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Transitions to chaos induced by additive and multiplicative noise

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For a class of multistable systems, deterministic and stochastic chaos are closely related mathematically; a necessary condition for the occurrence of noise-induced chaos with sensitive dependence on initial conditions can be derived from the generalized Melnikov function. Proof that this condition is applicable requires the approximate representation of the noise in the form of a modified Shinozuka process or other uniformly continuous and uniformly bounded processes. Additive and/or multiplicative Gaussian noise with any spectral density can be accommodated, as can other types of noises, including shot noise and non-Gaussian noise. We review recent results, including a successful verification of our Melnikov-based approach against results based on a solution of the Fokker-Planck equation. We conclude by briefly describing ongoing research.

1.INTRODUCTION

Multistable systems subjected to noise excitation can exhibit irregular jumps between regions of phase space associated with the competing attractors of the noise-free systems. Such behavior has been referred to as "motion with noise-induced jumps" [1], "basin hopping" [2], or "stochastic chaos" [3].

Motion with noise-induced jumps can be visually indistinguishable from deterministic chaos, as can be seen in Figs. 1a and 1b, which represent the displacements of a Duffing-Holmes oscillator forced by a realization of a broadband stochastic process and by a harmonic excitation, respectively. The question has therefore arisen of whether certain certain apparently chaotic motions (e.g., hydroelastic motions of a double galloping oscillator [4], or the time evolution of Belousov-Zhabotinsky (BZ) reactions [3]) are stochastic or deterministic. Recently it has been shown that the correlation dimension [5], the spectral density [2, 6], or the largest Lyapounov exponent [7] can fail to discriminate between the two types of motion.

We show in Section 2 that, in a wide class of stochastic dynamical systems, sensitivity

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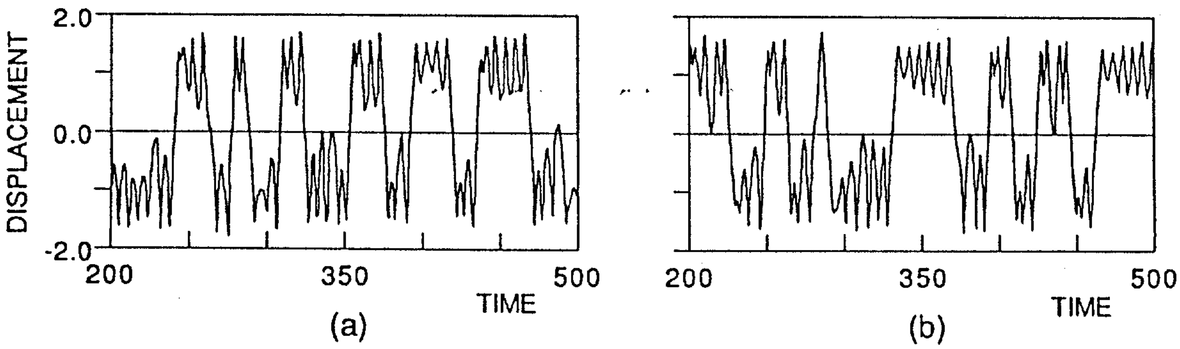

Figure 1. Time histories of motions of Duffing-Holmes oscillator subjected to: (a) broadband stochastic forcing; (b) harmonic forcing.

to initial conditions (SIC) can be induced by additive and/or multiplicative noise, just as in the deterministic counterparts of those systems it can be induced by a change in the perturbation parameters. [7]. In Section 3 the approach used in Section 2 is verified for a particular case against results obtained by using the Fokker-Planck equation. Section 4 outlines current research.

2. ONE-DEGREE-OF-FREEDOM SYSTEMS

In this section we consider first one-degree-of-freedom systems

$$\ddot{z}(t) = -V'(z) + \epsilon[\gamma g(t) + \rho G(t) - \beta \dot{z}] \tag{1}$$

where V is an energy potential, g and G represent deterministic and stochastic forcing functions, respectively. g is assumed to be bounded and uniformly continuous. The parameters ρ, γ and β are nonnegative and fix the relative amounts of external forcing and damping in the system. The unperturbed counterpart of the system ($\epsilon=0$) is assumed to have two hyperbolic fixed points connected by a heteroclinic orbit $(z_s(t),\dot{z}_s(t))$. If the two hyperbolic points coincide the orbit is homoclinic. To show that $\epsilon\rho G(t)$ can induce SIC we represent the noise process so that expressions for the generalized Melnikov function [8] can be written for the noisy system.

