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Deterministic and Stochastic Chaos

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ABSTRACT

Stochastic differential equations and classical techniques related to the Fokker-Planck equation are standard bases for the analysis of nonlinear systems perturbed by noise. An alternative, complementary approach applicable to systems featuring heteroclinic or homoclinic orbits uses phase space flux as a measure of noise-induced chaotic dynamics. We continue our development of this method, extending our previous treatment of additive noise to the more general case of multiplicativ noise. This extension is used with a new model of shot noise to treat the Duffing osc...tor with shot noise-like dissipation.

INTRODUCTION

Two fundamental paradigms are used to account for the seemingly unpredictable and erratic motions exhibited by many dynamical systems: the first is that of a dynamical system perturbed by noise - a differential equation, say, driven by white noise or jump noise. This type of motion is called stochastic chaos and is typically studied using stochastic differential equations and classical techniques related to the Fokker-Planck equation. Stochastic chaos exhibited by nonlinear systems is an ongoing object of intense interest [1, 2, 3, 4, 5]. Noise-induced state transitions in nonlinear systems, in particular, have received much recent attention [6, 7, 8, 9].

The second paradigm of erratic motion posits a purely deterministic dynamical system with 1) an uncertainty (perhaps very, very small) in its initial state and 2) a flow structure admitting intersecting stable and unstable manifolds. Such a system is capable of bounded motion which becomes increasingly unpredictable with the time evolution of the system. In effect, the special structure of the system flow amplifies the uncertainty in the initial state of the system to the point that system states sufficiently far in the future are essentially unpredictable. Such systems are said to be sensitively dependent on initial conditions and the motion they exhibit is termed deterministic chaos.

Understanding of each of these paradigms has now matured to the point where they are being actively compared and contrasted in many ways. We mention three important studies of this type. Arecchi, Badii and Politi [10], investigating the effect of noise on the forced Duffing oscillator in the region of parameter space where different attractors coexist, found that the noise may be viewed as inducing jumps between attractors with the noise-induced transitions obeying simple kinetic laws. In related work, Kautz [11] obtained basin of attraction rate of escape results for a Josephson junction excited by thermal noise. Taking a different approach, Kapitaniak [12] studied numerical solutions of the Fokker-Planck equation obtained for randomly and periodically forced nonlinear oscillators. Choosing oscillator parameters known to result in deterministic chaos, he found that the probability density function of the motion exhibited multiple spiked maxima.

Deterministic chaos and stochastic chaos are not mutually exclusive; one may have a system sensitively dependent upon initial conditions which is randomly perturbed by noise. Indeed, the presence of noise is inevitable in any real system. This fact undermines any attempt to identify system dynamics as simply deterministic chaos or stochastic chaos. Nevertheless, in a line of work beginning with Sigeti and Horsthemke [13], the spectrum of the system dynamics has been investigated as a way to distinguish deterministic chaos from noise-driven stochastic chaos. Sigeti and Horsthemke argued that the two types of motion could be distinguished by the order of the rate of spectral decay. The practical limitation of this approach has been quantified in a series of investigations of weakly forced systems with attracting homoclinic/heteroclinic orbits. Brunsden and coworkers [14, 15] found the power spectrum of the chaotic motion for the noiseless case of this class of systems, basing their derivation on the crucial assumption (affirmed by numerical simulations) that the time history of the motion could be represented as a random superposition of deterministic structures. Corresponding results were obtained by Stone and Holmes [16] and Stone [17] for white noise perturbation and then by Simiu and Frey [18] for colored noise plus periodic forcing, the conclusion in each case being that the spectrum of noise-induced motion and that of deterministic chaos are essentially (to first order) indistinguishable. A recent nonspectral approach is that of Kennel and Isabelle [19]. They propose a computational method of distinguishing between deterministic chaos and stochastic chaos with similar spectra based on the short-term nonlinear predictability of the system dynamics.

Model systems in which the motion is a combination of both deterministic and stochastic chaos are ideal for investigating the relationship between stochastic and deterministic chaos. Grassberger and Procaccia [20] and Ben-Mizrachi, Procaccia and Grassberger [21] have proposed a scheme based on the correlation dimension to investigate this relationship. They theorize that in the case of noise-perturbed systems the correlation dimension increases with increases in the embedding dimension while for deterministic chaos it is constant. For deterministic chaos perturbed by weak noise they predict that the correlation dimension will increase with the embedding dimension to a point and then stop. However, examples contradicting this have been reported by Fichthorn, Gulari and Ziff [22] and by Chen [23].

In systems whose motion is a combination of deterministic and stochastic chaos the role of noise in the supression or promotion of deterministic chaos is an area of active study. Typically, systems must exceed a certain parametric threshold for deterministic chaos to occur. Noise-induced changes to this threshold were first considered for cases of discrete-time systems. Mayer-Kress and Haken [24] and Crutchfield and Huberman [25] found that for the logistic map external noise broadened the power spectrum and caused the maximal Liapounov exponent to switch from negative to positive - two indications of a transition to chaos. Tsuda and Matsumoto [26, 27], looking for similar behavior in the B-Z map, found that the introduction of external noise produced spikes in the spectrum, indicating the suppression of chaos. More recently, Kapitaniak [12] defined a random maximal Liapounov exponent and reported that, for the class of continuous-time systems he considered, the introduction of weak noise tended to decrease this quantity. Interpreting Kapitaniak's work to suggest that weak noise may suppress deterministic chaos that would otherwise occur in the absence of noise, Bulsara, Schieve and Jacobs [28], [29] formulated a theory to account for this using the Melnikov function - a key quantity from Melnikov's theory of separated manifolds. These investigators redefined the Melnikov function to address the presence of weak noise and found that it acted to raise the parametric threshold, suppressing deterministic chaos. In an analysis of this work, Simiu, Frey and Grigoriu [30] concluded that this redefined Melnikov function addressed only one second-order effect of the noise. Taking a different tack which obviated the need to redefine the Melnikov function, Simiu et al. concluded that in the weak noise limit the parametric threshold for chaos was, for a wide class of continuous-time systems, never raised by the presence of noise. This conclusion was further developed by Frey and Simiu in [31] using the notion of phase space flux transport. An application of this methodology to the technologically interesting case of systems perturbed by noise with finite-tailed marginal distributions (e.g., wave heights limited by physical factors) was given by Simiu and Grigoriu in [32].

This brief survey of recent work in stochastic and deterministic chaos shows that many fundamental questions remain unresolved and it will, we hope, stimulate further interest in the subject. The remainder of this chapter is divided into five sections. In the first two sections we briefly review the calculation in [31] of phase space flux for systems perturbed by additive near-Gaussian noise and present a new calculation of the flux factor in the case of the Duffing oscillator with additive near Gaussian noise. In the third section, the more general case of multiplicative noise is treated. Presented in the following section is a new model of shot noise tailored to the requirements of Melnikov's method. This shot noise model is analogous to the modified Shinozuka noise model used to represent Gaussian noise in [31]. In the last section, we treat the Duffing oscillator with shot noise-like dissipation as a system with multiplicative noise and calculate the flux factor.

