JOURNAL OF COMPUTATIONAL PHYSICS **136,** 580–584 (1997)
ARTICLE NO. CP975782

A Fast Spherical Filter with Uniform Resolution

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Received September 4, 1996; revised May 27, 1997

This paper introduces a fast algorithm for obtaining a uniform resolution representation of a function known at a latitude–longitude grid on the surface of a sphere, equivalent to a triangular, isotropic truncation of the spherical harmonic coefficients for the function. The proposed *spectral truncation method*, which is based on the fast multipole method and the fast Fourier transform, projects the function to a space with uniform resolution while avoiding surface harmonic transformations. The method requires $O(N^2 \log N)$ operations for $O(N^2)$ grid points, as opposed to $O(N^3)$ operations for the standard spectral transform method, providing a reduced-complexity spectral method obviating the pole problem in the integration of time-dependent partial differential equations on the sphere. The filter's performance is demonstrated with numerical examples. © 1997 Academic Press

1. INTRODUCTION

The standard spectral transform algorithm used in weather and climate modeling for spherical geometry is not of optimal computational complexity. For a function tabulated at $O(N^2)$ points on the sphere, the spectral transform algorithm to obtain the corresponding surface harmonic coefficients requires $O(N^3)$ operations. The inverse transformation, from coefficients to grid values, is of like cost. At high resolution, the surface harmonics transformation is the most expensive operation of the spectral transform method; other operations are of asymptotic complexity of at most $O(N^2 \log N)$. Orszag [11] has described an algorithm for fast transformation with asymptotic cost $O(N^2 \log^2 N/\log \log N)$. His algorithm, however, based on a low-order WKB method, is unlikely to be effective in applications requiring high accuracy. Driscoll and Healy [3] and, more recently, Healy, Moore, and Rockmore [7] have proposed a fast surface harmonics transformation algorithm of asymptotic complexity $O(N^2 \log^2 N)$, but some questions remain regarding its efficiency and stability. Alpert and Rokhlin [1] have described a fast algorithm for transforming Legendre polynomial expansions, but it appears not to generalize to the spherical case. Swarztrauber [12] has reviewed other transformation algorithms.

In this paper we introduce a technique for avoiding the surface harmonics transformation, while retaining the benefits of using surface harmonics representations. The technique is based on a fast algorithm for the orthogonal projection of a function defined on the sphere and known at a latitude–longitude grid onto the space spanned by a truncated surface harmonic expansion. Appropriate choice of the truncation gives representations with uniform resolution. The algorithm uses the Christoffel–Darboux formula [14] for the summation of products of orthogonal polynomials, in combination with the fast multipole method [5] and the fast Fourier transform. Semi-implicit integration of partial differential equations (PDE) on the sphere can be accomplished by computing products of functions in the physical domain, computing differential operators and the inverse Laplacian in a Fourier representation [10], and truncating to uniform resolution at each time step.

In this paper we restrict our attention to the definition and performance of the filter itself, while leaving the demonstration of its use in the integration of PDE on the sphere to a later paper. In Section 2 we define the filter mathematically and describe the filtering operation via the proposed algorithm, as well as via the standard transform algorithm. In Section 3 we show accuracy and performance results for both methods.

2. THE SPHERICAL FILTER

This section consists of three subsections: in the first we mathematically define the spherical filter; in the second we summarize the standard spectral transform algorithm; and in the third we describe the novel spectral truncation method.

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0021-9991/97 $25.00

2.1. *Isotropic Spherical Truncation*

The surface harmonics $Y_n^m(\phi, \theta)$ of degree $n = 0, 1, 2, \ldots$ and order $m = -n, -n+1, \ldots, n-1, n$ at longitude ϕ and latitude θ are defined by

$$Y_n^m(\phi, \theta) = \frac{1}{\sqrt{2\pi}} \overline{P}_n^m (\sin \theta) e^{im\phi}, \tag{1}$$

where $\overline{P}_n^m$ denotes the normalized associated Legendre functions,

$$\overline{P}_n^m(\mu) = \sqrt{\left(n + \frac{1}{2}\right) \frac{(n-m)!}{(n+m)!}} (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu), \tag{2}$$

for $m \geq 0$ and $\mu \in [-1, 1]$, where P_n is the Legendre polynomial of degree n. For $m < 0$ and $|m| \leq n$, $\overline{P}_n^m(\mu) = \overline{P}_n^{-m}(\mu)$.

