &= -\frac{\pi}{2\eta\beta_1^2} \int_{-\infty}^{\infty} \left\{ \frac{\sqrt{2}e^{\sqrt{2}r_1\mathcal{Q}/2}[\mathcal{P} \cos(\frac{1}{2}\sqrt{2}r_1\mathcal{P}) - \mathcal{Q} \sin(\frac{1}{2}\sqrt{2}r_1\mathcal{P})]}{2\sqrt{\eta}|\beta_1||\xi_2|\sqrt{(\xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2}} \right. \\ &\quad \left. + \frac{e^{r_1\sqrt{\xi_2^2 + \beta_1^2}}[r_1(\xi_2^2 + \beta_1^2) - \sqrt{\xi_2^2 + \beta_1^2}]}{(\xi_2^2 + \beta_1^2)^2} \right\} e^{-ir_2\xi_2} d\xi_2, \end{aligned} \tag{5.12}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_2^4 e^{-i\boldsymbol{\xi} \cdot \mathbf{r}}}{[(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2](\xi_1^2 + \xi_2^2 + \beta_1^2)^2} d\xi_1 d\xi_2 \\ &= -\frac{\pi}{2\eta\beta_1^2} \int_{-\infty}^{\infty} \left\{ \frac{\sqrt{2}|\xi_2|e^{\sqrt{2}r_1\mathcal{Q}/2}[\mathcal{P} \cos(\frac{1}{2}\sqrt{2}r_1\mathcal{P}) - \mathcal{Q} \sin(\frac{1}{2}\sqrt{2}r_1\mathcal{P})]}{2\sqrt{\eta}|\beta_1|\sqrt{(\xi_2^2 + \beta_1^2)^2 + \eta\beta_1^2\xi_2^2}} \right. \\ &\quad \left. + \frac{\xi_2^2 e^{r_1\sqrt{\xi_2^2 + \beta_1^2}}[r_1(\xi_2^2 + \beta_1^2) - \sqrt{\xi_2^2 + \beta_1^2}]}{(\xi_2^2 + \beta_1^2)^2} \right\} e^{-ir_2\xi_2} d\xi_2, \end{aligned} \tag{5.13}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1^2 \xi_2^2 e^{-i\xi \cdot r}}{[(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi_2^2](\xi_1^2 + \xi_2^2 + \beta_1^2)^2} d\xi_1 d\xi_2 \\ &= -\frac{\pi}{\eta \beta_1^2} \int_{-\infty}^{\infty} \left\{ \frac{\sqrt{2} e^{\sqrt{2} r_1 \mathcal{Q}/2} [\mathcal{P} \cos(\frac{1}{2} \sqrt{2} r_1 \mathcal{P}) - \mathcal{Q} \sin(\frac{1}{2} \sqrt{2} r_1 \mathcal{P})]}{2\sqrt{\eta} |\beta_1| |\xi_2|} \right. \\ & \quad \left. - \frac{e^{r_1 \sqrt{\xi_2^2 + \beta_1^2}} [r_1 (\xi_2^2 + \beta_1^2) + \sqrt{\xi_2^2 + \beta_1^2}]}{2(\xi_2^2 + \beta_1^2)} \right\} e^{-ir_2 \xi_2} d\xi_2. \end{aligned} \tag{5.14}$$

Examining each single Fourier integral in equations (5.12)–(5.14) together with the asymptotics of \mathcal{P} provided in (5.11), one readily sees that the double Fourier integral in (5.7) is indeed non-singular.

(c) *Numerical evaluation of u_1^{ns}*

Implementing the FGM Green function in a boundary-integral analysis will obviously require computation of the integrals expressing the non-singular terms. The purpose of this subsection is to illustrate that there is no fundamental difficulty in achieving a reliable evaluation of u_1^{ns} in equation (5.1). However, the important issue of what is an efficient algorithm is left for future work.

The results of numerical integration of the non-singular term u_1^{ns} using fast Fourier transform algorithms (Brigham 1974; Walker 1991) are presented in this subsection. In order to compute the Fourier-type integral

$$H(r) = \int_0^{\infty} h(\xi) e^{-i\xi r} d\xi$$

numerically, we approximate it by its discrete version,

$$H\left(\frac{2\pi k}{L}\right) \approx \sum_{n=0}^{N-1} h\left(\frac{nL}{N}\right) e^{-i2\pi nk/N} \frac{L}{N}, \quad k = 1, 2, \dots, N,$$

where $[0, L]$ is some appropriate truncation interval and N is the number of consecutive sampled values. In practice, N is often chosen to be a power of 2, and this will reduce the operation count from $O(N^2)$ to $O(N \log_2 N)$ (Cooley & Tukey 1965).