ADDITIVE EXCITATION

We consider the integrable, two-dimensional, one-degree-of-freedom Newtonian dynamical system [33] with energy potential V governed by the equation of motion

$$\ddot{x} = -V'(x), \quad x \in \mathcal{R}. \tag{1}$$

System (1) is assumed to have two hyperbolic fixed points connected by a heteroclinic orbit $\vec{x}_s = (x_s(t), \dot{x}_s(t))$. If the two hyperbolic fixed points coincide, then $\vec{x}_s$ is homoclinic. A perturbative component is introduced into system (1), giving

$$\ddot{x} = -V'(x) + \varepsilon w(x, \dot{x}, t). \tag{2}$$

The perturbative function $w : \mathcal{R}^2 \times \mathcal{R} \rightarrow \mathcal{R}$ is assumed to satisfy the Meyer-Sell uniform continuity conditions [34] and only the near-integrable case, $0 < \varepsilon \ll 1$, is considered. In this section we restrict our attention to the case of additive excitation and linear damping treated in [31]. For this case,

$$w(x, \dot{x}, t) = \gamma g(t) + \rho G(t) - \kappa \dot{x} \tag{3}$$

and system (2) takes the form

$$\ddot{x} = -V'(x) + \varepsilon[\gamma g(t) + \rho G(t) - \kappa \dot{x}]. \tag{4}$$

Here g and G represent deterministic and stochastic forcing functions, respectively. g is assumed to be bounded, $|g(t)| \leq 1$, and uniformly continuous (UC). The parameters ρ, γ and κ are nonnegative and fix the relative amounts of damping and external forcing in the model.

The random forcing G in (3) is taken to be a randomly weighted modification of the Shinozuka noise model [35], [36],

$$G(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \frac{\sigma}{S(\nu_n)} \cos(\nu_n t + \varphi_n). \tag{5}$$

where $\{\nu_n, \varphi_n; n = 1, 2, \ldots, N\}$ are independent random variables defined on a probability space $(\Omega, \mathcal{B}, P)$, $\{\nu_n; n = 1, 2, \ldots, N\}$ are nonnegative with common distribution Ψ_o, $\{\varphi_n; n = 1, 2, \ldots, N\}$ are identically uniformly distributed over the interval $[0, 2\pi]$ and N is a fixed parameter of the model.

Let $\mathcal{F}$ denote the linear filter with impulse response $h(t) = \dot{x}_s(-t)$ where $\dot{x}_s(t)$ is the velocity component of the orbit $\vec{x}_s$ of system (1). $\mathcal{F}$ is called the system orbit filter and its output is $\mathcal{F}[u] = u * h$ where $u = u(t)$ is the filter input and $u * h$ is the convolution of u and h. S in (5) is then defined to be modulus $S(\nu) = |H(\nu)|$ of the orbit filter transfer function

$$H(\nu) = \int_{-\infty}^{\infty} h(t)e^{-j\nu t}dt \tag{6}$$

and σ in (5) is

$$\sigma = \int_{0}^{\infty} S^2(\nu)\Psi(d\nu).$$

Let the distribution Ψ_o of the angular frequencies ν_n in (5) have the form

$$\Psi_o(A) = \frac{1}{\sigma^2}\int_A S^2(\nu)\Psi(d\nu) \tag{7}$$

where A is any Borel subset of $\mathcal{R}$. S is assumed to be bounded away from zero on the support of Ψ, $S(\nu) > S_m > 0$ a.e. Ψ. Under this condition S is said to be Ψ-admissible. If S is Ψ-admissible, then it is also bounded away from zero on the support of Ψ_o and $1/S(\nu_n) < 1/S_m$ a.s. Ψ_o. We have the following results for G and its filtered counterpart $\mathcal{F}[G]$.

Fact G1: G and $\mathcal{F}[G]$ are each zero-mean and stationary.

Fact G2: If S is Ψ-admissible then G is uniformly bounded with $|G(t,\omega)| \leq \sqrt{2N/S_m}$ for all $t \in \mathcal{R}$ and $\omega \in \Omega$.

Fact G3: The marginal distribution of $\mathcal{F}[G]$ is that of the sum

$$\sigma\sqrt{\frac{2}{N}}\sum_{n=1}^{N}\cos U_n$$

where $\{U_n; n = 1, \ldots, N\}$ are independent random variables uniformly distributed on the interval $[0, 2\pi]$.

Fact G4: G and $\mathcal{F}[G]$ are each asymptotically Gaussian in the limit as $N \to \infty$. In particular, the random variables $G(t)$ and $\mathcal{F}[G](t)$ are, for each t, asymptotically Gaussian.

Fact G5: The spectrum of G is $2\pi\Psi$ and G has unit variance.

Fact G6: The spectrum of $\mathcal{F}[G]$ is $2\pi\Psi_o$ and its variance is σ^2.

Fact G7: Let the spectrum Ψ of G be continuous. Then $\mathcal{F}[G]$ is ergodic.

Proof of the first six of these results can be found in [31]. Fact G7 is related to the fact that Gaussian processes with continuous spectra are ergodic [37, 38]. Five realizations of G with bandlimited spectrum are shown for comparison in Figure 1 together with