The set of surface harmonics $Y = \{Y_n^m\}$ form an orthonormal basis for the space $L^2(S)$ of square-integrable functions on the surface of the unit sphere S. Furthermore, each triangular truncation of Y possesses uniform resolution on the sphere. In particular, if a function $f: S \to \mathbf{C}$ is given by

$$f(\phi, \theta) = \sum_{n=0}^{N} \sum_{m=-n}^{n} f_n^m \cdot Y_n^m(\phi, \theta) \tag{3}$$

for some N, then any rotation of f can be represented by an expansion of the same degree with appropriately transformed coefficients. This property of uniform resolution, in addition to spectral convergence and the simple representation of differential operators, is the reason for the popularity of the surface harmonics for the integration of partial differential equations on the sphere.

The spectral transform method relies on an algorithm for the orthogonal projection of a function f known at a latitude–longitude grid onto the space spanned by surface harmonics Y_n^m of degree $n \leq N$ (see, for example, [13]). In particular, if f is band-limited of degree K, it can be expanded as

$$f(\phi, \theta) = \sum_{n=0}^{K} \sum_{m=-n}^{n} f_n^m \cdot Y_n^m(\phi, \theta) \tag{4}$$

Suppose the surface discretization (grid)

$$(\phi_i, \theta_j), \quad i = 1, \ldots, I; \quad j = 1, \ldots, J, \tag{5}$$

has the property that the function values

$$f(\phi_i, \theta_j), \quad i = 1, \ldots, I; \quad j = 1, \ldots, J,$$

uniquely determine the coefficients f_n^m for $|m| \leq n \leq N \leq K$. The *truncation* of degree N is the linear transformation

$$\{f(\phi_i, \theta_j)\} \to \{\tilde{f}(\phi_i, \theta_j)\}, \tag{6}$$

where

$$\tilde{f}(\phi, \theta) = \sum_{n=0}^{N} \sum_{m=-n}^{n} f_n^m \cdot Y_n^m(\phi, \theta). \tag{7}$$

The standard algorithm uses a particular, convenient choice for the grid $\{(\phi_i, \theta_j)\}$ (presented in the next subsection). We have implemented the new algorithm for this grid, but the algorithm will also work for other choices of $\{\theta_j\}$ and even for one choice of $\{\theta_j\}$ for f and another for $\tilde{f}$. This flexibility is expected to provide some computational advantages when the filter algorithm is used for the integration of PDE.

2.2. *Spectral Transform Method*

The spectral transform method performs a forward and backward spherical transform for computing the truncation (6).

First, the Fourier coefficients $f^m(\theta)$ of the function $f(\phi, \theta)$ are determined for $m = -N, \ldots, N$, for latitudes $\theta_1, \ldots, \theta_J$ (defined below) by the formula

$$\begin{aligned} f^m(\theta) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\phi, \theta) e^{-im\phi}\, d\phi \\ &= \frac{\sqrt{2\pi}}{I} \sum_{i=1}^{I} f(\phi_i, \theta) e^{-im\phi_i}, \end{aligned} \tag{8}$$

where $\phi_i = 2\pi i/I$ for $i = 1, \ldots, I$. The number I of gridpoints in the east–west direction is chosen to allow this Fourier analysis to be exact; for band limit K, this restriction implies $I \geq K + N + 1$. The equispaced longitudes ϕ_i allow use of the fast Fourier transform.

