"How to determine the TIME DELAYS for robust stability?"

Prof. Nejat Olgac
Mech. Eng. Dept., University of Connecticut, Storrs, CT 06269

Abstract— An intriguing perspective is developed on the assessment of stability robustness of systems with multiple independent and uncertain delays. It is based on a holographic mapping, which is implemented over the domain of the delays. This mapping considerably alleviates the problem, which is known to be N-P hard. It creates a dramatic reduction in the dimension of the problem from infinity to manageably small number. Ultimately the process is reduced to studying the problem within a finite dimensional cube with edges of length 2π in the new domain, what we name the **building block**. In essence, the mapping collapses the entire set of potential stability switching points onto a small (upperbounded) number of building hypersurfaces. We further demonstrate that these building hypersurfaces can be implicitly defined and they are completely isolated within the above mentioned cube. It is also shown that the exhaustive detection of these building hypersurfaces is necessary and sufficient in order to arrive at the complete stability robustness picture. As a consequence, this concept yields a very practical and efficient procedure for the stability assessment of such systems. This novel perspective serves very well for the preparatory steps of the authors' earlier contribution in the area, **Cluster Treatment of Characteristic Roots (CTCR).** We elaborate on this combination, which forms the main contribution of the paper. Several experimental validation studies of this new concept have been reported in the recent archival literature.

In this project, we consider linear time-invariant, multiple time-delayed systems (LTI-MTDS). The general state-space form of this class of systems is given as,

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \sum_{j=1}^{p} \mathbf{B}_j\, \mathbf{x}(t-\tau_j) \qquad (1)$$

where $\mathbf{x} \in \Re^n$, $\mathbf{A}$, $\mathbf{B}_j$, $j = 1 \ldots p$ are all constant matrices in $\Re^{n \times n}$ and the vector of time delays $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_p) \in \Re^{p+}$ of which the elements are positive and rationally independent from each other.

The determination of the stability robustness against delay uncertainties of this system is our main objective. This problem has been studied for over four decades resulting in some respectable volume of literature. It is known to be of complexity class $\mathcal{N}$-$\mathcal{P}$ hard. Our novel paradigm, **Cluster Treatment of Characteristic Roots (CTCR),** suggests a practical and numerically efficient procedure. Indeed, CTCR produces a complete stability robustness tableau within the domain of the delays, $\boldsymbol{\tau} \in \Re^{p+}$. The numerical efficiency of CTCR is tested for cases $n = 3$, $p = 2$, and $n = 6$, $p = 3$ very favorably.

The key procedure that enables the CTCR paradigm is a holographic mapping of the delays (known as Rekasius substitution). This mapping successfully converts spectrally infinite dimensional problem into a finite dimensional one. We can metaphorically describe this operation as looking at the problem under special optics. To improve the process further we developed an additional perspective, which further confines the domain of analysis into a finite dimensional cube. This cube is called the "**building block**".

CTCR and two key propositions (which were unrecognized until our findings)

(i) The system in (1) is infinite dimensional.

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Its characteristic equation is

$$CE(s, \tau_1, \ldots, \tau_p) = \det\left[s\mathbf{I} - \mathbf{A} - \sum_{j=1}^{p} \mathbf{B}_j e^{-\tau_j s} \right] \qquad (2)$$

and it has infinite spectra.

(ii) The continuous variations of such settings, in $\boldsymbol{\tau} \in \Re^{p+}$, are the only possible locations where the stability switching can take place: call them the "**switching hypersurfaces**".

(iii) One must determine, *exhaustively* all those hypersurfaces in $\boldsymbol{\tau} \in \Re^{p+}$ space, where an imaginary spectrum exists.

(iv) However, there are still countably infinite number of such hypersurfaces.

(viii) Therefore, one must introduce a feature-based discipline, what we call "**the clustering**" procedure, to those imaginary root crossings in order to bring the analysis to a manageable size.. CTCR procedure achieves precisely this objective.