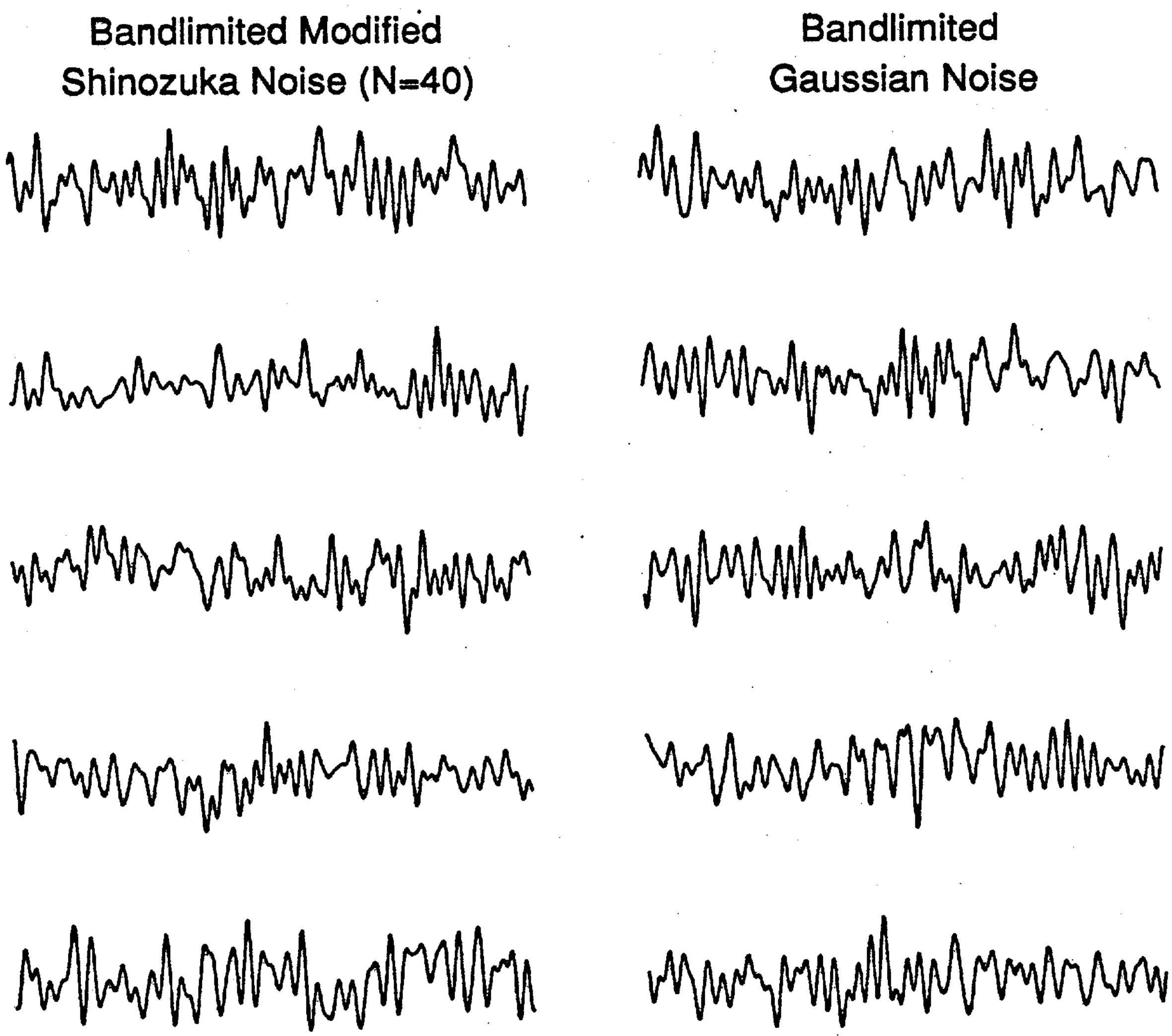

Figure 1. Realizations of modified Shinozuka and Gaussian noise processes with identical bandlimited spectra and $S(\nu) = \mathrm{sech}\,\nu$.

five realizations of Gaussian noise with the same spectrum. $S(\nu) = \text{sech}\nu$ is used in this example.

The system orbit filter $\mathcal{F}$ enters into the construction of G in two ways. First, S appears as a random scaling factor in (5) and, second, S appears in (7) in the expression for the distribution Ψ_o of the frequencies ν_n. Concerning the effect on the mean, covariance and spectrum of G, these two uses of S completely cancel one another. Effectively, S is a free parameter of the model subject only to the constraint that it be Ψ-admissible. Although the choice of S has no effect on G, it does significantly affect the filtered process $\mathcal{F}[G]$. Choosing the parameter S to be the modulus of the orbit filter makes the process $\mathcal{F}[G]$ ergodic.

Let us now consider the effect of the perturbation $\varepsilon w(x, \dot{x}, t)$ on the global geometry of (1). For sufficiently small perturbations, the hyperbolic fixed points of (1) persist and remain hyperbolic and the stable and unstable manifolds associated with the orbit of (1) separate [39]. The distance between the separated manifolds is expressible as an asymptotic expansion $\varepsilon M + O(\varepsilon^2)$ where M is a computable quantity called the Melnikov function. The separated manifolds may intersect transversely and, if such intersections occur, they are infinite in number and define lobes marking the transport of phase space [40]. The amount of phase space transported, the phase space flux, is a measure of the chaoticity of the dynamics [41]. The lobes defined by the intersecting manifolds generally have twisted, convoluted shapes whose areas are difficult to determine, making analytical calculation of the flux difficult, if not impossible. For the case of small perturbations, however, the phase space flux can be expressed in terms of the Melnikov function. The average phase space flux has the asymptotic expansion $\varepsilon\Phi + O(\varepsilon^2)$ where Φ, here called the flux factor, is a time average of the Melnikov function:

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} M^+(\theta_1 - t, \theta_2 - t)dt \tag{8}$$

where M^+ is the maximum of M and 0.

To apply Melnikov theory to a deterministic excitation g, g must be bounded and UC. In the case of random perturbations G, the theory requires that G be uniformly bounded and uniformly continuous across both time and ensemble. The noise model G in (4) is uniformly bounded as noted in Fact G2. However, G does not necessarily have the needed degree of continuity.

We define a stochastic process X to be ensemble uniformly continuous (EUC) if, given any $\delta_1 > 0$, there exists $\delta_2 > 0$ such that if $t_1, t_2 \in \mathcal{R}$ and $|t_1 - t_2| < \delta_2$ then $|X_{t_1}(\omega) - X_{t_2}(\omega)| < \delta_1$ for all $\omega \in \Omega$. A stochastic process can have UC paths and fail to be EUC. The derivative $G'(\omega)$ of the noise path $G(\omega)$ is bounded,

$$|G_t'(\omega)| \leq \frac{\sigma}{S_m}\sqrt{\frac{2}{N}} \sum_{n=1}^{N} \nu_n(\omega)$$

for all $t \in \mathcal{R}$. Thus G is EUC if the sum of its angular frequencies $\{\nu_1, \ldots, \nu_N\}$ is bounded. This sum is bounded if and only if G is bandlimited. Thus G is EUC if it is bandlimited.

Conditions on the perturbation function w sufficient for the Melnikov function to exist are [34]: for every compact set $\mathcal{K} \in \mathcal{R} \times \mathcal{R}$, (i) w is UC on $\mathcal{K} \times \mathcal{R}$ and (ii) there is a constant k such that

$$|w(x_1, \dot{x}_1, t) - w(x_2, \dot{x}_2, t)| \leq k(|x_1 - x_2| + |\dot{x}_1 - \dot{x}_2|)$$

for all $t \in \mathcal{R}$ and $(x_1, \dot{x}_1), (x_2, \dot{x}_2) \in \mathcal{K}$. These conditions are met in (3) provided g is UC and G is EUC. Under these conditions, the Melnikov function for system (4) is given by the Melnikov transform $\mathcal{M}[g, G]$ of g and G:

$$\begin{aligned} M(t_1, t_2) &= \mathcal{M}[g, G] \\ &= -\kappa \int_{-\infty}^{\infty} \dot{x}_s^2(t)dt + \gamma \int_{-\infty}^{\infty} \dot{x}_s(t)g(t + t_1)dt \\ &\quad + \rho \int_{-\infty}^{\infty} \dot{x}_s(t)G(t + t_2)dt. \end{aligned} \tag{9}$$

Recall that $h(t) = \dot{x}_s(-t)$ is the impulse response of the orbit filter $\mathcal{F}$. Denoting the integral of $\dot{x}_s^2$ by I, we obtain

$$M(t_1, t_2) = -I\kappa + \gamma\mathcal{F}[g](t_1) + \rho\mathcal{F}[G](t_2). \tag{10}$$

Substituting (10) into (8) we obtain

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\gamma\mathcal{F}[g](\theta_1 - s) + \rho\mathcal{F}[G](\theta_2 - s) - I\kappa]^+ ds. \tag{11}$$

Existence of the limit in (11) depends on the nature of the excitations g and G and their corresponding convolutions $\mathcal{F}[g] = g * h$ and $\mathcal{F}[G] = G * h$.