Second, the spherical harmonic coefficients f_n^m, for $|m| \leq n \leq N$, are determined by a surface spherical transform, using the Gaussian quadrature

$$\begin{aligned} f_n^m &= \int_{-\pi/2}^{\pi/2} f^m(\theta) \overline{P}_n^m(\mu) \cos\theta\, d\theta \\ &= \sum_{j=1}^{J} f^m(\theta_j) \overline{P}_n^m(\mu_j)\, w_j, \end{aligned} \tag{9}$$

where $\mu = \sin\theta$ and $\theta_1, \ldots, \theta_J$ denote the roots of $P_J(\sin\theta)$ and $w_1, \ldots, w_J$ denote the corresponding Gaussian quadrature weights (see, for example, Szegö [14, p. 47]). The number J of grid points in the north–south direction is

TABLE I

Parameters Used in the Filter Implementations

N	K	I	J	V
15	30	48	24	1
31	62	96	48	2
42	84	128	64	2
63	126	192	96	4
79	158	240	120	4
85	170	256	128	4
95	190	288	144	8
106	212	320	160	8
119	238	360	180	8
127	254	384	192	8
143	286	432	216	8
159	318	480	240	8
170	340	512	256	8
190	380	576	288	16
213	426	640	320	16
239	478	720	360	16
255	510	768	384	16
319	638	960	480	16
341	682	1024	512	16

Note. The truncation index N, band limit K, east-west grid size I, and north-south grid size J are chosen so that I allows efficient FFTs (has only prime factors 2, 3, and 5), J is even, and the exact reconstruction constraints $I \geq K + N + 1$ and $J \geq (K + N + 1)/2$ are satisfied. The number V of intervals used for the FMM is also shown.

chosen to allow this spectral analysis to be exact; for band limit K, this requires $J \geq (K + N + 1)/2$.

Third, the filtered Fourier coefficients $\tilde{f}^m(\theta_j)$ are computed by a backward surface harmonic transform, using only the spherical coefficients up to degree N,

$$\tilde{f}^m(\theta) = \sum_{n=|m|}^{N} f_n^m \cdot \overline{P}_n^m(\mu). \tag{10}$$

Fourth and last, the filtered grid values $\tilde{f}(\phi_i, \theta_j)$ of the function are computed by a backward fast Fourier transform, again using only the Fourier coefficients up to degree N,

$$\tilde{f}(\phi, \theta) = \sqrt{2\pi} \sum_{m=-N}^{N} \tilde{f}^m(\theta) e^{im\phi}. \tag{11}$$

The Fourier coefficients $f^m(\theta_j)$ are computed by the first step in $O(IJ \log I)$ operations. The spherical harmonic coefficients f_n^m are computed by the second step in $O(JN^2)$ operations, while the third step requires $O(JN^2)$ operations to compute Fourier coefficients $\tilde{f}^m(\theta_j)$. Finally, the filtered function values $\tilde{f}(\phi_i, \theta_j)$ are obtained in the fourth step in $O(IJ \log I)$ operations. If $K = 2N$ (to allow representation of products of functions) and I and J are chosen as small as possible, then $I = 3N + 1$ and $J = (3N + 1)/2$ and the overall complexity of the spectral transform filter is of $O(N^3)$ for $O(N^2)$ grid values.

2.3. *Spectral Truncation Method*

In this subsection we present an algorithm for applying the truncation filter (6) in $O(N^2 \log N)$ operations for representations of size $O(N^2)$. An earlier presentation of this algorithm may be found in Jakob [8]. The procedure obviates the computation of surface harmonic transformations for integration of PDE on the sphere.

We suppose a function f defined on the sphere is given by the surface harmonic expansion (4). The spectral truncation algorithm combines steps 2 and 3 of the standard spectral transform algorithm, simplifies the sum with the Christoffel–Darboux formula for orthogonal polynomials, and evaluates the resulting sum with the fast multipole method.

