Abstract. – A simple dynamic pool model is used to examine the problem of stock-recruitment parameter uncertainty from a Bayesian perspective. Probabilities associated with different parameter values are used to weight the losses (i.e., opportunity costs to society) associated with any given fishing mortality rate. By choosing appropriate forms for the loss and probability density functions, the model is shown to result in an analytic solution. Because this solution gives the fishing mortality rate that maximizes the expected value of the logarithm of sustainable yield, it is denoted F_{MELSY}. The solution is a monotone-decreasing function of parameter uncertainty, converging on the fishing mortality rate corresponding to maximum sustainable yield as the degree of uncertainty approaches zero. As an empirical illustration, the model is applied to the eastern Bering Sea stock of rock sole *Pleuronectes bilineatus*.

Manuscript accepted 4 June 1992.
Fishery Bulletin, U.S. 90:561–573 (1992).

A Bayesian approach to management advice when stock-recruitment parameters are uncertain

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Exploiting a stock at the fishing mortality rate (F) associated with maximum sustainable yield (MSY) is a common fishery management strategy. For the most part, three simple propositions are sufficient to justify this strategy: (1) The stock exhibits a sustainable yield determined by the fishing mortality rate, (2) more sustainable yield is always preferable to less, and (3) the parameters underlying the stock's dynamics are known with certainty. However, parameters governing stock dynamics are typically not known with certainty, and in such cases it is possible to demonstrate that the appropriate F value may be less than the value corresponding to MSY (F_{MSY}).

The approach to be used in this demonstration is taken from Bayesian decision theory (e.g., Raiffa 1968, DeGroot 1970). Early applications of Bayesian theory to fisheries problems were presented by Rothschild (1972), Lord (1973, 1976), Walters (1975), and Walters and Hilborn (1976). Of the many more recent applications, those presented by Ludwig and Walters (1982), Clark et al. (1985), and Walters and Ludwig (1987) bear most closely on the present study.

For simplicity, it will be assumed here that stock dynamics are deterministic but governed by parameters which may be imprecisely estimated. This approach is distinct from the more common one of assuming that stock dynamics are the product of a deterministic system (with parameter values given and fixed) modified by a random error term. Important early examples of the latter approach include Ricker (1958), Larkin and Ricker (1964), and Tautz et al. (1969). Ludwig and Walters (1982) and Mangel and Clark (1983) incorporate both approaches in a systematic fashion which makes the distinction especially clear.

The basic model

Thompson (1992) developed a simple dynamic pool model which can be solved explicitly for F_{MSY}. In terms of biomass per recruit, the model is basically that of Hulme et al. (1947); thus, body weight is taken to be a linear function of age, with intercept a_0. The main departure from Hulme et al. is that biomass at recruitment age a_r is taken to be proportional to stock biomass raised to a power q (Cushing 1971). With these specifications, sustainable yield Y(F) can be written

$$Y(F) = F\left[\left(\frac{p}{M}\right)\left(\frac{1+K''+F'}{(1+F')^2}\right)\right]^{\frac{1}{1-q}}, \tag{1}$$

where M is the instantaneous rate of natural mortality, $F' = F/M$, p is the proportionality term in the Cushing stock-recruitment relationship, and $K'' = 1/[M(a_r - a_0)]$ (which can be interpreted in this model as the pristine ratio of growth to recruitment). The

Cushing exponent q is constrained to fall between 0 and 1. In the limiting case of $q=0$, recruitment is constant, while in the other limiting case of $q=1$, recruitment is proportional to biomass.

Differentiating Equation (1) with respect to F and setting the resulting expression equal to zero gives the following equation for F_{MSY}:

$$F'_{MSY} = \frac{-(q+1)K'' + 1 + \sqrt{(q+1)^2K''^2 + (6q-2)K'' + 1}}{2q} - 1, \qquad (2)$$

where $F'_{MSY} = F_{MSY}/M$.

A common rule of thumb is that F'_{MSY} should equal 1. The locus of parameter values for which this rule holds precisely is given by

$$K'' = \frac{1}{q} - 2. \qquad (3)$$

Analyzing the model in a Bayesian framework

Parameter estimates in any model are by definition associated with some degree of uncertainty. For example, parameters governing the stock-recruitment relationship are particularly difficult to estimate precisely (Larkin 1973, Paulik 1973, Ludwig and Walters 1981, Walters and Ludwig 1981 and 1987, Shepherd 1982, Clark 1985, Clark et al. 1985, Rothschild and Mullen 1985, Shepherd and Cushing 1990). In the presence of such uncertainty, a Bayesian approach would use the probabilities associated with different parameter values to weight the losses (i.e., opportunity costs to society) associated with choosing a particular fishing mortality rate. Following similar studies by Ludwig and Walters (1982), Clark et al. (1985), and Walters and Ludwig (1987), the present analysis will focus on the uncertainty surrounding a single parameter, in this case the stock-recruitment exponent q. This uncertainty takes the form of a probability density function (pdf) P(q) which describes the relative credibility of alternative q values.

To simplify notation, define z(F, q) as the ratio of Y(F) to MSY for an arbitrary value of q drawn from P(q). Then, the "Bayes decision" (DeGroot 1970) is the value of F that minimizes

$$E\{L[z(F,q)]\} = \int_0^1 L[z(F,q)]\, P(q)\, dq, \qquad (4)$$

where L[z(F, q)] represents the losses resulting from selection of a particular value of F given a particular value of q, and E{L[z(F, q)]} is the expected value of L[z(F, q)] (the "risk," DeGroot 1970). The minimum value of E{L[z(F, q)]} is referred to as the "Bayes risk" (DeGroot 1970). The integral is taken over the interval 0 to 1 because the Cushing stock-recruitment relationship constrains q to that range.