To ensure the existence of the limit in (11), we assume that g is asymptotic mean stationary (AMS): a stochastic process $X(t)$ is defined to be AMS if [42] the limits

$$\mu_X(A) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[1_A(X(t))]dt \tag{12}$$

exists for each real Borel set $A \in \mathcal{R}$. Here 1_A is the indicator function, $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ otherwise. If the limits in (12) exist then μ_X is a probability measure [43]. μ_X is called the stationary mean (SM) distribution of the process X.

The deterministic forcing function g is assumed to be AMS so, due to the linearity of $\mathcal{F}$, $\mathcal{F}[g]$ is also AMS and we denote the SM distribution of $\mathcal{F}[g]$ by $\mu_{\mathcal{F}[g]}$. Assume the spectrum of G is continuous. Then, according to Fact G7, $\mathcal{F}[G]$ is ergodic. Ergodicity implies asymptotic mean stationarity so $\mathcal{F}[G]$ is AMS also with SM distribution $\mu_{\mathcal{F}[G]}$. All AMS deterministic functions are ergodic so $\mathcal{F}[g]$, like $\mathcal{F}[G]$, is ergodic. Inasmuch as

$\mathcal{F}[g]$ is deterministic, $\mathcal{F}[g]$ and $\mathcal{F}[G]$ are jointly ergodic with SM distribution $\mu_{\mathcal{F}[g]} \times \mu_{\mathcal{F}[G]}$ [31]. Then the limit (11) exists and can be expressed in terms of the SM distributions $\mu_{\mathcal{F}[g]}$ and $\mu_{\mathcal{F}[G]}$.

Theorem 1 [31]: Suppose g is AMS and $\mathcal{F}[G]$ is ergodic. Then the limit in (11) exists, the flux factor Φ is nonrandom and

$$\Phi = E[(\gamma A + \rho B - I\kappa)^+]$$

where A is a random variable with distribution equal to the SM distribution $\mu_{\mathcal{F}[g]}$ of the function $\mathcal{F}[g]$, B is a random variable with distribution equal to the SM distribution $\mu_{\mathcal{F}[G]}$ of the process $\mathcal{F}[G]$ and A and B are independent.

Theorem 1 applies broadly to uniformly bounded and EUC noise processes G with ergodic filtered counterpart $\mathcal{F}[G]$. The modified Shinozuka process (5) belongs to this class provided it is Ψ-admissible with continuous, bandlimited spectrum. Moreover, G in (5) is stationary and $\mathcal{F}[G]$ is asymptotically Gaussian. Hence $\mu_{\mathcal{F}[G]}$ is the marginal distribution of $\mathcal{F}[G]$ and, for large N, B is approximately Gaussian with zero mean and variance σ^2.

Theorem 2 [31]: Suppose g is AMS and G is a Ψ-admissible modified Shinozuka process with continuous bandlimited spectrum. Then the flux factor Φ is approximately

$$\Phi \doteq E[(\gamma A + \rho\sigma Z - I\kappa)^+] \tag{13}$$

where Z is a standard Gaussian random variable. The error in this approximation decreases as N is made larger.

Most remarkable about (13) is the fact that, for Gaussian excitation without a deterministic component g, the detailed nature of the system is expressed in the flux factor Φ solely through the constant I and the scaling factor σ where

$$\sigma^2 = \int_0^\infty S^2(\nu)\Psi(d\nu)$$

represents the degree of "match" of the noise spectrum Ψ to the orbit filter. Further analysis of (13) is given in [31].

DUFFING OSCILLATOR WITH ADDITIVE NEAR-GAUSSIAN NOISE

The Duffing oscillator [44] is one of the simplest one-degree-of-freedom Newtonian dynamical systems capable of deterministic chaotic motion and has been extensively studied via mechanical laboratory and numerical computer models as well as analytically. The potential energy for this system is $V(x) = x^4/4 - x^2/2$. Consider the forced Duffing oscillator with additive noise and linear damping:

$$\ddot{x} = x - x^3 + \varepsilon[\gamma g(t) + \rho G(t) - \kappa\dot{x}]. \tag{14}$$

Here $\gamma \geq 0$, $\kappa \geq 0$ and $\rho \geq 0$ are constants, g is deterministic and bounded $|g(t)| \leq 1$, and G is the modified Shinozuka noise process reviewed in the previous section.

The unperturbed Duffing oscillator $\ddot{x} = x - x^3$ has a hyperbolic fixed point at the origin $(x, \dot{x}) = (0, 0)$ in phase space connected to itself by symmetric homoclinic orbits. These orbits are given by

$$\begin{pmatrix} x_s(t) \\ \dot{x}_s(t) \end{pmatrix} = \pm \begin{pmatrix} \sqrt{2}\mathrm{sech} t \\ -\sqrt{2}\mathrm{sech} t \tanh t \end{pmatrix}.$$

The impulse response h of the righthand (+) orbit is $h(t) = \dot{x}_s(-t) = \sqrt{2}\mathrm{sech} t \tanh t$. Thus $I = 4/3$.

The flux factor Φ for this system is given exactly in Theorem 1 and approximately in Theorem 2. The approximation in Theorem 2 was obtained by representing the marginal distribution $\mu_{\mathcal{F}[G]}$ of $\mathcal{F}[G]$ by a Gaussian distribution and is appropriate for large N. However, because the Gaussian distribution has infinite tails, Theorem 2 indicates that the flux factor is nonzero for all levels $\rho > 0$ of noise. We now present a different approximation to the flux factor based on the beta distribution which better describes the effect of the finite tails of $\mu_{\mathcal{F}[G]}$. We consider the case $\gamma = 0$. This is the case in which there is no deterministic forcing or, equivalently, the case in which the mass of the distribution $\mu_{\mathcal{F}[g]}$ is concentrated at zero by the orbit filter $\mathcal{F}$. The latter occurs, for instance, when the spectrum of g is located outside the passband of $\mathcal{F}$.

For the Duffing oscillator (14) with no deterministic forcing ($\gamma = 0$), linear damping and additive Shinozuka noise (5), we have, according to Theorem 1, $\Phi = E[(\rho\sigma B_N - 4\kappa/3)^+]$. Using Fact G3 we take

$$B_N = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos U_n$$

and define

$$\Phi' = \frac{3\Phi}{4\kappa}, \quad \rho' = \frac{3}{\sqrt{2}} \frac{\rho\sigma}{\kappa}, \quad B' = \frac{B_N + \sqrt{2N}}{2\sqrt{2N}}.$$

Then

$$\Phi' = E[(\rho'\sqrt{N}(B' - 1/2) - 1)^+].$$

The support of B' is the interval $(0, 1)$ and is approximately beta-distributed [48] with density

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha - 1}(1 - t)^{\beta - 1}, \quad 0 < t < 1$$

where the parameters $\alpha > 0$ and $\beta > 0$ of the distribution are chosen so that the mean and the variance of the beta distribution are the same as those of B'. The mean and the variance of the beta distribution with parameters α and β are, respectively,

$$\frac{\alpha}{\alpha + \beta}, \quad \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

and $E[B'] = 1/2$ and $Var[B'] = (8N)^{-1}$ so

$$\frac{\alpha}{\alpha+\beta} = \frac{1}{2}, \quad \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{8N}.$$