The Christoffel–Darboux formula [14, p. 42] for the associated Legendre functions has the form

TABLE II

Relative l_2 Truncation Error and l_2 Representation Error for the Spectral Transform and Spectral Truncation Algorithms

	l_2 truncation error		l_2 representation error	
N	Standard transform	Multipole truncation	Standard transform	Multipole truncation
15	8.80E − 14	9.10E − 14	1.00E − 01	1.00E − 01
31	5.36E − 13	5.31E − 13	1.33E − 02	1.33E − 02
42	7.08E − 13	6.91E − 13	6.07E − 03	6.07E − 03
63	1.20E − 12	1.27E − 12	1.97E − 03	1.97E − 03
79	5.21E − 12	5.14E − 12	1.22E − 03	1.22E − 03
85	5.52E − 12	5.68E − 12	9.33E − 04	9.33E − 04
95	8.62E − 12	8.81E − 12	7.09E − 04	7.09E − 04
106	9.11E − 12	9.05E − 12	5.72E − 04	5.72E − 04
106	9.11E − 12	9.05E − 12	5.72E − 04	5.72E − 04
119	2.71E − 12	2.74E − 12	4.19E − 04	4.19E − 04
127	1.37E − 11	1.37E − 11	3.63E − 04	3.63E − 04
143	2.13E − 11	2.15E − 11	2.63E − 04	2.63E − 04
159	7.61E − 12	7.47E − 12	1.97E − 04	1.97E − 04
170			1.66E − 04	1.66E − 04
190			1.29E − 04	1.29E − 04
213			9.86E − 05	9.86E − 05
239			7.43E − 05	7.43E − 05
255			6.22E − 05	6.22E − 05
319			3.53E − 05	3.53E − 05
341			3.03E − 05	3.03E − 05

Note. The test field for the truncation error was a random, but band-limited field. The truncation error is measured relative to a truncation computed with the standard transform with 128 bit arithmetic. The test field for representation error was the height field for the shallow water equations test case 1 (local cone with discontinuous second derivative) described in Williamson *et al.* [15] rotated by an angle $\alpha = \pi/2$. The error is measured relative to the analytic solution [9]. The truncation error at high resolutions is omitted due to memory constraints.

TABLE III

Execution Times for the Spectral Transform and Spectral Truncation Algorithms

	Execution time (s)		
N	Standard transform	Multipole truncation	Forward & back. FFTs
63	0.042	0.046	0.011
79	0.091	0.077	0.046
85	0.109	0.093	0.041
95	0.153	0.130	0.060
106	0.227	0.160	0.095
119	0.318	0.204	0.116
127	0.372	0.230	0.154
143	0.563	0.301	0.196
159	1.456	0.368	0.272
170	0.950	0.575	0.358
190	1.314	0.599	0.582
213	1.805	0.748	0.700
239	3.765	0.940	1.095
255	3.632	1.378	1.256
319	8.699	1.738	2.035
341	13.638	2.483	2.334

Note. The times are for the filters, excluding time for the forward and backward FFTs, which are shown separately, on a workstation with a double precision LINPACK performance rating of approximately 130 million floating-point operations per second and 512 Mbytes memory.

$$(\tilde{\mu} - \mu) \sum_{n=|m|}^{N} \overline{P}_n^m(\tilde{\mu}) \overline{P}_n^m(\mu) = \varepsilon_{N+1}^m [\overline{P}_{N+1}^m(\tilde{\mu}) \overline{P}_N^m(\mu) - \overline{P}_N^m(\tilde{\mu}) \overline{P}_{N+1}^m(\mu)], \tag{12}$$

where

$$\varepsilon_n^m = \sqrt{(n^2 - m^2)/(4n^2 - 1)}. \tag{13}$$

Combining (9), (10), and (12) we obtain for $\theta \neq \theta_i$ ($\mu \neq \mu_i$)

$$\begin{aligned} \tilde{f}^m(\theta) &= \sum_{n=|m|}^{N} \left(\sum_{i=1}^{J} f^m(\theta_i) \overline{P}_n^m(\mu_i) w_i \right) \overline{P}_n^m(\mu) \\ &= \sum_{i=1}^{J} f^m(\theta_i) w_i \sum_{n=|m|}^{N} \overline{P}_n^m(\mu_i) \overline{P}_n^m(\mu) \\ &= \sum_{i=1}^{J} f^m(\theta_i) w_i \varepsilon_{N+1}^m \cdot \frac{\overline{P}_{N+1}^m(\mu) \overline{P}_N^m(\mu_i) - \overline{P}_N^m(\mu) \overline{P}_{N+1}^m(\mu_i)}{\mu - \mu_i}. \end{aligned} \tag{14}$$

