Analytical Determination of the LLG Zero-Damping Critical Switching Field

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***Abstract*—Previous numerical studies based on the Landau-Lifshitz-Gilbert (LLG) equation have considered the magnetization reversal of a uniaxial, single-domain particle due to an applied field pulse with a short rise time. When the LLG damping constant $\alpha < 1$, these studies have observed coherent switching for applied field magnitudes below the Stoner-Wohlfarth limit. The switching field computed in these studies decreases as $\alpha \to 0$, with apparent convergence to a limiting value. In this paper, analytic methods determine the value of the switching field in the zero-damping limit for an applied field pulse with zero rise time. The locus of normalized switching fields in parametric form is $h_y = -\sin\theta(\cos\theta - 1)/2$; $h_z = -\cos\theta(\cos\theta + 1)/2$; $|\theta| \le 2\pi/3$. A non-parametric form is also derived. One surprising implication is that magnetization reversal may be caused by an applied field with easy axis component in the same direction as the initial magnetization ($h_z > 0$).**

***Index Terms*—Landau-Lifshitz dynamics, Stoner-Wohlfarth model, switching field, zero damping**

I. Problem Definition

The model considered in this paper is the same one previously investigated by He, Doyle, and Fujiwara [1]. In a Cartesian coordinate system with base vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, consider a uniaxial, single-domain particle with its easy axis parallel to $\hat{\mathbf{z}}$. Let the particle magnetization $\mathbf{M}$ be uniform with magnitude M_s (A/m). The anisotropy of the particle is represented by an effective field,

$$\mathbf{H}_k(\mathbf{M}) = \frac{2K}{\mu_0 M_s^2} M_z \hat{\mathbf{z}}, (\mathrm{A/m}) \tag{1}$$

where K (J/m^3) is the anisotropy energy density of the particle. Assume the magnetization begins at rest in the equilibrium value $\mathbf{M} = M_s\hat{\mathbf{z}}$ with zero applied field. At time $t = 0$ an external pulse field is applied in the y-z plane. After a rise time Δt_r (s), the pulse field reaches a constant value of $\mathbf{H}_a = \begin{bmatrix} 0 & H_y & H_z \end{bmatrix}$ (A/m). The external field remains constant for a pulse duration of τ_H (s), and then returns to zero. The question to be answered is under what conditions the external field pulse will cause the magnetization to reverse direction, so that its final equilibrium value is $\mathbf{M} = -M_s\hat{\mathbf{z}}$.

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II. Stoner-Wohlfarth Analysis

The magnetic energy W (J) of the particle in the presence of the applied field is

$$\begin{aligned} W(\mathbf{M}) &= -\mu_0 V \mathbf{H}_a \cdot \mathbf{M} - \frac{\mu_0 V}{2}\mathbf{H}_k \cdot \mathbf{M} & (2)\\ &= -\mu_0 V(H_y M_y + H_z M_z) - \frac{KV}{M_s^2} M_z^2, & (3) \end{aligned}$$

where V is the volume of the particle. An equilibrium magnetization is one which locally minimizes (3), subject to the constraint $|\mathbf{M}| = M_s$.

The optimization problem is more conveniently stated in terms of normalized quantities, $\mathbf{m} = \mathbf{M}/M_s$, $\mathbf{h} = (\mu_0 M_s/2K)\mathbf{H} = \mathbf{H}/H_k$, and $w = W/2KV$. Then, equilibrium values of $\mathbf{m}$ are those which locally minimize

$$w = -h_y m_y - h_z m_z - \frac{1}{2} m_z^2 \tag{4}$$

subject to the constraint $|\mathbf{m}| = 1$. The solutions are the well-known Stoner-Wohlfarth equilibria [2]. For small values of $\mathbf{h}$, there are two equilibrium magnetization values. When $h_y^{2/3} + h_z^{2/3} > 1$, there is only one equilibrium magnetization.

When Δt_r is large, so that the external field pulse is applied slowly, the particle magnetization will respond so as to keep an equilibrium value. A transition from two equilibria to one equilibrium is required for magnetization reversal. Thus, the external field pulse causes a reversal of the particle magnetization only when $h_z < 0$ and $h_y^{2/3} + h_z^{2/3} > 1$.

III. Landau-Lifshitz Dynamics

When Δt_r is sufficiently small, the dynamics of the particle magnetization in response to the external field pulse cannot be neglected [1]. The Landau-Lifshitz-Gilbert (LLG) equation [3], [4], also in normalized form,

$$\frac{d\mathbf{m}}{dt} = -|\gamma H_k|(\mathbf{m} \times \mathbf{h}_T) + \alpha(\mathbf{m} \times \frac{d\mathbf{m}}{dt}), \tag{5}$$

predicts the motion of $\mathbf{m}$ in response to the total magnetic field $\mathbf{h}_T = \mathbf{h}_a + \mathbf{h}_k$, where $\gamma = -2.21 \times 10^5$ (rad/s)/(A/m) is the gyromagnetic ratio and α is a dimensionless phenomenological damping parameter. During the interval $\Delta t_r < t < \Delta t_r + \tau_H$, the external field pulse has the constant value $\mathbf{h}_a$, and the rate of energy dissipation is

$$\frac{dw}{dt} = -\mathbf{h}_T \cdot \frac{d\mathbf{m}}{dt} = -\alpha \mathbf{h}_T \cdot \left(\mathbf{m} \times \frac{d\mathbf{m}}{dt} \right). \qquad (6)$$

Only the second term of (5), the damping term, contributes to the energy dissipation rate. The rate of energy dissipation can be controlled by selecting the value of the damping parameter α. When the values of both α and Δt_r are small, the increase in magnetic energy due to the application of the external field pulse outpaces the dissipation of energy and the trajectory of $\mathbf{m}$ is not confined to equilibrium values.