Therefore $\alpha = \beta = N - 1/2$ and

$$\begin{aligned} \Phi' &= \Phi'(\rho', N) \\ &\doteq \int_0^1 [\rho'\sqrt{N}(x-1/2)-1]^+ \frac{\Gamma(2N-1)}{\Gamma^2(N-1/2)} x^{N-3/2}(1-x)^{N-3/2}dx. \end{aligned} \tag{15}$$

Equation (15) shows that $\Phi' = 0$ for $\rho'\sqrt{N}/2 \leq 1$. In other words, $\Phi = 0$ for

$$\frac{\rho\sigma}{\kappa} < \frac{2}{3}\sqrt{\frac{2}{N}}.$$

Above this threshold,

$$\begin{aligned} \Phi' &\doteq \int_{\frac{1}{2}+\frac{1}{\rho'\sqrt{N}}}^1 [\rho'\sqrt{N}(x-1/2)-1] \frac{\Gamma(2N-1)}{\Gamma^2(N-1/2)} x^{N-3/2}(1-x)^{N-3/2}dx \\ &= \rho'\sqrt{N} \int_{\frac{1}{2}+\frac{1}{\rho'\sqrt{N}}}^1 \frac{\Gamma(2N-1)}{\Gamma^2(N-1/2)} x^{N-1/2}(1-x)^{N-3/2}dx \\ &\quad -(1+\frac{\rho'\sqrt{N}}{2}) \int_{\frac{1}{2}+\frac{1}{\rho'\sqrt{N}}}^1 \frac{\Gamma(2N-1)}{\Gamma^2(N-1/2)} x^{N-3/2}(1-x)^{N-3/2}dx \\ &= \frac{\rho'\sqrt{N}}{2}[1 - \text{Beta}(1/2 + 1/(\rho'\sqrt{N}); N+1/2, N-1/2)] \\ &\quad -(1+\frac{\rho'\sqrt{N}}{2})[1 - \text{Beta}(1/2 + 1/(\rho'\sqrt{N}); N-1/2, N-1/2)] \end{aligned} \tag{16}$$

where $\text{Beta}(x; \alpha, \beta)$ is the regularized incomplete Beta function

$$\text{Beta}(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0 < x < 1.$$

$\Phi' = \Phi'(\rho', N)$ is plotted in Figure 2 as a function of ρ' for various values of N using (16). For comparison, the limiting Gaussian noise case $N \to \infty$ is also plotted using the righthand side of (13).

MULTIPLICATIVE EXCITATION

We turn now to a more general form for w, the multiplicative excitation model:

$$w(x, \dot{x}, t) = \gamma(x, \dot{x})g(t) + \rho(x, \dot{x})G(t). \tag{17}$$

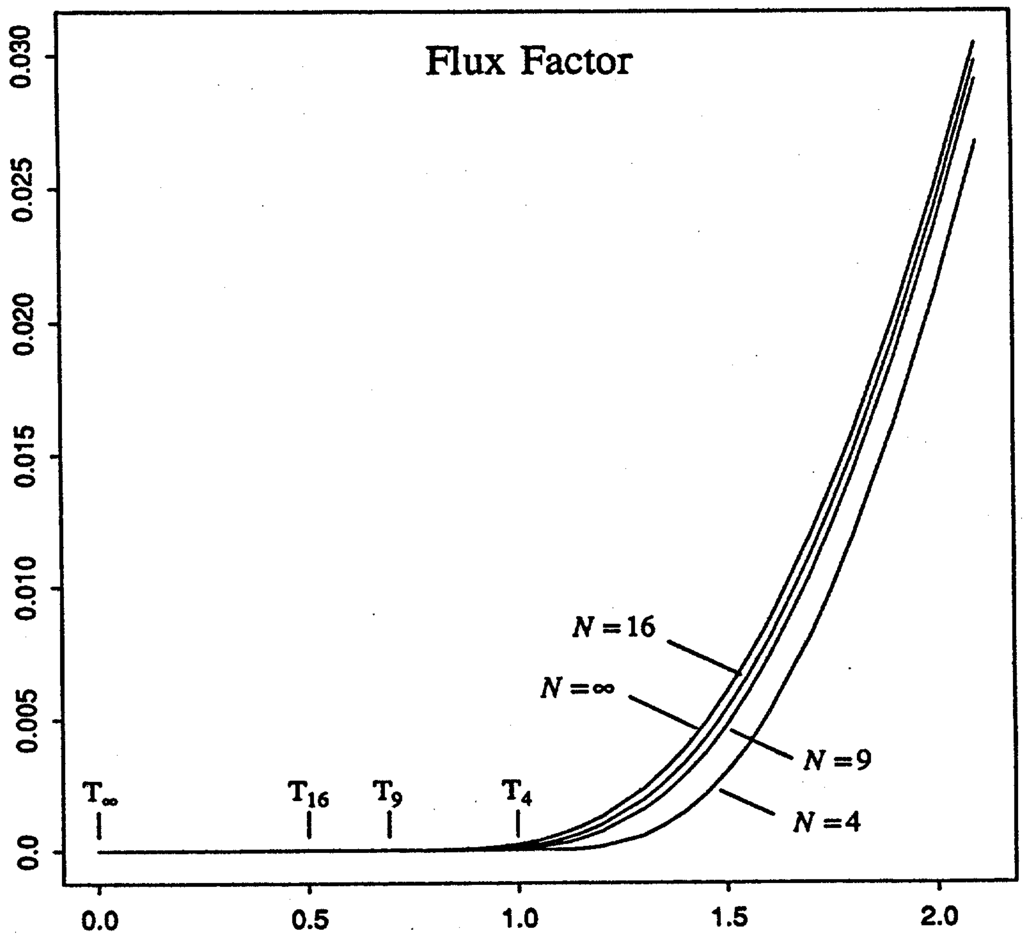

Figure 2. The flux factor Φ' as a function of the noise strength ρ' for various values of N. T_N is the threshold for positive flux.

As in the additive excitation model, the function g represents deterministic forcing while $G(t) = G(t,\omega)$, $\omega \in \Omega$ is a stochastic process representing a random forcing contribution.

The Melnikov function is calculated as in (9) to be

$$M(t_1,t_2) = \mathcal{M}[g,G] = \int_{-\infty}^{\infty} \dot{x}_s(t)[\gamma(x_s(t),\dot{x}_s(t))g(t+t_1) + \rho(x_s(t),\dot{x}_s(t))G(t+t_2)]dt.$$

We define orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ with impulse responses

$$h_1(t) = \dot{x}_s(-t)\gamma(x_s(-t),\dot{x}_s(-t)), \quad h_2(t) = \dot{x}_s(-t)\rho(x_s(-t),\dot{x}_s(-t))$$

and corresponding transfer functions $H_1(\nu)$ and $H_2(\nu)$. Then

$$M(t_1,t_2) = \mathcal{F}_1[g](t_1) + \mathcal{F}_2[G](t_2). \tag{18}$$

Generalizing the additive excitation model (3) by allowing the coefficients γ and ρ to depend on the state $(x,\dot{x})$ of the system has, according to (18), two significant consequences. First, the orbit filter $\mathcal{F}$ in the additive model is replaced in the multiplicative model by two different orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ and, second, the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are, like $\mathcal{F}$, linear, time-invariant and noncausal with impulse responses given solely in terms of the orbit $\vec{x}_s$ of the unperturbed system and the functions γ and ρ.