The Bayes decision can be derived by differentiating E{L[z(F, q)]} with respect to F and solving for the value that sets the derivative equal to zero. The validity of this procedure requires that all parameter values, including those describing P(q), remain constant into the future. The solution corresponding to such an assumption is sometimes known as a "myopic Bayes" solution (Ludwig and Walters 1982, Mangel and Clark 1983, Mangel and Plant 1985, Parma 1990). A more general alternative is to allow for the possibility that parameter estimates will be updated in the future, but this approach is vastly more difficult (Clark et al. 1985, Mangel and Plant 1985, Walters and Ludwig 1987).

Minimizing risk under a logarithmic loss function

Of course, specification of the functions L and P is crucial to this problem. Following Lord (1976) and Ludwig and Walters (1982), one possible choice is to assume that L is a linear function of z($L(z)=1-z$). Another common form is the quadratic $L(z)=(1-z)^2$, which has been used in the fisheries literature by Walters (1975), Hightower and Grossman (1987), and Charles (1988). One of the oldest alternatives is the logarithmic loss function, $L(z)=-\ln(z)$, dating back to the work of Bernoulli in 1738 (transl. 1954). Logarithmic loss (or, conversely, utility) seems first to have been used in the fisheries literature by Gleit (1978), followed by Lewis (1981, 1982), Mendelssohn (1982), Opaluch and Bockstael (1984), Ruppert et al. (1984, 1985), Deriso (1985), Walters (1987), Walters and Ludwig (1987), Getz and Haight (1989), Hightower and Lenarz (1989), Hightower (1990), Parma (1990), and Parma and Deriso (1990).

Linear, quadratic, and logarithmic loss functions are compared in Figure 1. As Figure 1 indicates, the logarithmic loss function corresponds to a "preservationist" viewpoint, in which extinction of the stock is absolutely unacceptable (i.e., the loss corresponding to extinction is infinite). Because the logarithmic loss function is clearly identifiable as a risk-averse alternative function (see Discussion), it is a good candidate for illustrating how a Bayesian approach can differ from more traditional approaches which do not incorporate uncertainty in an explicit fashion.

To incorporate the logarithmic loss concept into the model, first note that Equation (1) allows z(F, q) to be written

$$z(F, q) = \frac{Y(F)}{MSY} = \frac{F\left[\left(\frac{p}{M}\right)\left(\frac{1+K''+F'}{(1-F')^2}\right)\right]^{\frac{1}{1-q}}}{F_{MSY}\left[\left(\frac{p}{M}\right)\left(\frac{1+K''+F'_{MSY}}{(1+F'_{MSY})^2}\right)\right]^{\frac{1}{1-q}}} = \frac{F'}{F'_{MSY}}\left[\left(\frac{1+F'_{MSY}}{1+F'}\right)^2\left(\frac{1+K''+F'}{1+K''+F'_{MSY}}\right)\right]^{\frac{1}{1-q}}. \quad (5)$$

For an arbitrary value of q, the (logarithmic) loss associated with a given choice of F is thus

$$L[z(F, q)] = \ln(F'_{MSY}) - \frac{2\ln(1+F'_{MSY}) - \ln(1+K''+F'_{MSY})}{1-q} - \ln(F') + \frac{2\ln(1+F') - \ln(1+K''+F')}{1-q}. \quad (6)$$

Substituting Equation (6) into Equation (4), the risk can be written

$$E\{L[z(F, q)]\} = \int_0^1 P(q)\left(\ln(F'_{MSY}) - \frac{2\ln(1+F'_{MSY}) - \ln(1+K''+F'_{MSY})}{1-q}\right)dq$$

$$- \int_0^1 P(q)\left(\ln(F') - \frac{2\ln(1+F') - \ln(1+K''+F')}{1-q}\right)dq. \quad (7)$$

From Equation (2), it is clear that F'_{MSY} involves only K" and q. Thus, regardless of the form of P(q), the first integral on the right-hand side of Equation (7) is independent of F. Therefore, the problem of finding the Bayes decision is equivalent to minimizing the second integral on the right-hand side of Equation (7). Remembering that the integral (taken over the interval 0 to 1) of a constant multiplied by P(q) is equal to the constant itself, the following proxy objective function is obtained:

$$E_1\{L[z(F, q)]\} = -\ln(F') + [2\ln(1+F') - \ln(1+K''+F')]\int_0^1 \frac{P(q)}{1-q}dq. \quad (8)$$

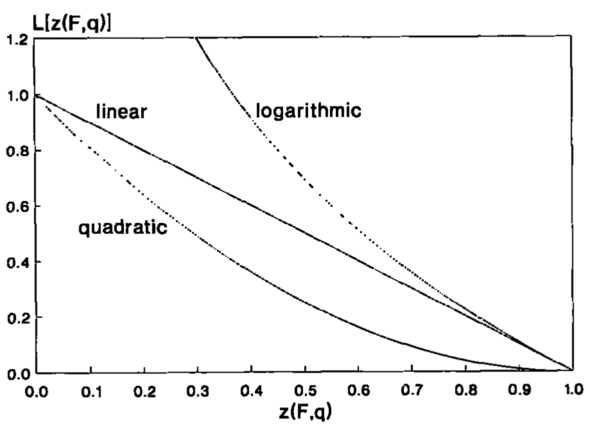

Figure 1
Three possible loss functions. Loss, or relative utility foregone, is plotted against the ratio of Y(F)/MSY for quadratic, linear, and logarithmic loss functions.