Definition 1: Kernel Hypersurface, $\wp_0$. Those hypersurfaces which consist of points in $\boldsymbol{\tau} \in \Re^{p+}$ complying with $\langle \boldsymbol{\tau}, \omega \rangle$ correspondence with the constraint that

$$0 < \tau_k < 2\pi/\omega\,,\ k = 1, \ldots, p \qquad (5)$$

are called the *Kernel hypersurfaces*.

<u>**Proposition I**</u>**.** There is a s**mall number of kernel hypersurfaces and the upperbound of this number is** n^2 **.**

Definition 2: Offspring Hypersurfaces, $\wp$. Those ∞^p hypersurfaces, which consist of the points with larger delays (than those of the kernel) but still resulting in the same imaginary spectrum as the *kernel*, are called the *offspring hypersurfaces*.

Definition 3. Root Tendency, RT. The root tendency indicates the direction of transition of the imaginary root (to C^- or to C^+) as we increase only one of the delays, τ_j, by ε ($0 < \varepsilon$ $<<1$) while all the others remain fixed.

<u>**Proposition II**</u>**. Root tendency invariance property.** Take an imaginary characteristic root, ωi, of equation (2) caused by any one of the infinitely many grid points in $\boldsymbol{\tau} \in \Re^{p+}$. The root tendency $RT\big|_{s=\omega i}^{\tau_k}$ remains invariant (-1 for **stablizing** and +1 for **destablizing**) so long as the grid points on different *'offspring hypersurfaces'* are obtained keeping all $p-1$ delays τ_j, $j = 1, \ldots, k-1, k+1, \ldots, p$, fixed.♦

Utilizing these two propositions the CTCR paradigm yields a unique stability robustness picture of the LTI-MTDS. For an effective description we offer a case study.

<u>*Example case study*</u>

Take a system characteristic equation second order system with two delays:

$$CE(s, \tau_1, \tau_2) =$$
$$s^2 + 3s + 8 + (3s+1)e^{-\tau_1 s} + (-s+8)e^{-\tau_2 s} + 5e^{-(\tau_1+\tau_2)s} = 0$$

Fig. 1 shows the kernel (red) and the offspring, and the complete stability picture. Fig. 2 displays the same picture but in "building block form (in the $\tau\,\omega$ space).

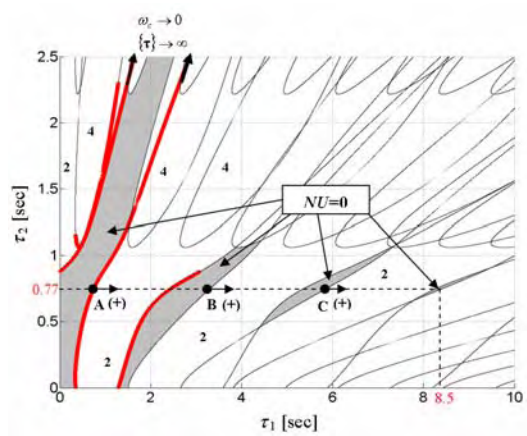

Fig. 1. The complete stability picture of the example system.

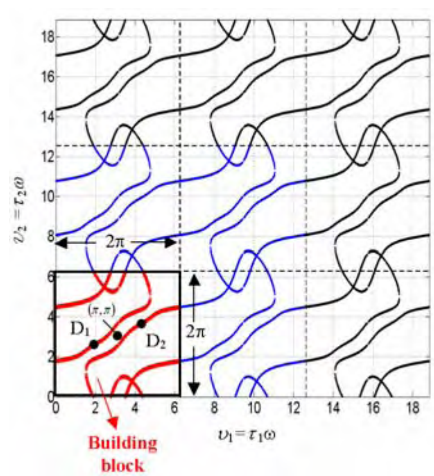

Fig. 2. Building block (thick) and its offspring.