Substituting (18) into (8) gives

$$\Phi = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\rho\mathcal{F}_1[g](\theta_1 - s) + \gamma\mathcal{F}_2[G](\theta_2 - s)]^+ ds. \tag{19}$$

Just as in the case of the additive excitation model, existence of the limit in (19) hinges on the joint ergodicity of the function $\mathcal{F}_1[g] = g * h_1$ and the process $\mathcal{F}_2[G] = G * h_2$.

Theorem 3: Consider system (2) with perturbation function w as in (17) such that g is AMS and $\mathcal{F}_2[G]$ is ergodic. Let $\mu_{\mathcal{F}_1[g]}$ and $\mu_{\mathcal{F}_2[G]}$ be the SM distributions of $\mathcal{F}_1[g]$ and $\mathcal{F}_2[G]$, respectively. Then the limit in (19) exists, the flux factor Φ is nonrandom and

$$\Phi = E[(\gamma A + \rho B)^+]$$

where A is a random variable with distribution $\mu_{\mathcal{F}_1[g]}$, B is a random variable with distribution $\mu_{\mathcal{F}_2[G]}$ and A and B are independent.

Theorem 3 is an extension of Theorem 1 to the general case of multiplicative excitation. Theorem 3 can in turn be extended to systems with more general planar vector fields than that of system (2) and to third and higher order one degree-of-freedom systems. Only the orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ change in these more general cases; the form of the flux factor Φ given in Theorem 3 remains the same.

BOUNDED, EUC SHOT NOISE

Shot noise is a frequently used model of physical noise phenomena. Well-known applications include noise modelling in electrical and electronic systems. Shot noise is also used to model impulsive loads on mechanical systems - automotive traffic on bridges, for instance. Potential new applications include models of friction and noise modelling for micromachines.

The usual model K of constant-rate shot noise is a stochastic process of the form [45], [46]

$$K(t) = \sum_{k \in \mathcal{Z}} r(t - T_k) \tag{20}$$

where $\mathcal{Z}$ is the set of integers, $\{T_k, k \in \mathcal{Z}\}$ are the epochs (shots) of a Poisson process with rate $\lambda > 0$ and r is bounded and square-integrable,

$$\int_{-\infty}^{\infty} r^2(t)dt < \infty.$$

r is call the shot response of the process K. Define

$$R(\nu) = \int_{-\infty}^{\infty} r(t)e^{-j\nu t}dt. \tag{21}$$

The shot noise K can be viewed as the output of the linear filter with transfer function R excited by a Poisson process with rate λ and, according to Campbell's theorem [45],

$$E[K(t)] = \lambda \int_{-\infty}^{\infty} r(\tau)d\tau \ = \ \lambda R(0),$$

$$Cov[K(t_1), K(t_2)] = \lambda \int_{-\infty}^{\infty} r(t_1 - \tau)r(t_2 - \tau)d\tau$$

and, in particular,

$$Var[K(t)] = \lambda \int_{-\infty}^{\infty} r^2(\tau)d\tau.$$

The spectrum Ψ of K is

$$\Psi(A) = \lambda \int_A Q^2(\nu)d\nu \tag{22}$$

where $Q(\nu) = |R(\nu)|$ is the modulus of the transfer function R.

The usual shot noise model K in (20) is neither bounded nor EUC and cannot be used in conjunction with Melnikov theory in calculating the phase space flux in chaotic systems. A modification of K is needed which approximates K and yet has the requisite path properties. We now describe a model of shot noise which meets these requirements.

Let K_N be a stochastic process of the form

$$K_N(t) = \sum_{j \in \mathcal{Z}} \sum_{k=1}^{2^N} r(t - T_{jkN} - A_j - T) \tag{23}$$

where N is a positive integer, $A_j = 2^N(j - 1/2)/\lambda$ and $\{T, T_{jkN}, j \in \mathcal{Z}, k = 1, \ldots, 2^N\}$ are independent random variables such that for each N and j, $\{T_{jkN}, k = 1, 2, \ldots, 2^N\}$

are identically uniformly distributed in the interval $(A_j, A_{j+1}]$ and T is uniformly distributed between 0 and $2^N/\lambda$. As in the usual shot noise model (20), λ is here again the rate of the process; it is the mean number of epochs (shots) T_{jkN} per unit time. We assume just as for K, that r in (23) is bounded and square-integrable. We further assume that r is UC and that the radial majorant

$$r^*(t) = \sup_{|\tau| \geq |t|} |r(\tau)|$$

of the shot response is integrable; i.e.

$$\int_{-\infty}^{\infty} r^*(t)dt < \infty.$$

According to this specification of K_N, realizations of the process are obtained by partitioning the real line into the intervals $(A_j, A_{j+1}]$ of length $2^N/\lambda$ with common random phase T and then placing 2^N epochs independently and at random in each interval. The random phase T eliminates the (ensemble) cyclic nonstationarity produced by the partitioning by $(A_j, A_{j+1}]$.

Let $\mathcal{F}$ be a linear, time-invariant filter with impulse response h. Define the transfer function H of $\mathcal{F}$ as in (6) and let $S = |H|$ be the modulus of H. We now list some important properties of K_N and $\mathcal{F}[K_N]$. Multivariate, multiparameter and time-varying shot rate generalizations of these results exist. The proofs of these more general results will be presented in a separate paper.

Fact K1: $E[K_N(t)] = \lambda \int_{-\infty}^{\infty} r(\tau)d\tau = \lambda R(0)$ for all N and t.

Fact K2: $E[\mathcal{F}[K_N](t)] = \lambda \int_{-\infty}^{\infty} (r * h)(\tau)d\tau = \lambda R(0)S(0)$ for all N and t.

Fact K3: K_N and $\mathcal{F}[K_N]$ are stationary processes.

Fact K4: Let $N \to \infty$. K_N converges in distribution [47] to the shot noise K with the same shot response r and rate λ. $\mathcal{F}[K_N]$ is also a shot noise of the form (22) with shot response $r * h$. Hence, $\mathcal{F}[K_N]$ converges in distribution to the shot noise K with shot response $r * h$ and rate λ.

Fact K5: The variances of K_N and $\mathcal{F}[K_N]$ converge, respectively, to those of K and $\mathcal{F}[K]$:

$$Var[K_N(t)] \to \lambda \int_{-\infty}^{\infty} r^2(\tau)d\tau,$$

$$Var[\mathcal{F}[K_N](t)] \to \lambda \int_{-\infty}^{\infty} (r * h)^2(\tau)d\tau.$$

Fact K6: The spectrum of K_N converges weakly [47] to the spectrum (21) of the shot noise K with the same shot response r and rate λ. Similarly, the spectrum of $\mathcal{F}[K_N]$ converges weakly to the spectrum of the shot noise K with shot response $r * h$ and rate λ.

Fact K7: K_N is uniformly bounded for all N, $|K_N(t)| \leq \lambda \int_{-\infty}^{\infty} r^*(t)dt$.

Fact K8: K_N is EUC for all N.