As an example, the single integral in equation (5.12) is considered. By the symmetry of the integrand in the argument of ξ_2 , we can rewrite the single integral as

$$\begin{aligned} f(r_1, r_2) = \int_0^{\infty} & \left\{ \frac{\sqrt{2} e^{\sqrt{2} r_1 \mathcal{Q}/2} [\mathcal{P} \cos(\frac{1}{2} \sqrt{2} r_1 \mathcal{P}) - \mathcal{Q} \sin(\frac{1}{2} \sqrt{2} r_1 \mathcal{P})]}{2\sqrt{\eta} |\beta_1| \xi \sqrt{(\xi^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi^2}} \right. \\ & \left. + \frac{e^{r_1 \sqrt{\xi^2 + \beta_1^2}} [r_1 (\xi^2 + \beta_1^2) - \sqrt{\xi^2 + \beta_1^2}]}{(\xi^2 + \beta_1^2)^2} \right\} \cos(r_2 \xi) d\xi, \end{aligned} \tag{5.15}$$

where \mathcal{P} and \mathcal{Q} are defined in equations (5.9) and (5.10), respectively.

In figure 1, we plot a convergence test for $f(r_1 = 0.20, r_2)$ as a function of r_2 , where the parameters were chosen as $\beta_1 = 0.25, \nu = 0.30$. For the truncation interval $[0, L], L = 2^8$, three sampled values $N = 2^{16}, 2^{17}$ and 2^{18} were employed. Thus, the integration interval sizes, $\Delta\xi = L/N$, are set to be $\frac{1}{256}, \frac{1}{512}$, and $\frac{1}{1024}$. Figure 1 shows the convergence of the numerical integrals evaluated by fast Fourier transform as $\Delta\xi$ gets finer. In figure 2, a 3D surface plot is given for $f(r_1, r_2)$ as a function of (r_1, r_2) .

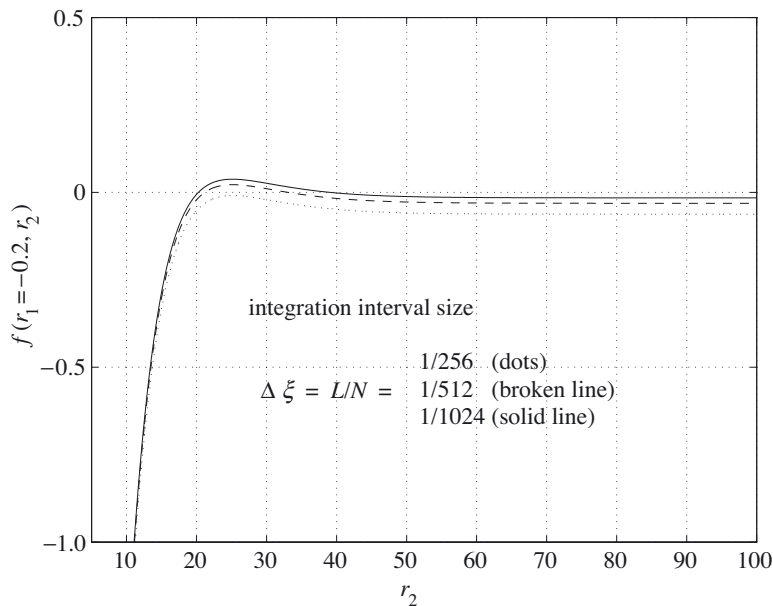


Figure 1. Convergence of the numerical integration. The parameters were chosen as $\beta_1 = 0.25$, $\nu = 0.30$.