Solutions to (5) in terms of the spherical coordinates of $\mathbf{m}$ [5] were computed numerically by He, Doyle, and Fujiwara [1] for several values of $\Delta t_r < 10$ ns, and several values of α. They determined that the non-equilibrium trajectories of $\mathbf{m}$ allowed magnetization reversal by coherent rotation to occur at smaller applied field magnitudes than predicted by Stoner-Wohlfarth analysis. For $\alpha > 1$, the energy dissipation rate is rapid enough that Stoner-Wohlfarth analysis computes the correct value for the critical magnitude of the external field pulse required for magnetization reversal for each applied field direction. For each rise time Δt_r and applied field direction, as α decreases < 1, the critical field magnitude required for magnetization reversal decreases. As $\alpha \to 0$, a limiting value for the critical switching field is reached. In the next section, analytic methods are used to determine that limiting value, the zero-damping critical switching field.

IV. Zero-Damping Switching Field

When $\alpha = 0$ in (5), the damping term disappears and the first term, the precession term, may be expanded,

$$\begin{aligned} \frac{1}{|\gamma H_k|}\frac{d\mathbf{m}}{dt} &= (m_z h_y - m_y h_z - m_y m_z)\hat{\mathbf{x}} \qquad (7) \\ &+ (m_x h_z + m_x m_z)\hat{\mathbf{y}} - m_x h_y \hat{\mathbf{z}}. \end{aligned}$$

As the magnetization follows the trajectory determined by (7), there is no energy dissipation, so all points on that trajectory satisfy

$$w(t) = -h_y m_y(t) - h_z m_z(t) - \frac{1}{2}m_z^2(t) = w(\Delta t_r) \qquad (8)$$

for $\Delta t_r < t < \Delta t_r + \tau_H$.

Assume $\Delta t_r = 0$. Recall that $m_z = 1$ at $t = 0$, so (8) becomes

$$-h_y m_y(t) - h_z m_z(t) - \frac{1}{2}m_z^2(t) = -h_z - \frac{1}{2}, \qquad (9)$$

or

$$m_y = -\frac{1}{2h_y}(m_z - 1)(m_z + 2h_z + 1), \qquad (10)$$

the equation of a parabola, considered as a function of m_z. The precession trajectory of constant energy is that portion of the intersection of this parabola with the sphere $|\mathbf{m}| = 1$ which includes the point $\mathbf{m} = \hat{\mathbf{z}}$. The intersection can take the form of two closed curves, each surrounding one Stoner-Wohlfarth equilibrium, or a single closed curve, similar to the stitches on a baseball, which surrounds both Stoner-Wohlfarth equilibria. Of course, when the applied field is large enough that there is only one Stoner-Wohlfarth equilibrium, the intersection is a single closed curve which surrounds that equilibrium.

For any small, positive value of α, the magnetization trajectory will depart the constant energy trajectory and spiral toward a Stoner-Wohlfarth equilibrium surrounded by the constant energy trajectory. Assume a small positive value for α and a value for τ_H large enough that it does not influence the final value of $\mathbf{m}$. When there is only one equilibrium surrounded by the constant energy trajectory, there is no uncertainty about the final magnetization value. However, when both equilibria are surrounded by the constant energy trajectory, the spiraling magnetization trajectory could lead to either one, depending on the precise value of α. Thus, magnetization reversal is possible whenever the constant energy trajectory surrounds both Stoner-Wohlfarth equilibria.

If the constant energy trajectory surrounds both Stoner-Wohlfarth equilibria, it must also surround the saddle point in the energy surface which separates them. That saddle point occurs at a magnetization $\mathbf{m}^* = \begin{bmatrix} m_x^* & m_y^* & m_z^* \end{bmatrix}$, which is a stationary point of the equations of motion. Thus,

$$\begin{aligned} m_z^* h_y - m_y^* h_z - m_y^* m_z^* &= 0, \qquad (11) \\ m_x^* h_z + m_x^* m_z^* &= 0, \qquad (12) \\ m_x^* h_y &= 0. \qquad (13) \end{aligned}$$

To determine whether the constant energy trajectory surrounds one or two equilibria, we may test whether it surrounds the saddle point. Those constant energy trajectories which surround the saddle point are separated from those which do not by the boundary case of those constant energy trajectories which intersect the saddle point. Clearly the saddle point lies in the y-z plane ($m_x^* = 0$), and we may write

$$\begin{aligned} m_y^* &= \sin\theta, \qquad (14) \\ m_z^* &= \cos\theta. \qquad (15) \end{aligned}$$

The values of $\mathbf{h}$ for which the constant energy trajectory intersects the saddle point are determined by finding the common solutions of (9) and (11),

$$\begin{aligned} h_y &= -\sin\theta(\cos\theta - 1)/2, \qquad (16) \\ h_z &= -\cos\theta(\cos\theta + 1)/2, \qquad (17) \end{aligned}$$

where $|\theta| < 2\pi/3$.