Fact K9: K_N and $\mathcal{F}[K_N]$ are each ergodic for all N.

Facts K1-6 establish that for large N the shot noise K_N closely approximates the standard shot noise model K in all important respects. Facts K7-9 show that K_N, unlike K, can be used in Melnikov's method-type calculations of the flux factor. Five realizations of K_N with Gaussian shot response $r(t) = \exp(-t^2)$ are shown for comparison in Figure 3 together with five realizations of K with the same shot response and shot rate.

DUFFING OSCILLATOR WITH SHOT NOISE-LIKE DISSIPATION

As an example of a system with multiplicative shot noise, we consider the Duffing oscillator with weak forcing and non-autonomous damping:

$$\ddot{x} = x - x^3 + \varepsilon[\gamma g(t) - \kappa(K_N(t) + \eta)\dot{x}]. \tag{24}$$

Here $\gamma \geq 0$, $\kappa \geq 0$ and $\eta \geq 0$ are constants, g is deterministic and bounded $|g(t)| \leq 1$, and K_N is the shot noise model introduced in the previous section. The perturbation in (24) is a particular case of the multiplicative excitation model (17) with $\gamma(x, \dot{x}) = \gamma$, $\rho(x, \dot{x}) = -\kappa\dot{x}$, and $G(t) = K_N(t) + \eta$. $\kappa(K_N(t) + \eta)$ in (24) serves as a time-varying damping factor and plays the same role as the constant κ in (4). The two terms $\kappa\eta$ and κK_N represent, respectively, viscous and shot noise-like damping forces. The contribution of the viscous term to the flux factor has already been considered. We therefore choose $\eta = 0$ and only consider the shot noise-like component of the damping. We also assume the shot response r of K_N to be nonnegative in this example so that the factor κK_N is nonnegative.

According to Theorem 3, the Melnikov function for this example is

$$M(t_1, t_2) = \mathcal{F}_1[g](t_1) + \mathcal{F}_2[G](t_2)$$

where

$$h_1(t) = \gamma\dot{x}_s(-t) = \gamma\sqrt{2}\text{sech}t\tanh t$$

and

$$h_2(t) = -\kappa\dot{x}_s^2(-t) = -2\kappa\text{sech}^2 t\tanh^2 t.$$

The corresponding moduli of the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are

$$S_1(\nu) = \sqrt{2}\pi\gamma\nu\text{sech}\frac{\pi\nu}{2}$$

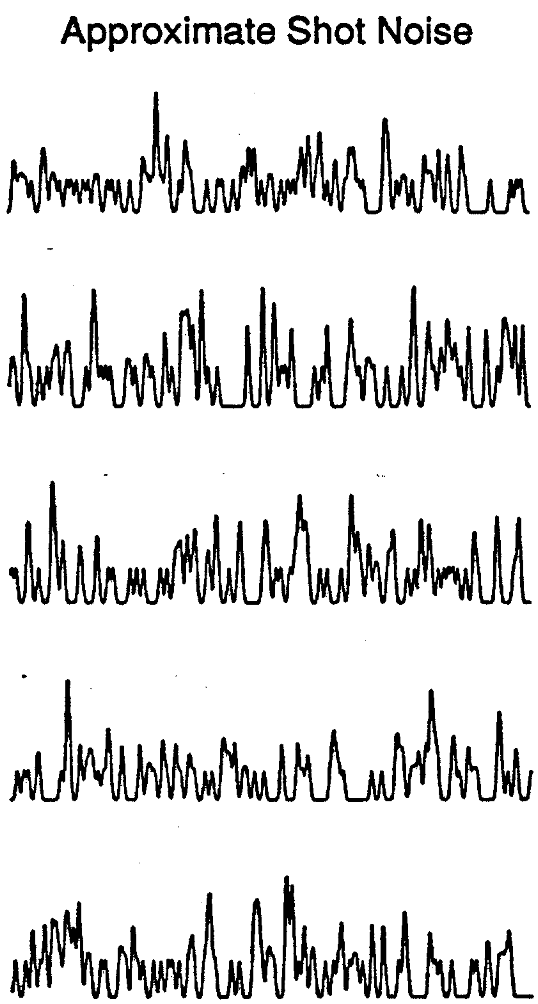

Figure 3. Realizations of approximate shot noise and true shot noise with identical shot responses $r(t)=\exp(-t^2)$.

and

$$S_2(\nu) = 4\kappa \int_0^\infty \mathrm{sech}^2 t \tanh^2 t \cos \nu t dt.$$

We have $S_1(0) = 0$ so the d.c. component (if any) of g is completely removed by $\mathcal{F}_1$ and has no effect on the Melnikov function. K_N does have a d.c. component; K_N is ergodic so its d.c. component is $E[K_N] = \lambda R(0)$ where

$$R(0) = \int_{-\infty}^\infty r(t)dt > 0.$$

$S_2(0) = 4\kappa/3 > 0$ so the d.c. component of K_N passed by $\mathcal{F}_2$ is

$$E[\mathcal{F}_2[K_N]] = E[K_N]S_2(0) = \frac{4\kappa\lambda R(0)}{3}.$$

Assume the deterministic forcing function g is AMS. K_N is uniformly bounded and EUC and $\mathcal{F}_2[K_N]$ is ergodic. Thus $\mathcal{F}_1[g]$ and $\mathcal{F}_2[K_N]$ are jointly ergodic. By Theorem 3, the flux factor Φ exists and

$$\Phi = E[(A - B_N)^+] \tag{25}$$

where the distribution of A is $\mu_{\mathcal{F}_1[g]}$, the distribution of B_N is $\mu_{\mathcal{F}_2[K_N]}$ and A and B_N are independent.

The distribution of $\mathcal{F}_2[K_N]$ is, for large N, approximately that of the shot noise $\mathcal{F}_2[K]$ as noted in Fact K4. This is the basis for the following theorem.

Theorem 4: The flux factor Φ for the Duffing oscillator (24) with weak forcing and shot noise damping coefficent κK_N is approximately

$$\Phi \doteq E[(A - B)^+]$$

where A is $\mu_{\mathcal{F}_1[g]}$-distributed, B is $\mu_{\mathcal{F}_2[K]}$-distributed, A and B are independent and K is the shot noise (20). This approximation improves as N increases.