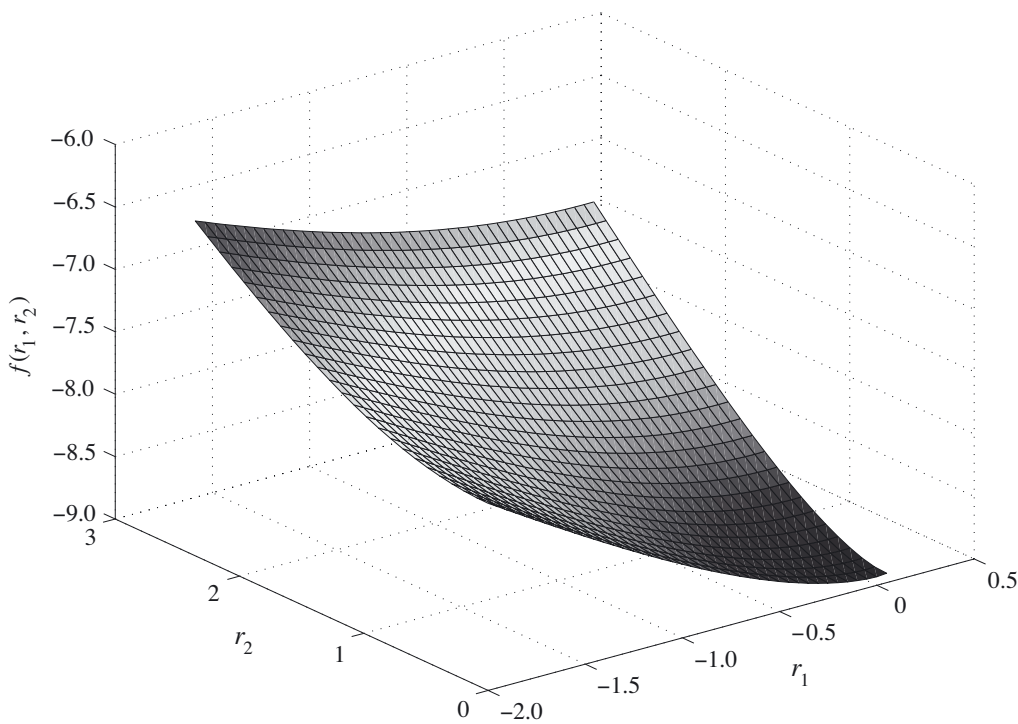


Figure 2. A three-dimensional plot of the function $f(r_1, r_2)$ defined in equation (5.15); $\beta_1 = 0.25$ and $\nu = 0.30$.

(d) *Splitting-off the Kelvin solution*

In this subsection we show that if the 2D Kelvin solution is subtracted from u_1 , according to equation (1.3), then the remaining term u_1^g is not bounded at infinity. The expression for u_1^g can be derived directly by inverting the Fourier transform of (4.11):

$$u_1^g(\mathbf{x}; \mathbf{x}') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1^g(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \quad (5.16)$$

where

$$U_1^g(\xi_1, \xi_2) = \frac{1}{\mu_0(\kappa + 1)} \times \left[\frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{\Delta} - \frac{(\kappa - 1)\xi_1^2 + (\kappa + 1)\xi_2^2}{|\boldsymbol{\xi}|^4} \right]. \quad (5.17)$$

To show that the double Fourier integral (5.17) is unbounded, express U_1^g as

$$U_1^g(\xi_1, \xi_2) = \frac{-|\boldsymbol{\beta}|^2}{\mu_0(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)|\boldsymbol{\xi}|^2} + \frac{2}{\mu_0(\kappa + 1)} \left[\frac{\xi_1^2}{|\boldsymbol{\xi}|^4} - \frac{\xi_1^2 + \beta_1^2}{\Delta} - \frac{2(3 - \kappa)(\beta_2\xi_1 - \beta_1\xi_2)^2}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)\Delta} \right], \quad (5.18)$$

and evaluate each integral in the expression. By using formulae (A 1) and (A 5) in Appendix A, the first fraction on the right-hand side of equation (5.18) has logarithmic unboundedness at infinity after the double Fourier integral,

$$\frac{1}{2\pi\mu_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2} - \frac{1}{|\boldsymbol{\xi}|^2} \right) e^{-i\boldsymbol{\xi} \cdot \mathbf{r}} d\xi_1 d\xi_2 = \frac{1}{\mu_0} [K_0(|\boldsymbol{\beta}||\mathbf{r}|) + \log|\mathbf{r}|]. \quad (5.19)$$

The first fraction inside the square brackets on the right-hand side of equation (5.18) becomes

$$\frac{1}{\pi\mu_0(\kappa + 1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1^2}{|\boldsymbol{\xi}|^4} e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2 = -\frac{1}{\mu_0(\kappa + 1)} \left[\log|\mathbf{r}| + \frac{(x_1 - x_1')^2}{|\mathbf{r}|^2} \right], \quad (5.20)$$

where we have used formula (A 3). The remaining fractions inside the brackets on the right-hand side of equation (5.18) can be shown to have finite double Fourier integrals by using residue calculus to evaluate the integral with respect to ξ_1 (see § 5 b).

Summarizing all the above, we conclude that u_1^g is not bounded at infinity. As indicated in the previous subsections, the key point is to take away a modified Bessel function $K_0(|\boldsymbol{\beta}||\mathbf{r}|)$, instead of the Kelvin solution.