Incorporating a beta probability density function

The next step in determining the Bayes decision is to select a form for the pdf P(q). Bayesian decision theory frequently makes use of the beta family of pdfs (e.g., DeGroot 1970, Holloway 1979). The beta distribution would seem to be a natural candidate for P(q), since it constrains q to the necessary (0,1) range. In its standard form, the beta distribution can be written

$$P(q) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right) q^{\alpha-1}\,(1-q)^{\beta-1}, \tag{9}$$

where α and β are positive constants and $\Gamma(\cdot)$ is the gamma function, which, except for $\Gamma(1)=1$, can be described in terms of the recursion formula

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1). \tag{10}$$

By Equations (9) and (10), then, the integral in Equation (8) can be evaluated as follows:

$$\int_0^1 \frac{P(q)}{1-q}\,dq = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right)\int_0^1 q^{\alpha-1}\,(1-q)^{\beta-2}\,dq = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right)\left(\frac{\Gamma(\alpha)\,\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)}\right)$$

$$= \left(\frac{\Gamma(\beta-1)}{\Gamma(\beta)}\right)\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-1)}\right) = \frac{\alpha+\beta-1}{\beta-1}. \tag{11}$$

Substituting Equation (11) into Equation (8) then gives

$$E_1\{L[z(F,\,q)]\} = -\ln(F') + \frac{[2\ln(1+F') - \ln(1+K''+F')]\,(\alpha+\beta-1)}{\beta-1}. \tag{12}$$

Differentiating Equation (12) with respect to F′ and setting the resulting expression equal to zero yields the quadratic expression

$$\alpha F'^2 + [K''(2\alpha+\beta-1) + \alpha - \beta + 1]\,F' - (\beta-1)\,(K''+1) = 0. \tag{13}$$

Before solving Equation (13), it would be helpful to cast the solution in terms of parameters which are more intuitive than α and β, for example the mean and variance of $P(q)$. The beta distribution has mean m and variance v as follows:

$$m = \frac{\alpha}{\alpha+\beta} \qquad \text{and} \qquad v = \frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)}. \tag{14 and 15}$$

Conversely, Equations (14) and (15) can be solved simultaneously to describe α and β in terms of m and v:

$$\alpha = \left(\frac{m(1-m)}{v} - 1\right) m \qquad \text{and} \qquad \beta = \left(\frac{m(1-m)}{v} - 1\right)(1-m). \tag{16 and 17}$$

Unlike the normal distribution, the variance of the beta distribution exhibits a maximum possible value for a given mean. Remembering that α and β are constrained to be positive, the maximum possible value of v can be derived from either Equation (16) or Equation (17) by setting the left-hand side equal to 0 and solving for v. This exercise results in a maximum v equal to $m(1-m)$. Thus, α and β can be written in terms of the mean and a scaled variance $v'\left(=\dfrac{v}{m(1-m)}\right)$ as follows:

$$\alpha = \left(\frac{1}{v'} - 1\right) m \quad \text{and} \quad \beta = \left(\frac{1}{v'} - 1\right)(1-m). \qquad (18) \text{ and } (19)$$

For a given set of K″, m, and v′ values, Figure 2 shows the risk (depicted by the area under a particular curve) associated with three possible F′ values.

Fishing mortality at maximum expected log-sustainable yield

Substituting Equations (18) and (19) into Equation (13) and solving for F′ gives the value that minimizes risk. Because of the form used for the loss function, this process is equivalent to finding the level of F′ that maximizes the expected value of the logarithm of sustainable yield. It is thus convenient to refer to this value as F'_{MELSY} (for "maximum expected log sustainable yield"), which for this particular model can be written

$$F'_{MELSY} = \frac{[(m+2)\,K''-2]\,v' - (m+1)\,K'' + 1 + \sqrt{k_2\,v'^2 - k_1\,v' + k_0}}{2\,m(1-v')} - 1, \qquad (20)$$

where $k_2 = (m+2)^2\,K''^2 + (12m-8)\,K'' + 4,$
$k_1 = (2m^2+6m+4)\,K''^2 + (18m-8)\,K'' + 4,$ and
$k_0 = (m+1)^2\,K''^2 + (6m-2)\,K'' + 1.$

Figure 3 illustrates how F'_{MELSY} varies with K″, m, and v′. A few special cases are of particular interest. For example, when q is known with certainty, i.e., m = q and v′ = 0, Equation (20) reduces to Equation (2). Equation (2) is thus the "certainty equivalent" solution (Ludwig and Walters 1982). The ratio between F_{MELSY} and F_{MSY} is illustrated in Figure 4. Differences in K″ tend to have less influence on this ratio than differences in either m or v′.

Other important special cases of Equation (20) include the limits as K″ approaches zero and infinity, which are shown respectively below:

$$\lim_{K''\to 0} F'_{MELSY} = \frac{1-m(1-v') - 2v'}{m(1-v')} \quad \text{and} \quad \lim_{K''\to\infty} F'_{MELSY} = \frac{1-m(1-v') - 2v'}{1+m(1-v') - 2v'}. \qquad (21) \text{ and } (22)$$

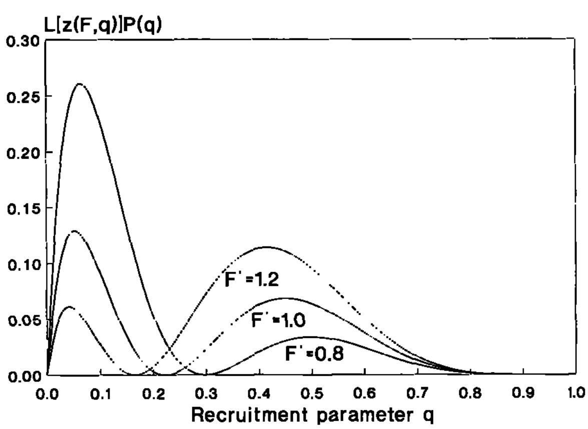

Figure 2
Risk under different F levels. The area under a curve is the risk associated with the F level that defines the particular curve. Parameter values used to generate these curves were K″ = 2.5, m = 0.2, and v′ = 1/11.