Φ can be calculated numerically as follows for given system parameters ν, γ and κ and shot parameters λ and r. Make the following definitions:

$$\Phi' = \frac{\Phi}{\gamma S_1(\nu)}, \quad A' = \frac{A}{\gamma S_1(\nu)}, \quad B' = \frac{B_N}{\gamma S_1(\nu)}, \quad \lambda' = \frac{16\lambda R^2(0)}{9J}, \quad \kappa' = \frac{3\kappa J}{4\gamma S_1(\nu)R(0)}$$

where

$$J = \int_{-\infty}^\infty (r * h)^2(t)dt.$$

Then

$$\Phi' = E[(A' - B')^+].$$

The random variable B' is approximately gamma-distributed [48] with density

$$\frac{t^{\alpha-1}e^{-t/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, \quad t > 0$$

where the parameters α and β are determined by the condition that $E[B']$ and $Var[B']$ equal the mean and the variance, respectively, of the gamma distribution. The mean and variance of the gamma distribution are $\alpha\beta$ and $\alpha\beta^2$, respectively, so

$$\alpha = \frac{E^2[B']}{Var[B']} = \frac{E^2[B_N]}{Var[B_N]} \doteq \frac{(4\lambda\kappa R(0)/3)^2}{\kappa^2\lambda J} = \frac{16}{9}\frac{\lambda R^2(0)}{J} = \lambda'$$

and

$$\beta = \frac{Var[B']}{E[B']} = \frac{1}{\gamma S_1(\nu)}\frac{Var[B_N]}{E[B_N]} \doteq \frac{1}{\gamma S_1(\nu)}\frac{3\lambda\kappa^2 J}{4\lambda\kappa R(0)} = \frac{3\kappa J}{4\gamma R(0)S_1(\nu)} = \kappa'.$$

The gamma approximation performs well for large α, and hence for $\lambda \gg J/R^2(0)$. For $g(t) = \sin\nu t$, the random variable A' is equal in distribution to $\cos U$ where U is uniformly distributed over the interval $[0, \pi]$. Thus,

$$\begin{aligned}
\Phi' &= \Phi'(\kappa',\lambda') \doteq \frac{1}{\pi\Gamma(\lambda')(\kappa')^{\lambda'}}\int_0^{\pi}\int_0^{\infty}(\cos u - t)^{+}t^{\lambda'-1}e^{-t/\kappa'}dtdu \\
&= \frac{1}{\pi\Gamma(\lambda')(\kappa')^{\lambda'}}\int_0^1\int_0^{\mathrm{Cos}^{-1}t}(\cos u - t)t^{\lambda'-1}e^{-t/\kappa'}dudt \\
&= \frac{1}{\pi(\kappa')^{\lambda'}\Gamma(\lambda')}\int_0^1(\sqrt{1-t^2} - t\mathrm{Cos}^{-1}t)t^{\lambda'-1}e^{-t/\kappa'}dt. \qquad (26)
\end{aligned}$$

(26) shows in particular that, in dimensionless units, the flux factor Φ depends only on κ and λ - the shape of the shot response r has only a scaling effect. $\Phi' = \Phi'(\kappa', \lambda')$ is shown in Figure 4.

Figure 4 shows that for large λ (the regime where the gamma approximation is most accurate), Φ is nearly linear for small κ with a turning point after which Φ falls off exponentially with κ. In fact, this large-κ exponential decay is an artifact of the gamma approximation - for our approximate shot noise K_N, the flux factor can be shown by a further analysis to actually fall to zero at some finite threshold value of κ. As $N \to \infty$, however, this threshold is pushed higher and higher to infinity.

Φ in (25) is nonzero if $P\{A > B_N\} > 0$. $|g(t)| \leq 1$ so A has bounded support. For sinusoidal forcing, for example, the support of A is $[-\gamma S_1(\nu), \gamma S_1(\nu)]$. Let $[-\gamma a_l, \gamma a_u]$ be the support of A. Then $P\{A > B_N\} > 0$ if $P\{B_N < \gamma a_u\} > 0$. For N sufficiently large, this latter probability is indeed positive for any given arbitrarily small value of γ. Thus, in the shot noise limit $K_N \to K$, $\Phi > 0$ for all values of the parameters κ, λ and r of the shot noise-like dissipation. This conclusion is analogous to that reached earlier for additive Gaussian excitation for which we showed that for bounded, approximately

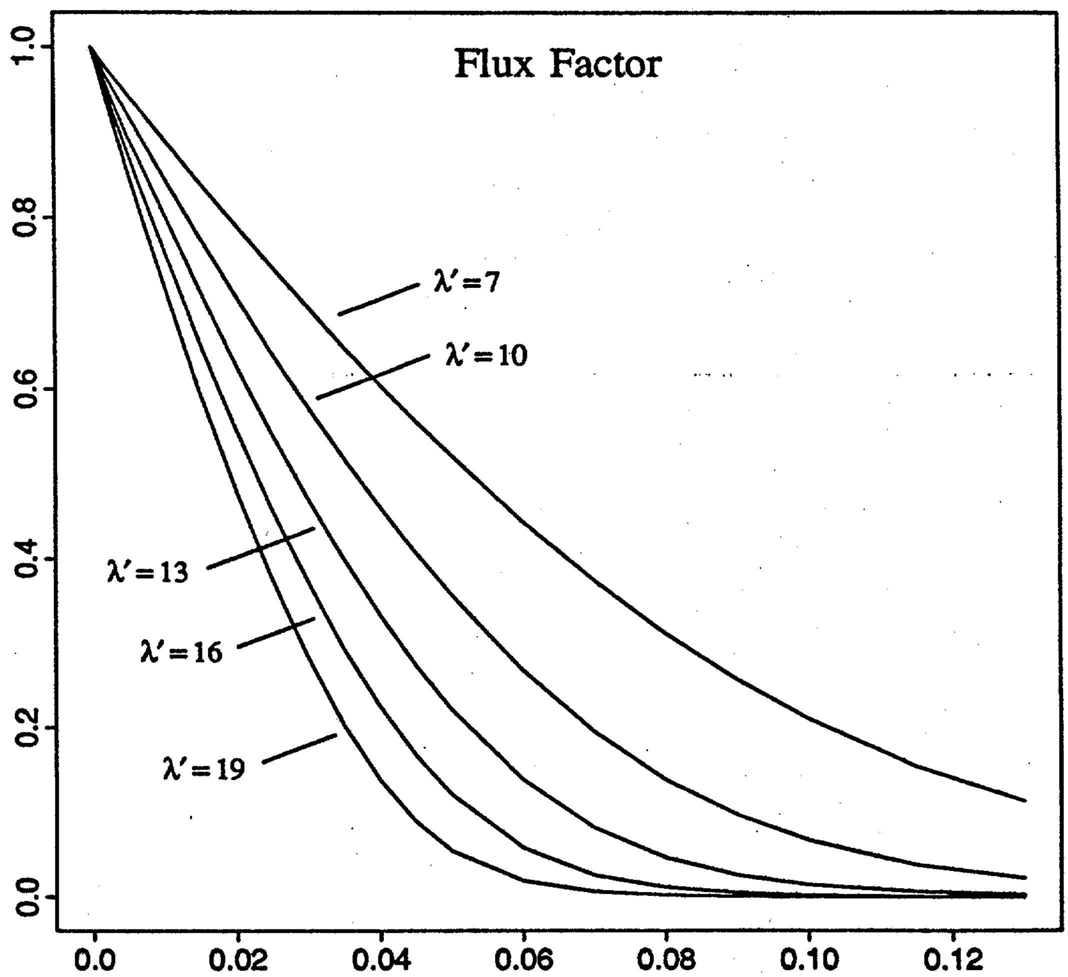

Figure 4. The flux factor Φ' as a function of the damping constant κ' for various shot rates λ'.

Gaussian noise, the flux factor drops to zero below a certain threshold. This threshold moves closer and closer to zero as the Gaussian approximation improves, leading to the conclusion that in the limit $N \to \infty$ of true Gaussian excitation the flux factor is nonzero for arbitrarily low levels of forcing.

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