6. The Green function solution for v_1

The shear term

$$v_1^0(\mathbf{x}; \mathbf{x}') = \frac{1}{2\pi\mu_0(\kappa + 1)} \left[\frac{(x_1 - x_1')(x_2 - x_2')}{|\mathbf{r}|^2} \right]$$

of the 2D Kelvin solution is not singular. It is therefore expected that the Green function solution v_1 for the exponentially graded materials can be written as

$$v_1(\mathbf{x}; \mathbf{x}') = e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')} [v_1^0(\mathbf{x}; \mathbf{x}') + v_1^g(\mathbf{x}; \mathbf{x}')], \quad (6.1)$$

where the grading part v_1^g is a linear combination of single Fourier-type integrals (see § 6*b*).

(a) *Double Fourier integral*

By inverting the Fourier transform of equation (4.3), and after some algebra, we obtain

$$v_1(\mathbf{x}; \mathbf{x}') = \frac{e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')}}{2\pi^2 \mu_0 (\kappa + 1)} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i(2 - \kappa)(\beta_2 \xi_1 - \beta_1 \xi_2) - (\xi_1 \xi_2 + \beta_1 \beta_2)}{\Delta} e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2. \quad (6.2)$$

If equation (4.11) is chosen to derive the Green function solution, then, after inverting the Fourier transform, we obtain

$$v_1^g(\mathbf{x}; \mathbf{x}') = \frac{1}{2\pi^2 \mu_0 (\kappa + 1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_1^g(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \quad (6.3)$$

where

$$V_1^g(\xi_1, \xi_2) = \frac{\xi_1 \xi_2}{|\boldsymbol{\xi}|^4} - \frac{\xi_1 \xi_2 + \beta_1 \beta_2}{\Delta} + i \frac{(2 - \kappa)(\beta_2 \xi_1 - \beta_1 \xi_2)}{\Delta}. \quad (6.4)$$

The difference between the integrands in equations (6.2) and (6.3) yields the Kelvin solution v_1^0 exactly, i.e.

$$\frac{1}{2\pi^2 \mu_0 (\kappa + 1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{\xi_1 \xi_2}{|\boldsymbol{\xi}|^4} e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2 = \frac{1}{2\pi \mu_0 (\kappa + 1)} \frac{(x_1 - x'_1)(x_2 - x'_2)}{|\mathbf{r}|^2}. \quad (6.5)$$

We have used equation (A 2) in Appendix A for deriving the above double integral (Mura 1987, p. 17). It will be shown in the next subsection that the double Fourier integral (6.2) is finite.

(b) *Contour integral for v_1*

Similarly to the previous contour integral analysis for u_1 , the double Fourier transform will be reduced to a single Fourier integral by integrating out the variable ξ_1 . In this case, the individual integrals that comprise equation (6.2) are

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\boldsymbol{\xi} \cdot \mathbf{x}}}{(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi_2^2} d\xi_1 d\xi_2 \\ &= \frac{\sqrt{2}\pi}{2\sqrt{\eta}|\beta_1|} \int_{-\infty}^{\infty} \frac{e^{\sqrt{2}x_1 \mathcal{Q}/2} [\mathcal{P} \cos(\frac{1}{2}\sqrt{2}x_1 \mathcal{P}) - \mathcal{Q} \sin(\frac{1}{2}\sqrt{2}x_1 \mathcal{P})]}{|\xi_2|(\xi_2^2 + \beta_1^2)} e^{-ix_2 \xi_2} d\xi_2, \end{aligned} \quad (6.6)$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1 e^{-i\boldsymbol{\xi} \cdot \mathbf{x}}}{(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi_2^2} d\xi_1 d\xi_2 \\ &= \frac{\pi i}{\sqrt{\eta} |\beta_1|} \int_{-\infty}^{\infty} \frac{e^{\sqrt{2}x_1 \mathcal{Q}/2} \sin(\frac{1}{2}\sqrt{2}x_1 \mathcal{P})}{|\xi_2|} e^{-ix_2 \xi_2} d\xi_2, \end{aligned} \quad (6.7)$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_2 e^{-i\boldsymbol{\xi} \cdot \mathbf{x}}}{(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi_2^2} d\xi_1 d\xi_2 \\ &= \frac{\sqrt{2}\pi}{2\sqrt{\eta} |\beta_1|} \int_{-\infty}^{\infty} \frac{e^{\sqrt{2}x_1 \mathcal{Q}/2} [\mathcal{P} \cos(\frac{1}{2}\sqrt{2}x_1 \mathcal{P}) - \mathcal{Q} \sin(\frac{1}{2}\sqrt{2}x_1 \mathcal{P})]}{\operatorname{sgn}(\xi_2)(\xi_2^2 + \beta_1^2)} e^{-ix_2 \xi_2} d\xi_2, \end{aligned} \quad (6.8)$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1 \xi_2 e^{-i\boldsymbol{\xi} \cdot \mathbf{x}}}{(\xi_1^2 + \xi_2^2 + \beta_1^2)^2 + \eta \beta_1^2 \xi_2^2} d\xi_1 d\xi_2 \\ &= \frac{\pi i}{\sqrt{\eta} |\beta_1|} \int_{-\infty}^{\infty} \frac{e^{\sqrt{2}x_1 \mathcal{Q}/2} \sin(\frac{1}{2}\sqrt{2}x_1 \mathcal{P})}{\operatorname{sgn}(\xi_2)} e^{-ix_2 \xi_2} d\xi_2. \end{aligned} \quad (6.9)$$