Fig. 1 illustrates the critical switching fields for the two extreme cases. When Δt_r is large or $\alpha > 1$, the Stoner-Wohlfarth assumptions are valid, and the normalized external field pulse must lie in the region below the solid curve in order to cause magnetization reversal. When $\Delta t_r = 0$, τ_H is large, and the normalized external field pulse lies below the dashed curve, there exists some small

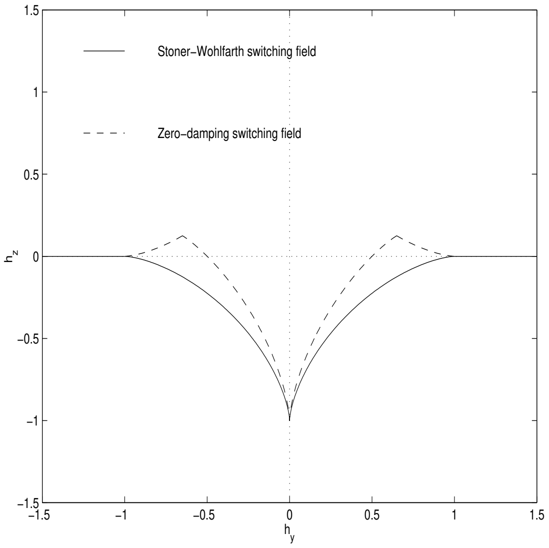

Fig. 1. For slowly applied field pulses, or large damping, Stoner-Wohlfarth analysis predicts magnetization reversal only for normalized applied fields below the solid curve. For an instantly applied field pulse and zero damping, magnetization reversal may occur for normalized applied fields below the dashed curve.

positive value of α for which magnetization reversal will occur. Intermediate cases of short, but non-zero, Δt_r or larger values of α still less than 1 have switching fields between these two limiting curves, and can be calculated by the numerical methods of He, Doyle, and Fujiwara [1]. As observed in other numerical computations by He and Doyle [6], [7], when α is small the magnetization may oscillate back and forth across the x-y plane several times before reaching an equilibrium. Whether or not reversal occurs depends on which side of the x-y plane the magnetization is on when its energy falls below that of the energy barrier. Different values of α will yield different outcomes.

One surprising conclusion to be drawn from Fig. 1 is that magnetization reversal may be caused by an external field pulses of appropriate magnitude applied on the same side of the hard plane as the initial magnetization, provided the field is applied at an angle greater than $\tan^{-1} 3\sqrt{3}$ from the easy axis.

V. Non-Parametric Expression

Given an applied field (h_y, h_z), it is not easy to use (16) and (17) to determine if the switching field is exceeded because they are in terms of the unknown parameter θ. A non-parametric expression for the zero-damping switching field is desirable.

Consider the squared magnitude of (7). This quantity is non-negative, and equals zero only at stationary points of the motion of $\mathbf{m}$. Substitution of the constraints $|\mathbf{m}| = 1$ and the constant energy trajectory (9) yields an expression,

$$\begin{aligned} \left| \frac{1}{\gamma H_k} \frac{d\mathbf{m}}{dt} \right|^2 &= \left(h_y^2 - h_z - \frac{1}{4} \right) + 2 h_z m_z \\ &- \left(h_z - \frac{1}{2} \right) m_z^2 - \frac{1}{4} m_z^4. \end{aligned} \tag{18}$$

The zero-damping switching fields are those values of (h_y, h_z) for which the constant energy trajectory intersects a saddle point. That is, the constant energy trajectory contains a point at which (18) must equal zero. This point is clearly a minimum of (18), which is easily computed. An equation for the zero-damping switching field is the equation which sets the minimum value of (18) to zero. The solution is

$$h_y^2 = \frac{1}{8} + \frac{5}{2} h_z - h_z^2 + \frac{1}{8} (1 - 8 h_z)^{3/2}, \tag{19}$$

for $-1 < h_z < 1/8$.

VI. Conclusions

Analytic methods have confirmed the finding of previous numerical studies that coherent magnetization reversal can be predicted by the LLG equation for applied field pulses with magnitude below the Stoner-Wohlfarth limit. The necessary conditions are short rise time of the applied field pulse, and sufficiently small damping parameter that the rate of energy increase due to the applied field outpaces the rate of energy dissipation due to the LLG damping term. Both parametric and non-parametric expressions for the switching field in the limit of zero rise time and zero damping have been derived. These analytical solutions have utility as test cases for the verification of numerical solvers.

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