"Accurate Simulation of Random Processes"

Jonathan Goodman
Courant Mathematics and Computing Laboratory (CMCL)
Courant Institute, New York University

Summary

Stochastic models describe the evolution of dynamical systems influenced by noise or uncertainty. Stochastic differential equations are particularly helpful idealizations of random systems, just as ordinary differential equations help model systems not subject to random noise. Over the past 50 years, computational scientists have developed sophisticated computational methods for solving ordinary differential equations. These are far more accurate, efficient, and versatile than naïve methods. Theory has not yet had much impact on computational practice in solving stochastic differential equations. Our work explores alternative theoretical frameworks that already have led to alternative simulation methods.

There is a large body of knowledge on how to solve ordinary and partial differential equations. Much of that grew out of mathematical analysis of issues related to accuracy and stability. By contrast, the theory of simulation of stochastic differential equations is less successful. It has had less influence on computational practice.

The first ingredient in a theoretical study of a computational method is a measure of accuracy. In a non-random problem, this seems straightforward (but often is not). You are trying to compute X and you actually get Y. The error is $X - Y$. In a random system, the value of X is not uniquely determined, nor is Y. It may not make sense to ask that X be close to Y. What matters is that the probability distribution describing the frequencies of different possible values of X should be close to that of Y. This is called *weak accuracy*.

This seems straightforward when X is a simple random variable, but is problematic when $X(t)$ is a whole random path. The solution of a differential equation, stochastic or non-random, is not a single number but a path, with $X(t)$ representing the state of the system at each time t. For this reason, much of the theory has focused on strong accuracy, which roughly asks that the approximate path, $Y(t)$, should be close to the corresponding exact path $X(t)$ for every value of the time variable t. After decades of study, improving on the most rudimentary methods in the strong sense requires very elaborate and time-consuming algorithms.

Several years ago, a group of three researchers began a search for less rigid error measures that allow for a greater

*212-998-3326, goodman@cims.nyu.edu

variety of approximation methods. Besides Goodman, the group consisted of Peter Glynn, Professor Management Science and Engineering at Stanford, and Jose Antonio Perez, then a graduate student in Mathematics at the Courant Institute. The error measure they devised, which they call the *Microscopic Total Variation* (*MTV*) measure, captures fine details of random paths, but only in a statistical sense. If a simulation method is accurate in the MTV sense, a single approximate path $Y(t)$ may not be close to a given exact path $X(t)$. But if you generate a large number of approximate paths, the collection will accurately reproduce even fine statistical properties of the exact path distribution.

The MTV definition (or any other) is useful for theory only if it is possible to make a theoretical analysis of the accuracy of approximation methods. The Goodman group found general methods that allowed them to evaluate the MTV accuracy of commonly used approximation methods. Their analysis allowed them to devise new approximation methods that have statistical properties similar to those of more complicated methods that are accurate in the strong sense. These new methods have far less computational complexity. What made this possible is the extra freedom that comes from not having to compare a specific approximate path to a specific exact path, but only the statistical properties of approximate and exact paths.

This work is developing in several directions. Goodman is collaborating with scientists at Lawrence Berkeley Laboratory to analyze and improve computational methods for a class of partial differential equations subject to noise. In particular, they are studying simulations of compressible gas on length scales small enough that thermal noise is important but large enough that atomistic simulation is impossible. There are several challenging features of this application, one being that the model of thermal noise leads to a very rough source term in the equation that grows as the length scales get smaller. Another is that this noise term affects the gas velocity directly but the density only indirectly.

Goodman also will begin working with a new Courant gradate student, Sang-Min Lee, on finding even more accurate approximation methods within the MTV framework. Preliminary analysis showed that straightforward generalization of the methods of Goodman, Glynn and Perez would be difficult. However, a new idea, an extra degree of randomization, should provide the extra flexibility to it possible. This would lead to considerable improvements in the efficiency of approximation methods for stochastic differential equations.

For further information on this subject contact:

Prof. Jonathan Goodman
Department of Mathematics
Courant Institute of Mathematical Sciences,
New York University
goodman@cims.nyu.edu
212-998-3326

Or

Dr. Anil Deane
Applied Mathematics Research Program
Office of Advanced Scientific Computing
Phone: 301-903-1465
deane@ascr.doe.gov