By inspecting each single Fourier integral, one can see that all the integrals are non-singular. In these formulae $\operatorname{sgn}(x)$ denotes the signum function,

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

7. Summary

We summarize the above discussion by listing each of the components of \mathbf{G} which are needed for a numerical implementation using the BEM.

$$\begin{aligned} & u_1(\mathbf{x}; \mathbf{x}') \\ &= \frac{e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')}}{4\pi\mu_0(\kappa + 1)} \\ & \quad \times \left[2\kappa K_0(|\boldsymbol{\beta}||\mathbf{r}|) + |\boldsymbol{\beta}| \frac{(x_1 - x'_1)^2 - (x_2 - x'_2)^2}{|\mathbf{r}|} K_1(|\boldsymbol{\beta}||\mathbf{r}|) \right. \\ & \quad \left. - \kappa |\boldsymbol{\beta}||\mathbf{r}| K_1(|\boldsymbol{\beta}||\mathbf{r}|) + \frac{(\kappa - 1)\beta_1^2 + (\kappa + 1)\beta_2^2}{\sqrt{\beta_1^2 + \beta_2^2}} |\mathbf{r}| K_1(|\boldsymbol{\beta}||\mathbf{r}|) + u_1^{\text{ns}}(\mathbf{x}; \mathbf{x}') \right], \end{aligned} \quad (7.1)$$

$$v_1(\mathbf{x}; \mathbf{x}') = e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')} \left[\frac{1}{2\pi\mu_0(\kappa + 1)} \frac{(x_1 - x'_1)(x_2 - x'_2)}{|\mathbf{r}|^2} + v_1^{\text{g}}(\mathbf{x}; \mathbf{x}') \right], \quad (7.2)$$

$$u_2(\mathbf{x}; \mathbf{x}') = e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')} \left[\frac{1}{2\pi\mu_0(\kappa + 1)} \frac{(x_1 - x'_1)(x_2 - x'_2)}{|\mathbf{r}|^2} + u_2^{\text{g}}(\mathbf{x}; \mathbf{x}') \right], \quad (7.3)$$

$$\begin{aligned}
v_2(\mathbf{x}; \mathbf{x}') &= \frac{e^{-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')}}{4\pi\mu_0(\kappa + 1)} \\
&\times \left[2\kappa K_0(|\boldsymbol{\beta}||\mathbf{r}|) + |\boldsymbol{\beta}| \frac{(x_2 - x'_2)^2 - (x_1 - x'_1)^2}{|\mathbf{r}|} K_1(|\boldsymbol{\beta}||\mathbf{r}|) \right. \\
&\quad \left. - \kappa|\boldsymbol{\beta}||\mathbf{r}|K_1(|\boldsymbol{\beta}||\mathbf{r}|) + \frac{(\kappa + 1)\beta_1^2 + (\kappa - 1)\beta_2^2}{\sqrt{\beta_1^2 + \beta_2^2}} |\mathbf{r}|K_1(|\boldsymbol{\beta}||\mathbf{r}|) + v_2^{\text{ns}}(\mathbf{x}; \mathbf{x}') \right], \tag{7.4}
\end{aligned}$$

where the non-singular terms u_1^{ns} , v_1^{g} , u_2^{g} and v_2^{ns} , in the form of a double Fourier integral, are listed in Appendix C. For numerical evaluation, those non-singular terms can be expressed as the linear combinations of single Fourier-type integrals (see §§ 5 *b* and 6 *b*). Moreover, it is easy to see that the classical 2D Kelvin solution is recovered as $\beta_1 \rightarrow 0$ and $\beta_2 \rightarrow 0$.