For $\mu = \mu_i$ the quotient is evaluated with l'Hôpital's rule. If f^m is known at $\theta_1, ..., \theta_J$, the evaluation of $\tilde{f}^m$ at a (possibly different) set of nodes $\tilde{\theta}_1, ..., \theta_J$ requires two applications of the fast multipole method (FMM) of size J:

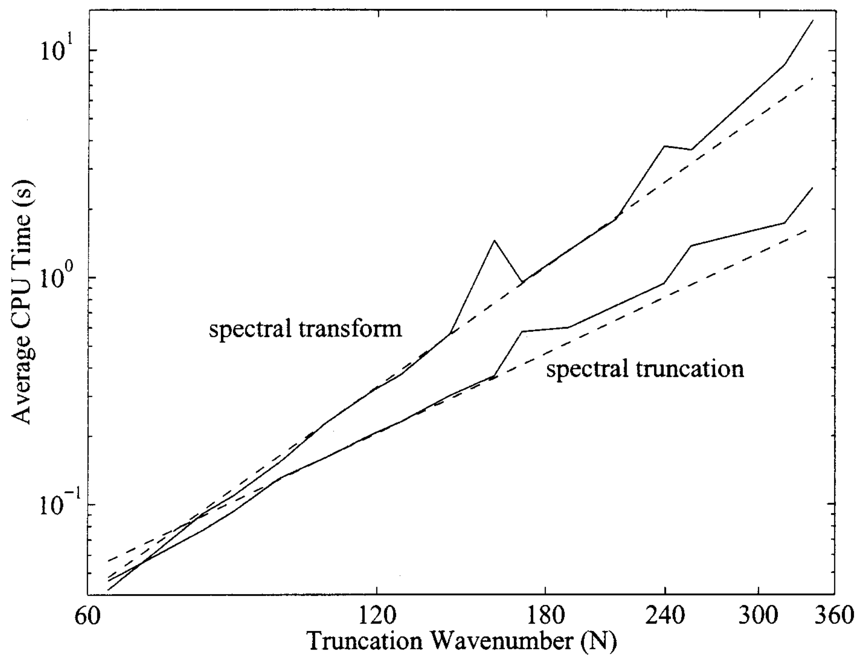

FIG. 1. Log–log plot of execution times for the spectral transform and spectral truncation implementations, as a function of truncation index N (from Table III). The dashed lines show exact $O(N^2)$ and $O(N^3)$ time complexities, with intercepts chosen to match the actual times at size $N = 106$.

$$\frac{\tilde{f}^m(\tilde{\theta}_j)}{\varepsilon^m_{N+1}} = \overline{P}^m_{N+1}(\mu_j) \sum_{i=1}^{J} \frac{f^m(\theta_i) w_i \overline{P}^m_N(\mu_i)}{\tilde{\mu}_j - \mu_i} - \overline{P}^m_N(\tilde{\mu}_j) \sum_{i=1}^{J} \frac{f^m(\theta_i) w_i \overline{P}^m_{N+1}(\mu_i)}{\tilde{\mu}_j - \mu_i}. \qquad (15)$$

The FMM allows the evaluation of the J filtered function values for a specific Fourier wavenumber m with $O(J)$ operations. The total operation cost for steps 2 and 3 is thus $O(N^2)$. The fast Fourier transform is still used for steps 1 and 4.

If we assume instead that f^m is known at non-Gaussian nodes, then a single application of the FMM can be used to interpolate f^m (see Section 2.3) to the nodes $\theta_1, ..., \theta_J$ in preparation for the truncation (see, for example, Boyd [2]).

3. NUMERICAL ACCURACY AND PERFORMANCE

The spherical filter was implemented using both the spectral transform method and the new spectral truncation method to compare the two algorithms' accuracy and cost. The spectral transform implementation was derived from the implementation of Hack and Jakob [6], which resides in Netlib. The spectral truncation implementation used an improved FMM, as developed by Dutt and Rokhlin [4], employing the singular value decomposition with 11 singular vectors, obtained from a 27-point Chebychev interpolation. Both implementations are in double-precision 64-bit arithmetic and share the code for the computation of the spherical harmonics. The implementation parameters are shown in Table I, truncation and representation errors are shown in Table II, and execution times are shown in Table III. The timings are graphed in Fig. 1.

