The Angle Between Null Spaces of the Radon and Related Transforms

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

by
Michael J. Donahue

*****

The Ohio State University
1991

Dissertation Committee

Boris Mityagin

Ranko Bojanic

Stephen Milne

Stanislav Rokhlin

Approved by

Adviser
Department of Mathematics

Vita

May 31, 1962	born—Columbus, Ohio
1981–1983	Undergraduate Research Assistant, University of Dayton Research Institute, Dayton, Ohio
1984	B.S. Electrical Engineering, The Ohio State University, Columbus, Ohio
1984–1987	Graduate Teaching Associate, Department of Mathematics, The Ohio State University
1985	M.S. Mathematics, The Ohio State University
1987–Present	Graduate Research Associate, Department of Welding Engineering, The Ohio State University

Publications

1. M. J. Donahue, A. P. Sprague and S. I. Rokhlin, "Point Matching Method for Flaw Detection in Printed Circuit Boards," in *Review of Progress in Quantitative NDE*, D. O. Thompson and D. E. Chimenti eds., **8B**, 1233-1240 (Plenum Press, New York, 1989).

Field of Study

Major Field: Mathematics

Table of Contents

List of Figures

FIGURES **PAGE**

CHAPTER I

Introduction

Consider a square integrable function f on the unit ball in $\mathbf{R}^N$. Fix a direction $\sigma \in S^{N-1}$ and a distance $t \in \mathbf{R}^1$, and integrate f over the $(N-1)$-dimensional hyperplane $\{x \in \mathbf{R}^N \mid \langle x, \sigma \rangle = t\}$. The resulting function of σ and t, denoted $Rf(\sigma, t)$, is the Radon transform of f. I will typically consider σ to be fixed and denote the resulting function of t by $R_\sigma f(t)$. In the general $(N-k)$-plane transforms, the integration over $(N-1)$-dimensional hyperplanes is replaced with integration over $(N-k)$-dimensional affine subspaces, σ is replaced with an orthogonal matrix and t is replaced with an element of $\mathbf{R}^k$. The special case $N-k=1$ (integration along lines) is known as the X-ray transform.

Early work on the inversion of transforms of this type dates back to Funk [1] and Radon [2]. Formulae for the inversion of the Radon (and also the related transforms) require the transformed function to be known for all σ and t. In practical reconstructions, however, $Rf(\sigma, t)$ is known at only a finite number of points. Of particular concern is the limited number of σ at which the transform is evaluated. There are several different approximate inversion techniques, and in each of them the angle between the null spaces of the transforms R_σ (for varying σ) is important. (Natterer's book [3] provides a good overview of the subject with references.) In Chapter II I present a brief introduction (covering known results) on the concept of the angle between subspaces of a Hilbert space .

The focus of this dissertation is the study of the angle between the null spaces of the general $(N-k)$-plane transforms on spaces of square integrable functions. Hamaker and Solmon [4] studied this problem on $L^2(\Omega^2)$, the space of square integrable functions on the unit disk in R^2. (Here the general $(N-k)$-plane transform necessarily has $N-k=1=N-1$, so the Radon transform is the only example here.) They showed that the angle between the null spaces of R_{σ_1} and R_{σ_2} is

$$\inf_{n\in\mathbf{N}} \arccos\left(\frac{|\sin(n+1)\theta|}{(n+1)\sin\theta}\right), \tag{1.1}$$

where θ is the angle between σ_1 and σ_2. Davison and Grunbaum [5], also working on $\mathbf{R}^2$, introduced weighting functions and showed the angle to be

$$\inf_{n\in\mathbf{N}} \arccos\left(C_n^{(\alpha)}(\cos\theta)/C_n^{(\alpha)}(1)\right), \tag{1.2}$$

where $C_n^{(\alpha)}$ is the Gegenbauer polynomial of degree n with parameter α. The value of α depends upon the weighting function. The case with $\alpha=1$ reduces to the problem of Hamaker and Solmon.

In Chapter III I present new results on the angle between null spaces of the general $(N-k)$-plane transform on R^N applied to square integrable functions on the unit ball $(L^2(\Omega^N))$. In particular, I reduce the problem to finding the supremum of the eigenvalues of a collection of (explicitly given) finite dimensional matrices. If $1 < N-k < N-1$, the dimensions of the matrices are not bounded. However, if $N-k = N-1$ or $N-k = 1$ (the Radon and X-ray transforms, respectively), then each of these matrices is triangular, and so explicit formula for the matrix entries, (3.33), provides the eigenvectors directly.