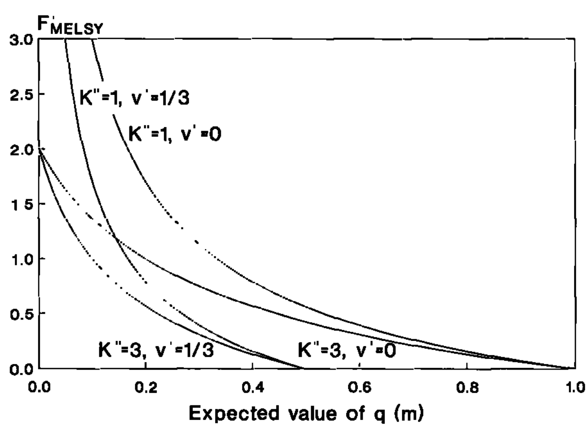

Figure 3
Values of F'_{MELSY} resulting from different combinations of parameter levels. F'_{MELSY} tends to decrease as K″, m, or v′ increases.

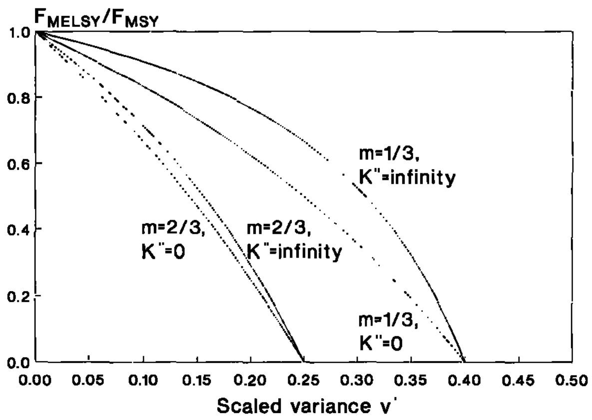

Figure 4
Ratio of F'_{MELSY} to F'_{MSY} under different combinations of parameter levels. The ratio tends to decrease as K'' decreases or as m or v′ increases.

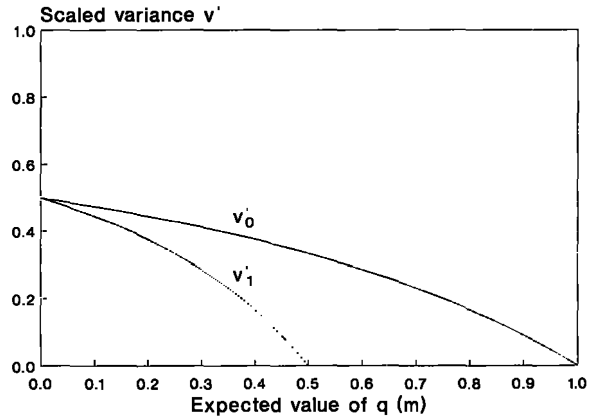

Figure 5
Limiting values of v′. The solid curve shows v'_0, the locus at which $F'_{MELSY}=0$. The dashed curve shows v'_1, the locus limiting the parameter subspace for which F'_{MELSY} can exceed 1. For (m, v′) combinations below the v'_1 curve, F'_{MELSY} can take any value, depending on K''. For (m, v′) combinations between the two curves, F'_{MELSY} can range between 0 and 1, again depending on K''. For (m, v′) combinations on or above the v'_0 curve, $F'_{MELSY}=0$.

Equation (20) also implies that F'_{MELSY} falls to zero whenever v′ reaches a critical value v'_0 defined as

$$v'_0 = \frac{1-m}{2-m}. \tag{23}$$

By Equation (19), v'_0 corresponds to a β value of 1. Whenever $\beta \leqslant 1$, the right-hand tail of the beta distribution fails to reach zero, implying a non-zero probability that $q=1$. When $q=1$, any positive F value causes the stock to go extinct. Given the preservationist attitude implicit in the logarithmic loss function, any possibility of extinction is unacceptable, so F'_{MELSY} drops to zero in this case. Note that F'_{MELSY} is never positive for values of v′ greater than 0.5.

Just as Equation (2) could be solved to determine the locus of parameter values under which F'_{MSY} takes on the special value of 1 (Eq. 3), Equation (20) can be solved to determine the following locus of parameter values under which $F'_{MELSY}=1$:

$$K'' = \frac{1-2v'}{m(1-v')} - 2. \tag{24}$$

In the certainty equivalent case, Equation (24) reduces to Equation (3). As K'' approaches zero, Equation (24) defines an upper limit on v′ (v'_1) for the special case where $F'_{MELSY}=1$:

$$v'_1 = \frac{1-2m}{2-2m}. \tag{25}$$

Under Equation (3), F'_{MSY} could exceed 1 only if q were less than 0.5. While Equation (25) implies essentially the same property (replacing F'_{MSY} with F'_{MELSY} and q with m), it adds a similar restriction on v′, namely that F'_{MELSY} can exceed 1 only if v′ is less than 0.5. [Note that this is a weaker version of the restriction implied by Equation (23). Equations (23) and (25) are compared in Figure 5.]