The accuracy of the two algorithms is identical for practical purposes and largely determined by the quality (orthogonality and orthonormality) of the computed spherical harmonics on the grid. For band-limited functions, the relative l_2 error (Table II, columns 2 and 3) and l_∞ error (not shown) of the truncation is below the orthogonality and orthonormality errors in Table IV of the reference solutions by Jakob-Chien, Hack, and Williamson [9]. For functions which are not band-limited, the convergence depends on the number of continuous derivatives of the function (Table II, columns 4 and 5). Both algorithms perform according to the theory; the computational complexities of $O(N^3)$ and $O(N^2)$, respectively, can be seen in the timings of Table III and in Fig. 1. The new spectral truncation algorithm starts to be significantly faster than the spectral transform algorithm at truncation index $N = 106$. The longer execution time for the spectral transform implementation at $N = 159$ is machine dependent and was not observed on other workstations. The spectral truncation algorithm also significantly reduces the memory requirements of the filter. Instead of $JN^2/2$ words for the spherical harmonic basis functions in the spectral transform algorithm, the multipole-based spectral truncation algorithm requires only $4JN$ spherical harmonics coefficients at the truncation limit.

ACKNOWLEDGMENTS

The authors thank Jim Hack and Ron Wittmann for their interest and for numerous helpful discussions.

REFERENCES

1. B. K. Alpert and V. Rokhlin, A fast algorithm for the evaluation of Legendre expansions. *SIAM J. Sci. Statist. Comput.* **12,** 158 (1991).
2. J. P. Boyd, Multipole expansions and pseudospectral cardinal functions: A new generalization of the fast Fourier transform, *J. Comput. Phys.* **103,** 184 (1992).
3. J. Driscoll and D. Healy. Computing Fourier transforms and convolutions on the 2-sphere. *Adv. in Appl. Math.* **15,** 202 (1994).
4. A. Dutt and V. Rokhlin. Fast Fourier Transforms for Nonequispaced Data. *SIAM J. Sci. Comput.* **14,** 1368 (1993).
5. L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. *J. Comp. Phys.* **73,** 325 (1987).
6. J. J. Hack and R. Jakob, *Description of a Global Shallow Water Model Based on the Spectral Transform Method,* Technical Note TN-343+STR, National Center for Atmospheric Research, Boulder, CO, February 1992.
7. D. Healy, S. Moore, and D. Rockmore. An FFT for the 2-sphere and applications, preprint, Department of Mathematics, Dartmouth College, 1996.
8. R. Jakob, *Fast and Parallel Spectral Transform Algorithms for Global Shallow Water Models,* Ph.D. thesis, University of Colorado at Boulder, 1993.
9. R. Jakob-Chien, J. J. Hack, D. L. Williamson, Spectral transform solutions to the shallow water test set. *J. Comp. Phys.* **119,** 164 (1995).
10. S. A. Orszag. Fourier Series on Spheres. *Mon. Weather Rev.* **102,** 56 (1974).
11. S. A. Orszag. Fast eigenfunction transforms. in *Science and Computers, Advances in Mathematics Supplementary Studies, Vol. 10*, edited by G. C. Rota, p. 23 (Academic Press, New York, 1986).
12. P. N. Swarztrauber. The approximation of vector functions and their derivatives on the sphere. *SIAM J. Num. Anal.* **18**(2), 191 (1981).
13. P. N. Swarztrauber. Spectral transform methods for solving the shallow-water equations on the sphere. *Mon. Weather Rev.* **124,** 730 (1996).
14. G. Szegö. *Orthogonal Polynomials* (Am. Math. Soc., Providence, RI, 1939).
15. D. L. Williamson, J. B. Drake, J. J. Hack, R. Jakob, and P. N. Swarztrauber, A standard test set for numerical approximations to the shallow water equations in spherical geometry. *J. Comp. Phys.* **102,** 211 (1992).