8. Concluding remarks

Using a Fourier transform technique, the Green function for a 2D exponentially graded elastic medium has been derived. The Green function can be decomposed into a ‘modified Bessel function $K_0(|\boldsymbol{\beta}||\mathbf{r}|)$ + non-singular terms’, which is different in form from that found in three dimensions. In three dimensions (Martin *et al.* 2002), the singularity in the Green function is confined in the Kelvin solution, $1/|\mathbf{r}|$, and that singularity appears only as $|\mathbf{r}| \rightarrow 0$. In the 2D case, the Kelvin solution, $\log |\mathbf{r}|$, possesses singularity at both $|\mathbf{r}| \rightarrow 0$ and $|\mathbf{r}| \rightarrow \infty$.

In the 3D case, the non-singular terms can be obtained as single integrals over finite intervals of modified Bessel functions and double integrals over finite regions of elementary functions (Martin *et al.* 2002). Here, the non-singular terms have been expressed as single Fourier-type integrals which can be evaluated numerically by fast Fourier transform algorithms. It is not clear however that this is the best approach, and further work on numerical methods is required. Using the results in this paper and those by Martin *et al.* (2002), a complete set of Green’s functions for both two and three dimensions is available for boundary-integral solution of problems for exponentially graded materials. With the availability of these Green’s functions, the advantages that are inherent in boundary-integral methods can now be used in treating crack propagation, and numerical implementation is currently under way.

This research was supported by the Applied Mathematical Sciences Research Program of the Office of Mathematical, Information and Computational Sciences, US Department of Energy, under contract DE-AC05-00OR22725 with UT-Battelle, LLC. This research was supported in part by an appointment to the Oak Ridge National Laboratory Postdoctoral Research Associates Program administered jointly by the Oak Ridge Institute for Science and Education and the Oak Ridge National Laboratory. G.H.P. acknowledges support from NASA-Ames, Engineering for Complex Systems Program, and the NASA-Ames Chief Engineer (Dr Tina Panontin) through grant NAG 2-1424. He also acknowledges additional support from the National Science Foundation (NSF) under grant CMS-0115954 (Mechanics and Materials Program). The authors thank Paul Martin and John Berger for their valuable communications and suggestions.

Appendix A. Some useful formulae for deriving Green's function for a 2D exponentially graded elastic medium

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\xi_1^2 + \xi_2^2} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = -2 \log |\mathbf{x}|, \quad (\text{A } 1)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1 \xi_2}{(\xi_1^2 + \xi_2^2)^2} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = -\frac{x_1 x_2}{|\mathbf{x}|^2}, \quad (\text{A } 2)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1^2}{(\xi_1^2 + \xi_2^2)^2} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = -\log |\mathbf{x}| - \frac{x_1^2}{|\mathbf{x}|^2}, \quad (\text{A } 3)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_2^2}{(\xi_1^2 + \xi_2^2)^2} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = -\log |\mathbf{x}| - \frac{x_2^2}{|\mathbf{x}|^2}, \quad (\text{A } 4)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\boldsymbol{\xi} \cdot \mathbf{x}}}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)} d\xi_1 d\xi_2 = K_0(|\boldsymbol{\beta}||\mathbf{x}|), \quad (\text{A } 5)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\boldsymbol{\xi} \cdot \mathbf{x}}}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2} d\xi_1 d\xi_2 = \frac{1}{|\boldsymbol{\beta}|} |\mathbf{x}| K_1(|\boldsymbol{\beta}||\mathbf{x}|), \quad (\text{A } 6)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1^2}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = K_0(|\boldsymbol{\beta}||\mathbf{x}|) - |\boldsymbol{\beta}| \frac{x_1^2}{|\mathbf{x}|} K_1(|\boldsymbol{\beta}||\mathbf{x}|), \quad (\text{A } 7)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_2^2}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = K_0(|\boldsymbol{\beta}||\mathbf{x}|) - |\boldsymbol{\beta}| \frac{x_2^2}{|\mathbf{x}|} K_1(|\boldsymbol{\beta}||\mathbf{x}|), \quad (\text{A } 8)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1^2 + \xi_2^2}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = 2K_0(|\boldsymbol{\beta}||\mathbf{x}|) - |\boldsymbol{\beta}||\mathbf{x}| K_1(|\boldsymbol{\beta}||\mathbf{x}|), \quad (\text{A } 9)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\xi_1^2 - \xi_2^2}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2} e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\xi_1 d\xi_2 = |\boldsymbol{\beta}| \frac{x_2^2 - x_1^2}{|\mathbf{x}|} K_1(|\boldsymbol{\beta}||\mathbf{x}|). \quad (\text{A } 10)$$

Formulae (A 1)–(A 4) can be found on p. 17 of Mura (1987); formulae (A 5) and (A 6) can be derived from the table, item (20), on p. 24 of Erdélyi (1954)†; formulae (A 7) and (A 8) can be obtained by applying differential operators $(-\partial^2/\partial x_1^2)$ and $(-\partial^2/\partial x_2^2)$, respectively, to both sides of formula (A 6). Clearly, equations (A 9) and (A 10) are consequence of (A 7) and (A 8).