Biomass at MSY compared with biomass at MELSY

Dividing Equation (1) through by F gives equilibrium stock biomass. By substituting Equations (20) and (2) into this expression and setting $q=m$, the ratio of stock biomass at MSY to stock biomass at MELSY is given by

$$\frac{B(F_{MSY})}{B(F_{MELSY})} = \left[\left(\frac{F'_{MELSY}+1}{F'_{MSY}+1}\right)^2\left(\frac{K''+F'_{MSY}+1}{K''+F'_{MELSY}+1}\right)\right]^{\frac{1}{1-m}}, \tag{26}$$

with limits

$$\lim_{K''\to 0}\left(\frac{B(F_{MSY})}{B(F_{MELSY})}\right) = \left(\frac{1-2v'}{1-v'}\right)^{\frac{1}{1-m}} \quad \text{and} \quad \lim_{K''\to\infty}\left(\frac{B(F_{MSY})}{B(F_{MELSY})}\right) = \left(\frac{(1+m)(1-2v')}{(1+m)(1-v') - v'}\right)^{\frac{1}{1-m}}. \qquad (27) \text{ and } (28)$$

Equations (26–28) decline from a value of 1 at $v'=0$ to a minimum at $v'=v'_0$. The minimum value depends on K'' and m, but is never greater than $1/e$.

Estimating the parameters of the beta distribution

To fit the Cushing stock-recruitment curve to a set of n stock-recruitment data points, it seems reasonable to assume the following model:

$$y_i = \rho + qx_i + \varepsilon_i, \qquad (29)$$

where x_i represents the natural logarithm of the ith stock biomass datum, y_i represents the natural logarithm of the ith recruitment datum (lagged according to the age of recruitment), $\rho = \ln(p)$, and ε_i is an independent error term distributed as $N(0,\sigma^2)$.

Press (1989) presented a Bayesian approach to estimating the parameters of the pdf of q using Equation (29) as the underlying model. The following paragraphs summarize this presentation, which begins by rephrasing the problem in the form of Bayes' theorem:

$$h(q,\rho,\sigma \mid \mathbf{x}, \mathbf{y}) \propto \left(\prod_{i=1}^{n} f(y_i \mid x_i,q,\rho,\sigma)\right) g_1(q)\, g_2(\rho)\, g_3(\sigma), \qquad (30)$$

where $\mathbf{x}$ is the vector $(x_1, \ldots, x_n)'$; $\mathbf{y}$ is the vector $(y_1, \ldots, y_n)'$; $h(q,\rho,\sigma \mid \mathbf{x}, \mathbf{y})$ represents the posterior pdf of the parameters q, ρ, and σ; $f(y_i \mid x_i,q,\rho,\sigma)$ represents the conditional pdf of y_i given the observed value of x_i and any particular values of q, ρ, and σ; and $g_j(\cdot)$ represents the prior pdf of the jth parameter.

Given the assumptions implicit in Equation (29), $f(y_i \mid x_i,q,\rho,\sigma)$ can be written

$$f(y_i \mid x_i,q,\rho,\sigma) = \frac{\exp\left(\frac{-(y_i-\rho-qx_i)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}. \qquad (31)$$

A special case of interest is the one in which the $g_j(\cdot)$ are all "vague" (also called noninformative or indifference) priors. These are pdfs which reflect indifference regarding the probability of alternative parameter values. Press (1989) treated $g_1(q)$ and $g_2(\rho)$ as constants, implying that all values on the real line are equally likely in the prior distribution. Since σ is constrained to be positive, however, Press set $g_3(\sigma)=1/\sigma$, reflecting a uniform prior distribution for $\ln(\sigma)$.

Using Equation (31) and the priors specified by Press (1989), Eq. (30) gives a straightforward solution. The classical least-squares estimates of q and ρ ($\hat{q}$ and $\hat{\rho}$, respectively) obtain as the maximum-likelihood estimates. In their posterior pdf, q and ρ jointly follow a bivariate Student's t distribution, so that marginally the posterior pdf of q, $h_1(q \mid \mathbf{x}, \mathbf{y})$, follows a univariate 3-parameter t distribution with $n-2$ degrees of freedom:

$$h_i(q \mid \mathbf{x},\mathbf{y}) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\sqrt{\pi(n-2)\, s_q^2\left(1 + \frac{(q-\hat{q})^2}{(n-2)\, s_q^2}\right)^{n-1}}}, \qquad (32)$$

where s_q^2 is the estimated variance of $\hat{q}$ given by

$$s_q^2 = \frac{\sum_{i=1}^{n} (y_i - \hat{p} - \hat{q}x_i)^2}{(n-2)\sum_{i=1}^{n} (x_i - \bar{x})^2}. \tag{33}$$

For the present application, the solution given by Press (1989) needs to be modified in only one respect. His suggested form for $g_1(q)$ implies a uniform distribution over the entire real line, whereas here P(q) has been specified *a priori* to be zero for all values less than 0 or greater than 1. Given Equations (30) and (31), this implies that the suggested uniform shape for $g_1(q)$ should be truncated outside the range 0 to 1. This in turn implies that $h_1(q \mid \mathbf{x},\mathbf{y})$ should also be truncated outside the range 0 to 1 (and rescaled appropriately).

Strictly speaking, then, P(q) follows a truncated *t* distribution in this approach, rather than the hypothesized beta. However, a beta distribution can be made to approximate the truncated *t* by solving for m and v as follows:

$$m = \frac{\int_0^1 q\, h_1(q \mid \mathbf{x},\mathbf{y})\, dq}{\int_0^1 h_1(q \mid \mathbf{x},\mathbf{y})\, dq} \tag{34}$$

and

$$v = \frac{\int_0^1 (q-m)^2\, h_1(q \mid \mathbf{x},\mathbf{y})\, dq}{\int_0^1 h_1(q \mid \mathbf{x},\mathbf{y})\, dq}. \tag{35}$$

Applying the model to rock sole

As an illustration of the approach suggested above, the model can be applied to the eastern Bering Sea stock of rock sole *Pleuronectes bilineatus*. This stock is exploited by a multispecies flatfish fishery, and is also the target of an important roe fishery (Walters and Wilderbuer 1988).