2.1 Noise representations

The generalized Melnikov function [8] is meaningful provided that normally hyperbolic invariant sets and their stable and unstable manifolds persist under perturbation. This in turn requires the approximate representation of the noise G(t) as a uniformly continuous and uniformly bounded process. A modified Shinozuka representation meets this requirement. For zero mean noise with unit variance and one-sided spectral density $2\pi\Psi(\omega)$ its expression is [7]:

$$G_t = (2/N)^{1/2} \sum_{n=1}^{N} [\sigma/S(\omega_n)]\cos(\omega_n t+\phi_n) \tag{2}$$

where $\{\omega_n,\phi_n; n=1,..,N\}$ are independent random variables, $\{\omega_n; n=1,..,N\}$ are nonnegative with common distribution $\Psi_o(\omega)=S^2(\omega)\Psi(\omega)/\sigma^2$, $\{\phi_n; n=1,2,..,N\}$ are identically uniformly distributed over $[0,2\pi]$, N is a parameter of the model, $S(\omega)$ is the modulus of the Fourier transform of $\dot{z}_s(-t)$, and

$$\sigma^2 = \int_0^\infty S^2(\omega)\Psi(d\omega). \tag{3}$$

An expression for shot noise that also meets the requirements of uniform continuity and uniform boundedness was developed in [9].

2.2 Generalized Melnikov Function

The linear filter F with impulse response $\dot{z}_s(-t)$ is referred to as the orbit filter. The terms associated with the forcing in the generalized Melnikov function for Eq. 1, $M(t_1,t_2)$, may be interpreted as the output of the orbit filter with input $\gamma g(t)+\rho G_t$ [7]. The expectation of M is $-I\beta + \gamma F[g(t)]$, where

$$I = \int_0^\infty \dot{z}_s^2(t)dt, \tag{4}$$

and its variance is $\rho^2\sigma^2$. The necessary condition for the occurrence of chaos with SIC and with jumps due to chaotic transport is that M have simple zeros. Like G, $M(t_1,t_2)$ is a Gaussian process in the limit $N \to \infty$. Therefore, in the Gaussian limit, the presence of even vanishingly small noise causes the Melnikov function to have simple zeros. The state of the system is thus driven from one basin of attraction to that of the competing attractor. Such motion is interpretable as chaotic motion on a single strange attractor [7].

It is shown in [7] that the average flux factor -- a measure of chaotic transport in the perturbed system -- is approximately

$$\Phi = E[(\gamma A+\rho\sigma Z-I\beta)^+] \tag{5}$$

where Z is the standard Gaussian random variable. The error in this approximation decreases to zero as $N \to \infty$.

Similar results were obtained in [9] for systems with multiplicative noise.

distributions for certain lower bounds of exit times.

3. MEAN TIME BETWEEN PEAKS – BRUNDSEN-HOLMES OSCILLATOR

Consider the Brundsen-Holmes oscillator, defined by Eq. 1 in which $V(z) = -z+z^3$, $\gamma=0$, $G(t)$ denotes white noise, and β, instead of denoting a constant, denotes the function $\delta-kz^2$ where δ and k are constants [10]. The mean time between successive maxima of $|z(t)|$ can be shown to be

$$T = K - (1/\lambda_u)\ln(\mathbf{M}) \tag{6}$$

[10], where K is a constant, λ_u is the eigenvalue associated with the unstable manifold linearized about the saddle point, and **M** is proportional to the average Melnikov

distance separating the stable and unstable manifolds of the perturbed system. In Ref. 10 the following result was obtained from Eq. 6 through the application of the Fokker-Planck equation:

$$T = K_1 - (1/\lambda_u)\ln(\epsilon\rho) \tag{7}$$

For forcing given by Eq. 2, it follows from the definition of the generalized Melnikov function (Section 2.2) that **M** has the same distribution as the ensemble average of the modulus of the Melnikov function at time t = 0 [6]. For N → ∞ this yields

$$\mathbf{M} = \rho\sigma/(2\pi)1/2 \int_0^\infty |y|\exp[-(1/2)y^2]dy = \rho\sigma(2/\pi)^{1/2} \tag{8}$$

Substitution of Eq. 8 into Eq. 6 yields Eq. 7, i.e., the results yielded by the approach of Section 2.2 and the approach of Ref. 10 are the same.

4. ONGOING RESEARCH

Research on the following topics is currently in progress:

1. Probability that chaotic transport can occur during specified time interval.
2. Transitions to chaos for non-Gaussian and tail-limited noise [11].
3. Extension of results obtained for one-degree-of-freedom systems to (a) systems with higher dimension, and (b) spatially extended systems.
4. Application of our results to investigation of chaos induced by turbulent low-frequency wind fluctuations in model of quasi-geostrophic ocean flow proposed by Allen et al. [12].

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