

**Accurate Simulation of Optical Waveguides by
a Generalized Discontinuous Galerkin (GDG) Method**

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Summary: We have developed a new Generalized Discontinuous Galerkin (GDG) method for PDEs with large jump conditions based on a split distribution function and its unique integration by parts formula. The GDG method can be used for accurate simulations of optical waveguides.

In a paraxial approximation of optical wave propagations in optical fibers, the time harmonic Maxwell's equations is reduced to Schrödinger equations where the propagation direction is identified as the time axis. Due to the mismatch of refractive indices between the core and cladding of the fiber, the electromagnetic fields are discontinuous solutions to the Schrödinger equations, a property not shared by the probability wave functions of quantum mechanics.

In order to handle the discontinuous solutions for the Schrödinger equations, we reformulate the PDEs using distribution variables where Dirac δ functions are introduced as source terms to enforce the jump conditions. After introducing auxiliary distribution variables, discontinuous Galerkin projection of the distribution variables are used to obtain the numerical approximations.

Consider a model Schrödinger equation

$$ic \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} \quad (\text{Eq. 1})$$

where the complex wave function φ has jumps at τ , $[\varphi] = f, [\varphi'] = g$.

First, we rewrite equations (Eq. 1) using Dirac δ functions as follows

$$ic \frac{\partial \varphi(x,t)}{\partial t} = \frac{\partial^2 \varphi(x,t)}{\partial x^2} - g \delta(x-\tau) - f \delta'(x-\tau).$$

Next, introducing a distributional auxiliary variable

$$p = \frac{\partial \varphi}{\partial x} - f(t) \delta(x-\tau), \quad (\text{Eq. 2})$$

then, the Schrödinger equation (Eq. 1) becomes

$$ic \frac{\partial \varphi}{\partial t} = \frac{\partial p}{\partial x} - g(t) \delta(x-\tau). \quad (\text{Eq. 3})$$

To construct a Galerkin projection of the distributional equations (Eq. 2-3), we developed the following concept of split distribution and its unique integration by parts formula

Split Distribution: We define the evenly-split Dirac δ function, for $v(x) \in C^\infty[a,0]$

$$\int_{\pm a}^0 v(x) \delta(x) dx = \mp \frac{1}{2} v(0)$$

Integration by Parts for Split Distributions:

For a piecewise continuous function $\varphi(x)$ with a jump at $x=0$, $[\varphi] = \varphi(0^+) - \varphi(0^-)$, the distribution $\frac{\partial \varphi}{\partial x}$ has the following integration by part formula, for $v(x) \in C^\infty[a,0]$

$$\int_a^0 \frac{\partial \varphi}{\partial x} v(x) dx = \{\varphi\} v(0) - \varphi(a) v(a) - \int_a^0 \varphi(x) \frac{\partial v}{\partial x} dx$$

where the average $\{\varphi\} = \frac{1}{2}(\varphi(0^-) + \varphi(0^+))$ at $x=0$ has to be used (refer to [1] for details).

Then, we multiply (Eq. 2-3) with a test function $v(x)$ and integrate over *each closed finite element K* to obtain, after integration by part,

$$ic \int_K \frac{\partial \varphi}{\partial t} v dx = h_\varphi(x_{k+1}) v(x_{k+1}^-) - h_\varphi(x_k) v(x_k^+) - \int_K p \frac{dv}{dx} dx - g(t) \int_K \delta(x - \tau) v dx$$

$$\int_K p v dx = h_p(x_{k+1}) v(x_{k+1}^-) - h_p(x_k) v(x_k^+) - \int_K \varphi \frac{dv}{dx} dx - f(t) \int_K \delta(x - \tau) v dx$$

where the numerical fluxes are defined as for $x_k = \tau$,

$$h_\varphi(x_k \pm) = \{\varphi\} \pm \frac{1}{2}[\varphi], h_p(x_k \pm) = \{p\} \pm \frac{1}{2}[p]$$

for $x_k \neq \tau$,

$$h_\varphi(x_k \pm) = \{\varphi\}, h_p(x_k \pm) = \{p\}.$$

The 2-D scalar Schrödinger equations with jump conditions can be similarly formulated as

$$ic \frac{\partial \varphi}{\partial t} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} - \delta(\xi - \xi^*) |\nabla \xi|^2 g$$

$$p = \frac{\partial \varphi}{\partial x} - \delta(\xi - \xi^*) f \frac{\partial \xi}{\partial x}$$

$$q = \frac{\partial \varphi}{\partial y} - \delta(\xi - \xi^*) f \frac{\partial \xi}{\partial y}$$

where $f(x^*, y^*)$, $g(x^*, y^*)$ are the jumps of the solution and the normal derivative at an interface location (x^*, y^*) , respectively. (ξ, η) are the local coordinates along the normal and tangential directions at location (x^*, y^*) .

Figure 1 shows the exponential convergence of the GDG for a 2-D Schrödinger equation with a nonsmooth solution.

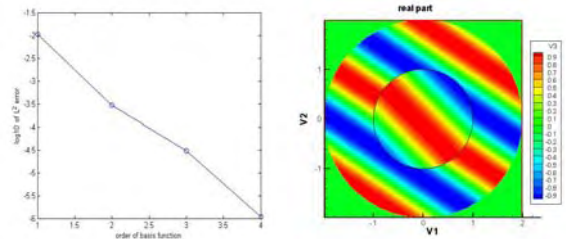


Figure 1. (Left) Exponential Convergence of GDG via order of basis functions. (Right) GDG solution of 2-D Scalar Schrödinger eq. with jumps.

References:

[1] K. Fan, W. Cai, X. Ji, A Generalized Discontinuous Galerkin Method for Schrödinger equations with non-smooth solution, submitted to *the Journal of Computational Physics*, Feb., 2007.

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