The parameters to be estimated are K″, m, and v′. Thompson (1992) estimated K″ for this stock at a value of 3.279, and described a set of stock and recruitment data ($n=7$) which can be used to estimate m and v′. Fitting Equation (29) to these data gives $\hat{q}=0.235$ and $s_q^2=0.114$ (Fig. 6). Substituting these parameters into Equations (34) and (35) gives $m=0.369$ and $v=0.057$, with $v'=0.243$. The relationship between the truncated *t* distribution defined by these values and the beta approximation is shown in Figure 7 ($R^2=0.97$).

With parameter values $K''=3.279$, $m=0.369$, and $v'=0.243$, Equation (20) gives $F'_{MELSY}=0.365$. Multiplying through by M (set at 0.2 by Walters and Wilderbuer 1988) gives $F'_{MELSY}=0.073$. Substituting m for q in Equation (2) yields $F'_{MSY}=0.607$, or $F_{MSY}=0.121$. This value of F_{MSY} differs somewhat from the value of 0.176 given by Thompson (1992), which was based on the least-squares estimate of q ($\hat{q}$) instead of the

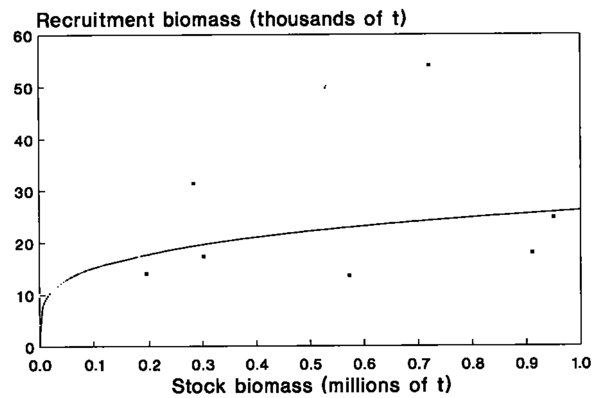

Figure 6
Stock-recruitment data and curve for eastern Bering Sea rock sole *Pleuronectes bilineatus*. Age-3 biomass (lagged 3 yr) is plotted against stock biomass for the years 1979–88. The curve is the least-squares fit.

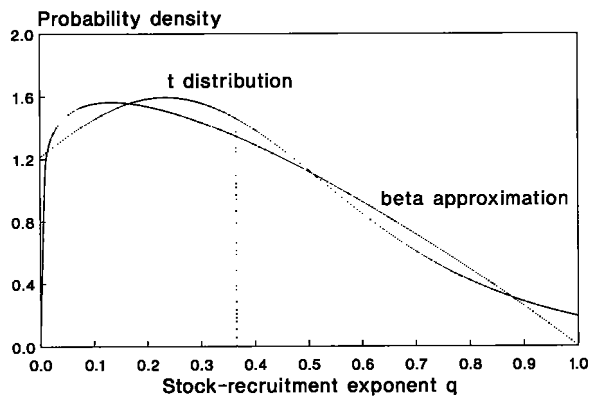

Figure 7
Comparison of truncated *t* and beta pdfs for the stock-recruitment exponent q in the eastern Bering Sea rock sole *Pleuronectes bilineatus* example.

Bayesian estimate (m). These two F_{MSY} values bracket the value of 0.155 which Walters and Wilderbuer (1988) derived from a surplus production model. Regardless of which F_{MSY} value is chosen, however, it exceeds F_{MELSY} by a significant amount.

Discussion

Evaluation of assumptions

The approach described here consists of three main components: the basic model represented by Equation (1), the logarithmic loss function, and the beta form for P(q). These components were chosen in part because they are tractable, making possible the analytic solution for F'_{MELSY} given by Equation (20). In addition, each has some degree of theoretical support, as described below.

The basic model The basic model was evaluated by Thompson (1992). In brief, the model includes terms for all of the requisite features of dynamic pool models (recruitment, growth, natural mortality, fishing mortality). The distinguishing features of the model (linear growth and a Cushing stock-recruitment relationship) satisfy the principal theoretical requirements for growth and stock-recruitment functions given by Schnute (1981) and Ricker (1975), respectively. Although the basic model is a simple one, it approximates more complicated models fairly well under a wide range of parameter values.

Logarithmic loss function The logarithmic loss function may require a bit more discussion. As mentioned earlier, this loss function is only one of several possibilities, two of the other most-common being the linear and quadratic forms. The principal argument against the linear loss function is that it implies strict risk neutrality, whereas most individuals tend to be at least somewhat risk-averse. Thus, if fishery managers tend to be risk-averse, a linear loss function would be inappropriate, except over a narrow range of yield values.

In contrast, the quadratic loss function implies a degree of risk aversion. In addition, the quadratic form has properties which prove convenient for a number of statistical applications. However, it has also been the subject of substantial criticism (Pratt 1964, Samuelson 1967, Box and Tiao 1973). Although the quadratic loss function does fall into the "risk-averse" category, this functional form manifests its risk aversion somewhat perversely by exhibiting increasing absolute risk aversion (Pratt 1964). In other words, a fishery manager using a quadratic loss function would be less willing to take risks as yields became higher.

The logarithmic loss function is another risk-averse alternative. It can be described as a special case of the isoelastic marginal loss function defined by $L(z)=(1-z^{\phi})/\phi$, where $\phi>0$ (the logarithmic case being obtained in the limit as ϕ approaches zero). Unlike the quadratic loss function, isoelastic marginal loss functions exhibit decreasing absolute risk aversion (Pratt 1964). Isoelastic marginal loss functions also display the convenient property of constant *relative* risk aversion R(z), defined as $-zL''(z)/L'(z)$ (Pratt 1964). Specifically, $R(z)=1-\phi$ for the isoelastic marginal loss family. The logarithmic case, where $R(z)=1$, thus represents a clear risk-averse alternative to the risk-neutral linear loss function, where $\phi=1$ and $R(z)=0$.

