Multiple Time Scales in Dynamical Systems

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Summary

Multiple time scales distort universal phenomena whose description has been a hallmark of dynamical systems theory. This research extends dynamical systems theory to encompass phenomena that are dominated by multiple time scales. The emphasis has been on relaxation processes in which a system alternates between slow and rapid changes. Different types of slow to fast and fast to slow transitions have been classified in generic systems with two time scales. This classification supports new numerical methods for computing bifurcations that occur in periodic relaxation oscillations. One product of the research is deeper insight into mechanisms that create chaotic attractors in dynamical systems.

Nonlinear differential equations are used throughout the sciences as dynamical models. Simulation and other numerical methods derive predictions from these models. Many of the models – in areas from combustion to climate change to cell biology – have multiple time scales that influence their dynamics and create technical problems in numerical studies. This project contributes to our understanding of dynamical phenomena associated with multiple time scales.

Singular perturbation theory examines limiting behavior of multiple time scale systems in which there is an infinite separation of fast and slow time scales. In this singular limit, solutions of a system tend to concatenations of segments that flow along slow manifolds on the slow time scale and segments that jump between sheets of the slow manifold on the fast time scale. The ratchet mechanism of a (mechanical!) clock is an example of a relaxation oscillation with such two time scale behavior.

One goal of this research is to classify the types of transitions between slow and fast dynamics found in generic systems with multiple time scales. Dynamical systems theory has been very successful in developing predictive models by focusing on generic, or typical, dynamics. This project uses this strategy to get insight into the behavior of systems with multiple time scales. Two recent discoveries are described below.

Chaotic dynamics in systems with a single time scale is often associated with "stretching and folding." Iterations of the quadratic function $f(x) = a + x^2$ are an intensively studied manifestation of this phenomenon. These iterations do not occur as a discretization of a continuous time dynamical system. However, two

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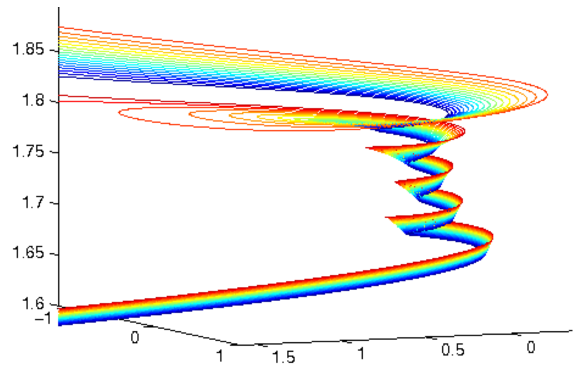
dimensional perturbations of this map do appear as “return maps” for vector fields. The appearance of chaotic attractors in these two dimensional maps has been the subject of intense interest. I have connected this research on chaotic attractors in two dimensional maps to singular perturbation theory in work with Martin Wechselberger and Lai Sang Young. We prove that one dimensional maps resembling the quadratic map appear naturally as the singular limits of return maps in singularly perturbed systems. These results link concrete models of multiple time scale differential equations to abstract, rigorous theories of discrete iterations.

Folded nodes occur in generic systems with two slow variables. They can be regarded as a constriction at the transition from slow to fast behavior. Large regions of trajectories converge at the folded node and jump from its vicinity. Surprisingly, the French mathematician Eric Benoît observed that trajectories twist and link each other as they pass through a folded node. The twisting of trajectories at a folded node provide a mechanism for creating mixed mode oscillations in which a physical system has complex oscillations which contain cycles of markedly differing amplitudes. This project has further explored the consequences of the trajectory twisting of folded nodes for the global dynamics of a system.

There are stable and unstable slow manifolds that determine how forward and backward slow trajectories meet at the folded node. The intersections of these manifolds are on canards, trajectories that remain on the unstable sheet of the slow manifold as they emerge from the folded node node. These canards separate groups of trajectories that make different numbers of twists as they emerge from the folded node, as shown in the figure.



A bundle of trajectories flowing through a folded node. A canard trajectory separates the two trajectories that make one fewer twist from the others.

The insight gained from these studies is being incorporated into numerical methods that compute aspects of dynamical behavior that inaccessible to direct simulation. These methods visualize geometric structures that underlie seemingly anomalous and discontinuous phenomena that appear in nonlinear dynamical systems with multiple time scales.

Reference: J. Guckenheimer, M. Wechselberger and Lai-Sang Young,
 Chaotic attractors of relaxation oscillators, Nonlinearity 19,701--720, 2006.

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