In Chapter IV I extend the known results (1.1) and (1.2) on $L^2(\Omega^2)$ to corresponding results on $L^2(\Omega^N)$ for both the Radon and the X-ray transforms. I show that the angle between null spaces of the Radon transform on $L^2(\Omega^N)$ is given by the formula of Davison and Grunbaum with $\alpha = N/2$, and that the angle between the null spaces of the X-ray transform on $L^2(\Omega^N)$ is given by the formula of Hamaker and Solmon (for all N). I also show the new result that the infimum over $n \in \mathbf{N}$ in both (1.1) and (1.2) can be replaced with the minimum over $n = 1, 2$.

In Chapter V I modify the general $(N-k)$-plane transform problem of Chapter III by replacing the domain $L^2(\Omega^N)$ with $L^2\left(\mathbf{R}^N, e^{-\|x\|^2}\right)$. This simplifies the problem considerable. The development in Chapter V parallels the Chapter III, but the resulting matrices are 1×1 for all $N-k$. The special case $N = 2$ has been solved previously by Davison and Grunbaum [5].

Funk's paper of 1916 [1] dealt with the inversion of the transform resulting from the integration over great circles on the unit sphere S^2. In Chapter VI I consider the transform resulting from the integration not over great circles but rather over the so-called “latitude” circles. This transform has as a parameter the choice of the “polar” axis. I develop an explicit formula for the angle between the null spaces of these transforms as a function of the angle between the “polar” axes.

Rounding out the paper are two appendices. Appendix A presents known formulae needed in the body of the work. These can be found in standard reference works (for example, [6] and [12]). Appendix B contains a proof of an “obvious” but inaccessible result which is needed in several places in the main body of the work.

CHAPTER II

Angles between subspaces in a Hilbert space

This chapter provides an introduction to the concept of the angle between subspaces of a Hilbert space, including a proof of a known result (Theorem 1) which I shall need throughout this paper.

Definition Let E, F be closed subspaces of a Hilbert space H. Then the angle between E and F, written $\gamma(E,F)$, is the scalar between 0 and $\pi/2$ satisfying

$$\cos(\gamma(E,F)) = \sup|\langle u,v\rangle|, \tag{2.1}$$

where the supremum is taken over all $u \in E \cap (E \cap F)^{\perp}$, $v \in F \cap (E \cap F)^{\perp}$, with $\|u\| = \|v\| = 1$.

Note that this implies that $\gamma(E,F) = \inf(\arccos(|\langle u,v\rangle|))$, so $\gamma(E,F)$ is the infimum of the angle between two vectors u and v, $u \in E \cap (E \cap F)^{\perp}$ and $v \in F \cap (E \cap F)^{\perp}$. In particular, if $E \cap F = \{0\}$, then $\gamma(E,F)$ is just the smallest angle (actually the greatest lower bound of the angle) between vectors u and v, $u \in E$ and $v \in F$.

Example Let $H = \ell^2$, $E = \{a \in \ell^2 \mid a_{2k} = 0 \text{ for } k = 0,1,2,\ldots\}$, and let $F = \{b \in \ell^2 \mid b_{2k+1} = b_{2k}/k \text{ for } k = 0,1,2,\ldots\}$. Note that $E \cap F = \{0\}$. Let $\{e^k\}_{k=0}^{\infty}$ be the standard basis, i.e., $(e^k)_l = \delta_{kl}$, $l = 0,1,2,\ldots$. Let $u_n = e^{2n} \in E$, $v_n = e^{2n} + e^{2n+1}/n \in F$. Then

$$\|u_n\| = 1, \qquad \|v_n\| = \sqrt{1 + 1/n^2} \tag{2.2}$$

and

$$\left\langle u_n, \frac{v_n}{\sqrt{1+1/n^2}} \right\rangle = \frac{1}{\sqrt{1+1/n^2}} \longrightarrow 1 \text{ as } n \to \infty. \tag{2.3}$$

Since $|\langle u, v\rangle| \leq \|u\|\|v\|$, it follows that

$$\sup_{\substack{u \in E, v \in F \\ \|u\| = \|v\| = 1}} |\langle u, v\rangle| = 1, \tag{2.4}$$

i.e., $\gamma(E, F) = 0$.

Note that in this example, the supremum in (2.1) is not attained. This cannot happen, of course, if H is finite dimensional, since then the unit sphere is compact.

Following is a theorem which I shall require thoughout the main body of this paper. This "obvious" result is well known, though the proof in the infinite dimensional case requires some care.

Theorem 1 *Let E, F be closed subspaces of a Hilbert space. Then* $\gamma(E, F) = \gamma(E^{\perp}, F^{\perp})$.

Proof: Since $(E^\perp)^\perp = E$ (and similarly for F), it suffices to show that $\gamma(E,F) \geq \gamma(E^\perp, F^\perp)$, since then $\gamma(E^\perp, F^\perp) \geq \gamma((E^\perp)^\perp, ((F^\perp)^\perp) = \gamma(E,F)$.