Appendix B. Asymptotic expansions of modified Bessel functions

For convenience in identifying singularities in the Green function, the well-known asymptotic behaviour of the modified Bessel functions $K_0(x)$ and $K_1(x)$ (at both $x \rightarrow 0$ and $x \rightarrow \infty$) is shown below. Note that the modified Bessel functions $K_0(x)$ and $K_1(x)$ satisfy the identities (Olver 1972)

$$\frac{d}{dx} K_0(x) = -K_1(x), \quad \frac{d}{dx} K_1(x) = -K_0(x) - \frac{K_1(x)}{x}.$$

† Based, in part, on notes left by Harry Bateman, and compiled by the staff of the Bateman Manuscript Project.

(i) As $x \rightarrow 0$,

$$K_0(x) \sim -\log x + \log 2 - \gamma + \frac{1}{4}(1 + \log 2 - \gamma - \log x)x^2 + O(x^4); \quad (B 1)$$

$$K_1(x) \sim \frac{1}{x} - \frac{1}{4}(1 + 2 \log 2 - 2\gamma - 2 \log x)x - \frac{1}{16}(\log 2 + \frac{5}{4} - \gamma - \log x)x^3 + O(x^5), \quad (B 2)$$

where $\gamma (\approx 0.577 216)$ is Euler's constant.

(ii) As $x \rightarrow \infty$,

$$K_0(x) \sim \frac{\sqrt{2\pi}}{2}e^{-x} \left[\frac{1}{\sqrt{x}} - \frac{1}{8} \frac{1}{x^{3/2}} + \frac{9}{128} \frac{1}{x^{5/2}} + O\left(\frac{1}{x^{7/2}}\right) \right]; \quad (B 3)$$

$$K_1(x) \sim \frac{\sqrt{2\pi}}{2}e^{-x} \left[\frac{1}{\sqrt{x}} + \frac{3}{8} \frac{1}{x^{3/2}} - \frac{15}{128} \frac{1}{x^{5/2}} + O\left(\frac{1}{x^{7/2}}\right) \right]. \quad (B 4)$$

Appendix C. Non-singular terms

In this appendix we list the formulae for the non-singular components of the Green function. They are given here as the original double Fourier integrals even though, as shown above, they can be reduced to single integrals. The reason for this is that it is not clear at this point what is the best way to numerically evaluate these terms.

$$u_1^{ns}(\mathbf{x}; \mathbf{x}') = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1^{ns}(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \quad (C 1)$$

where

$$U_1^{ns} = \frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{\Delta} - \frac{(\kappa - 1)(\xi_1^2 + \beta_1^2) + (\kappa + 1)(\xi_2^2 + \beta_2^2)}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2};$$

$$v_1^g(\mathbf{x}; \mathbf{x}') = \frac{1}{2\pi^2 \mu_0 (\kappa + 1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_1^g(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \quad (C 2)$$

where

$$V_1^g = \frac{\xi_1 \xi_2}{|\boldsymbol{\xi}|^4} - \frac{\xi_1 \xi_2 + \beta_1 \beta_2}{\Delta} + i \frac{(2 - \kappa)(\beta_2 \xi_1 - \beta_1 \xi_2)}{\Delta};$$

$$u_2^g(\mathbf{x}; \mathbf{x}') = \frac{1}{2\pi^2 \mu_0 (\kappa + 1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_2^g(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \quad (C 3)$$

where

$$U_2^g = \frac{\xi_1 \xi_2}{|\boldsymbol{\xi}|^4} - \frac{\xi_1 \xi_2 + \beta_1 \beta_2}{\Delta} - i \frac{(2 - \kappa)(\beta_2 \xi_1 - \beta_1 \xi_2)}{\Delta};$$

$$v_2^{ns}(\mathbf{x}; \mathbf{x}') = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_2^{ns}(\xi_1, \xi_2) e^{-i\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')} d\xi_1 d\xi_2, \quad (C 4)$$

where

$$V_2^{ns} = \frac{(\kappa + 1)(\xi_1^2 + \beta_1^2) + (\kappa - 1)(\xi_2^2 + \beta_2^2)}{\Delta} - \frac{(\kappa + 1)(\xi_1^2 + \beta_1^2) + (\kappa - 1)(\xi_2^2 + \beta_2^2)}{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\beta}|^2)^2}.$$