The fact that the logarithmic loss function tends toward negative infinity as the resource approaches extinction may be viewed as problematic by some. On the other hand, Smith (1985) views this behavior as a requisite characteristic for any loss function to be used in the context of renewable resources, arguing that it "introduces a useful conservation motive into the decision making process." Opaluch and Bockstael (1984) go even further, stating, "It is well known that the log function exhibits the best properties of the simple functional forms. . . ."

Beta probability density function The principal justification for using the beta pdf to describe P(q) is that the beta is a natural choice for the pdf of any continuous variable which is constrained to fall within the 0 to 1 range. The fact that it allows for an explicit solution to Equation (7) is another argument in its favor.

Unfortunately, the method presented here for estimating the parameters of P(q) is based on a model (Press 1989) which yields a truncated t distribution, not a beta distribution. If this model is accepted as a true description of reality, then the beta form for P(q) is only an approximation. Of course, most functional forms used in modeling are only approximations, so the question is whether the advantages of increased tractability provided by the beta distribution outweigh any attendant losses of accuracy. Holloway (1979) argues in the affirmative after noting the difficulty of identifying natural processes which yield the beta distribution as a formal result.

In general, the effectiveness of Bayes decisions is relatively insensitive to small changes in the assumed pdf (DeGroot 1970). This being the case, the question really is whether the difference between the truncated t distribution and the beta approximation is typically small. To assess the magnitude of this difference, the goodness-of-fit between the truncated t and beta distributions was examined for a wide range of n, $\hat{q}$, and s_q^2 values (Fig. 8). Note that $R^2>0.95$ for a wide range of parameter values, indicating that the loss of accuracy resulting from the beta approximation is often small.

Another fact to keep in mind is that the model presented by Press (1989) is only one possibility. Despite the pessimism conveyed by Holloway (1979), it is conceivable that other models could yield the beta distribution as an exact result.

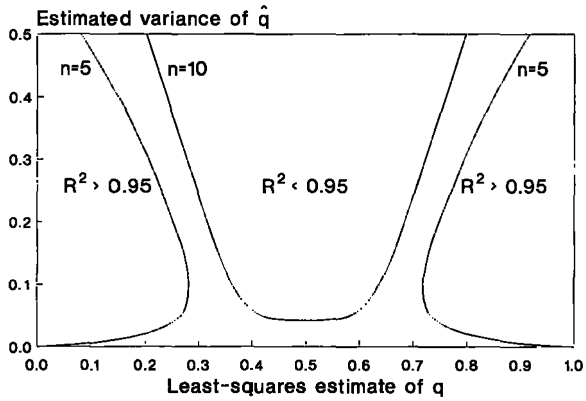

Figure 8

Loci of parameter values under which a beta approximation to the truncated t distribution gives an R^2 value of 0.95. R^2 was calculated by comparing the two distributions at q values of 0.01, 0.02, . . ., 0.99. For $n=5$, parameter combinations lying to the interior of the two curves correspond to R^2 values <0.95. For $n=10$, R^2 values <0.95 correspond to parameter combinations lying above the curve.

Comparison with previous studies

Of the many previous applications of Bayesian decision theory to fisheries, the studies by Ludwig and Walters (1982), Clark et al. (1985), and Walters and Ludwig (1987) are most closely related to the present work. The various features of the four approaches are outlined in Table 1. The three previous studies exhibit certain common features which distinguish them from the present study, namely: (1) use of a discrete time scale; (2) inclusion of an explicit adaptive management strategy; (3) inclusion of environmental stochasticity as well as parameter uncertainty; (4) inclusion of a positive discount rate in the objective function; (5) assumption of a normal form for the pdf of the uncertain parameter; and (6) inability to derive an exact analytic solution, even in the myopic case (except for one special instance considered by Clark et al.). The present study is also the only one of the group which includes both a biomass-based model and a risk-averse loss function.

Ludwig and Walters (1982) found that the deterministic optimum escapement level can be less than half the value of the Bayesian solution. Although the continuous form of the model used in the present study makes it difficult to talk about escapement per se, equilibrium stock size might serve as a suitable proxy

Table 1

Comparison of four studies describing Bayesian approaches to fishery management.

Feature	Ludwig and Walters (1982)	Clark et al. (1985)	Walters and Ludwig (1987)	This study
Time scale	discrete	discrete	discrete	continuous
Yield metric	numbers	biomass	numbers	biomass
Adaptive strategy included	yes	yes	yes	no
Age structure included	no (discrete generations)	yes	no (discrete generations)	yes
Discounting included	yes	yes	yes	no
Harvesting costs included	no	yes	no	no
Stochasticity included	yes	yes	yes	no
Loss function	linear	linear	logarithmic	logarithmic
Growth function	none	isometric von Bertalanffy	none	linear
Stock-recruitment function	Ricker (1954)	a) Cushing b) stock-independent c) linear-threshold	Cushing	Cushing
Uncertain parameter	Ricker exponent	a) ln(Cushing multiplier) b) mean ln(recruitment) c) mean ln(recruitment)[1]	Cushing exponent (q)	q
Pdf	normal	normal	normal	beta
Analytic solution obtained	no	case (b) (myopic only)	approximate[2] (myopic only)	yes

[1] Only recruitment data from stock sizes above the threshold were used to calculate the mean.
[2] Approximate solution valid only for pdfs with variance <0.01.

for comparison with the results of Ludwig and Walters. As Equations (26–28) indicate, a variety of parameter combinations allow for $B(F_{MSY})$ to be less than half of $B(F_{MELSY})$. Since the results presented by Ludwig and Walters (1982) were derived from a numbers-based model, Equation (27) is particularly relevant. Under this equation, a v′ value greater than 1/3 is sufficient to guarantee that the stock size at MSY will be less than half the stock size at MELSY, regardless of the value of m. At values of $m>0.5$, a v′ value of 0.227 is sufficient.

Clark et al. (1985) found that the relationship between the myopic Bayes and certainty-equivalent solutions depended on the model used. In the special case where recruitment is independent of stock size, for example, they found that the myopic Bayes solution always exceeded the certainty equivalent solution. For the same model, the authors also found that the myopic Bayes solution always increased with the level of uncertainty. These results are precisely the opposite of those obtained in the present study, where F_{MELSY} is always less than F_{MSY} and decreases monotonically with v′. In their "full cohort model" with a stock-recruitment relationship, however, Clark et al. (1985) obtained results similar to those of the present study. In one example, the myopic Bayes solution prescribed a 30–50% reduction in F relative to the certainty-equivalent solution. Using yet another model, Walters and Ludwig (1987) also found that the myopic Bayes solution was a monotone-decreasing function of uncertainty.

Conclusion

This paper describes an approach for treating the problem of parameter uncertainty in a systematic fashion. Although fisheries are often managed as though stock parameters are known with certainty, it would be preferable to develop a management approach more consistent with the fact that such certainty is the exception rather than the rule. Such an approach was developed here in the context of Bayesian decision theory. When applied to the particular model presented, this approach indicates that the optimal fishing mortality rate F_{MELSY} (Eq. 20) is always less than F_{MSY} (Eq. 2) except in the limiting case where q is known with certainty (Fig. 4).

This result provides formal support for the intuitive conclusion (e.g., Kimura 1988) that fishing mortality should be strongly constrained when the stock-recruitment relationship is uncertain. Similarly, Equation (25) indicates that if recruitment is highly dependent on stock size (specifically, if m exceeds 0.5), F_{MELSY} will always be less than the natural mortality rate.

The rock sole example illustrates the basic conservatism of the F_{MELSY} approach. In this example, F_{MELSY} was less than F_{MSY} by about 40%. Given that neither the F_{MSY} value (0.121) nor the fit from the stock-recruitment regression (Fig. 6) was atypical of groundfish stocks, the ratio between F_{MELSY} and F_{MSY} in this example provides a practical illustration of the extent to which an explicit accounting for uncertainty can influence management strategy. The magnitude of the effort reduction prescribed in this example is similar to results described by Ludwig and Walters (1982) and Clark et al. (1985). The confirmatory nature of these studies may suggest that the conventional wisdom regarding optimal exploitation rates should be reexamined. At the very least, the F_{MELSY} approach provides a low-end estimate of the maximum acceptable harvest rate and a warning against taking F_{MSY} estimates too seriously.

A great deal of the conservatism resulting from the F_{MELSY} approach as developed here stems from the assumption that all values of q are logically possible, despite the fact that a q value of 1 results in extinction under any level of fishing. One alternative might be to examine q in the context of life-history theory, to determine if it is possible to justify some other upper limit on the logically permissible range. A related alternative would be to use a nonuniform prior in estimating P(q). The assumption of a uniform prior may be overly pessimistic, since fishery biologists often have an intuitive feel for stock-recruitment parameters, even in the absence of data for a particular stock. Such information could be used to define an alternate prior pdf. Another possibility would be to establish an empirical prior based on the results of other stock-recruitment studies, but this would likely require a fairly elaborate weighting scheme so that stock-recruitment parameters from the most dissimilar stocks or environments would have the least influence on the form of the resulting pdf.

An additional factor which may add to the conservatism of the F_{MELSY} strategy as developed here is the use of the myopic Bayes solution rather than an actively adaptive solution. An actively adaptive solution would attempt to anticipate and make use of changes in available information resulting from alternative management actions (e.g., Walters and Hilborn 1976, Smith and Walters 1981, Ludwig and Walters 1982, Ludwig and Hilborn 1983, Clark et al. 1985, Walters 1986, Milliman et al. 1987, Walters and Ludwig 1987, Parma 1990, Parma and Deriso 1990). However, myopic Bayes (or similar) solutions often perform nearly as well as their actively adaptive counterparts (Mendelssohn 1980, Walters and Ludwig 1987, Parma 1990, Parma and Deriso 1990), and if the myopic Bayes solution is reestimated each year, the result is a passively adaptive strategy which is asymptotically optimal over time (Walters 1987). Most important for the purposes of the present study, though, is the fact that the myopic Bayes solution is computationally much simpler than the actively adaptive solution.

In conclusion, it should be stressed that while the approach suggested here was developed in the context of a particular model and particular loss and probability density functions, this development was meant primarily to illustrate the approach, not to limit it. More sophisticated applications—utilizing alternative assumptions, functional forms, and solution techniques—are certainly to be encouraged. In particular, future research might incorporate recruitment stochasticity, positive discount and cost rates, additional objective function components (e.g., yield variability), and uncertainty in other parameters and variables (e.g., the natural mortality rate, growth rate, and stock size).

Acknowledgments

I would like to thank James Balsiger, Nicholas Bax, Russell Kappenman, Daniel Kimura, Richard Methot, and Thomas Wilderbuer of the Alaska Fisheries Science Center for reviewing all or portions of this paper in various stages of development. Three anonymous reviewers also provided helpful suggestions. In addition, I would like to thank Robert Burr and Loveday Conquest of the University of Washington's Center for Quantitative Science for their assistance.

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