References

- Banerjee, P. K. 1994 *The boundary element methods in engineering*, 2nd edn. McGraw-Hill.
- Ben-Menahem, A. & Singh, S. J. 1981 *Seismic waves and sources*. Springer.
- Bonnet, M. 1995 *Boundary integral equation methods for solids and fluids*. Wiley.
- Brigham, E. O. 1974 *The fast Fourier transform*. Englewood Cliffs, NJ: Prentice-Hall.
- Cooley, J. W. & Tukey, J. W. 1965 An algorithm for the machine calculation of complex Fourier series. *Math. Comput.* **19**, 297–301.
- Cruse, T. A. 1988 *Boundary element analysis in computational fracture mechanics*. Dordrecht: Kluwer Academic.
- Delale, F. & Erdogan, F. 1983 The crack problem for a nonhomogeneous plane. *Trans. ASME J. Appl. Mech.* **19**, 609–614.
- Eischen, J. W. 1987 Fracture of non-homogeneous materials. *Int. J. Fracture* **34**, 3–22.
- Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1954 *Tables of integral transforms*, vol. 2. McGraw-Hill.
- Erdogan, F. 1995 Fracture mechanics of functionally graded materials. *Composites B* **5**, 753–770.
- Gray, L. J., Kaplan, T., Richardson, J. D. & Paulino, G. H. 2003 Green's functions and boundary integral analysis for exponentially graded materials: heat conduction. *Trans. ASME J. Appl. Mech.* **70**, 543–549.
- Hirai, T. 1996 Functionally gradient materials. In *Processing of ceramics: part II* (ed. R. J. Brook), Materials Science and Technology, vol. 17B, pp. 292–341. Weinheim: Wiley VCH.
- Kim, J.-H. & Paulino, G. H. 2002a Isoparametric graded finite elements for nonhomogeneous isotropic and orthotropic materials. *Trans. ASME J. Appl. Mech.* **69**, 502–514.
- Kim, J.-H. & Paulino, G. H. 2002b Finite element evaluation of mixed-mode stress intensity factors in functionally graded materials. *Int. J. Numer. Meth. Engng* **53**, 1903–1935.
- Kim, J.-H. & Paulino, G. H. 2003 T-stress, mixed-mode stress intensity factors, and crack initiation angles in functionally graded materials: a unified approach using the interaction integral method. *Comput. Meth. Appl. Mech. Engng* **192**, 1463–1494.
- Konda, N. & Erdogan, F. 1994 The mixed mode crack problem in a nonhomogeneous elastic medium. *Engng. Fract. Mech.* **47**, 533–545.
- Martin, P. A., Richardson, J. D., Gray, L. J. & Berger, J. R. 2002 On Green's function for a three-dimensional exponentially graded elastic solid. *Proc. R. Soc. Lond. A* **458**, 1931–1947.
- Miyamoto, Y., Kaysser, W. A., Rabin, B. H., Kawasaki, A. & Ford, R. G. 1999 *Functionally graded materials: design, processing applications*. Dordrecht: Kluwer Academic.
- Mukherjee, S. 1982 *Boundary element methods in creep and fracture*. London: Applied Science.
- Mura, T. 1987 *Micromechanics of defects in solids*. Dordrecht: Kluwer Academic.
- Olver, F. W. J. 1972 Bessel functions of integer order. In *Handbook of mathematical functions* (ed. M. Abramowitz & I. A. Stegun), ch. 9, pp. 355–433. New York: Dover.
- Paulino, G. H., Jin, Z.-H. & Dodds Jr, R. H. 2003 Failure of functionally graded materials. In *Comprehensive structural integrity* (ed. B. Karahaloo & W. G. Knauss), vol. 2, ch. 13. Elsevier.
- Sneddon, I. N. 1972 *The use of integral transforms*. McGraw-Hill.
- Suresh, S. & Mortensen, A. 1998 *Fundamentals of functionally gradient materials*. London: Institute of Materials.
- Sutradhar, A., Paulino, G. H. & Gray, L. J. 2002 Transient heat conduction in homogeneous and nonhomogeneous materials by the Laplace transform Galerkin boundary element method. *Engng Analysis Boundary Elements* **26**, 119–132.
- Sutradhar, A., Paulino, G. H. & Gray, L. J. 2003 Erratum to Transient heat conduction in homogeneous and nonhomogeneous materials by the Laplace transform Galerkin boundary element method. *Engng Analysis Boundary Elements* **27**, 639.
- Walker, J. S. 1991 *The fast Fourier transform*. Littleton, MA: CRC Press.