So let

$$\alpha = \sup |\langle u, v\rangle|, \tag{2.5}$$

where the supremum is taken over all $u \in E \cap (E\cap F)^\perp$, $v \in F \cap (E\cap F)^\perp$, $\|u\| = \|v\| = 1$. Let $0 < \epsilon < 1$ be fixed, and choose u_ϵ, v_ϵ as above with

$$|\langle u_\epsilon, v_\epsilon\rangle| \geq \alpha(1-\epsilon). \tag{2.6}$$

Now we want to construct $u^* = u^*_\epsilon \in E^\perp \cap (E^\perp \cap F^\perp)^\perp$, $v^* = v^*_\epsilon \in F^\perp \cap (E^\perp \cap F^\perp)^\perp$, such that

$$\left|\left\langle \frac{u^*}{\|u^*\|}, \frac{v^*}{\|v^*\|}\right\rangle\right| \to \alpha \text{ as } \epsilon \downarrow 0. \tag{2.7}$$

This will show that $\cos(\gamma(E^\perp, F^\perp)) \geq \alpha \Rightarrow \gamma(E^\perp, F^\perp) \leq \gamma(E,F)$. To this end, pick $r (= r_\epsilon) \in E$ to minimize $\|v_\epsilon - r\|$. This is possible since E is a closed, convex set in a Hilbert space (v_ϵ is fixed). Let $u^* = v_\epsilon - r$. Then $u^* \in E^\perp$ (by choice of r), but moreover, if $y \in E^\perp \cap F^\perp$, then $\langle u^*, y\rangle = \langle v_\epsilon - r, y\rangle = 0$ since $v_\epsilon \in F$ and $r \in E$. Thus

$$u^* \in E^\perp \cap (E^\perp \cap F^\perp)^\perp. \tag{2.8}$$

Next take $x \in E \cap F$. Then $\langle r, x\rangle = \langle v_\epsilon - u^*, x\rangle = 0$ since $v_\epsilon \in (E\cap F)^\perp$ and $u^* \in E^\perp$. Therefore

$$r \in E \cap (E \cap F)^\perp. \tag{2.9}$$

Also note that this shows that $\|u^*\|^2 + \|r\|^2 = \|v_\epsilon\|^2 = 1$ since $r \in E$ and $u^* \in E^\perp$, so

$$\|u^*\|^2 = 1 - \|r\|^2. \tag{2.10}$$

Furthermore, since $r/\|r\|$ is a unit vector in $E \cap (E \cap F)^\perp$, we have from (2.5) that

$$\alpha \geq |\langle v_\epsilon, r/\|r\| \rangle| = |\langle u^* + r, r/\|r\| \rangle| = |\langle r, r/\|r\| \rangle| = \|r\|. \tag{2.11}$$

Finally, (2.10) shows that minimizing $\|v_\epsilon - r\| = \|u^*\|$ corresponds to maximizing $\|r\| = |\langle v_\epsilon, r/\|r\| \rangle|$ by (2.11). In particular, $|\langle v_\epsilon, r/\|r\| \rangle| \geq |\langle v_\epsilon, u_\epsilon \rangle| \geq \alpha(1-\epsilon)$. Thus

$$\|r\| \geq \alpha(1-\epsilon). \tag{2.12}$$

Likewise, chose $s (= s_\epsilon) \in F$ to minimize $\|u_\epsilon - s\|$, and define $v^* = u_\epsilon - s$. Then we get the corresponding relations

$$v^* \in F^\perp \cap (E^\perp \cap F^\perp)^\perp \tag{2.13}$$

$$s \in F \cap (E \cap F)^\perp \tag{2.14}$$

$$\|v^*\|^2 = 1 - \|s\|^2 \tag{2.15}$$

$$\alpha(1-\epsilon) \leq \|s\| \leq \alpha. \tag{2.16}$$

Also note that

$$\langle v_\epsilon, u_\epsilon \rangle = \langle u^* + r, u_\epsilon \rangle = \langle r, u_\epsilon \rangle \tag{2.17}$$

since $u^* \in E^\perp$ (statement (ustarspace)) and $u_\epsilon \in E$. Likewise,

$$\langle v_\epsilon, u_\epsilon \rangle = \langle v_\epsilon, v^* + s \rangle = \langle v_\epsilon, s \rangle. \tag{2.18}$$

We are now ready to prove (2.7):

$$\left|\left\langle \frac{u^*}{\|u^*\|}, \frac{v^*}{\|v^*\|} \right\rangle\right| = \frac{1}{\|u^*\|\|v^*\|} |\, \langle v_\epsilon - r, u_\epsilon - s\rangle \,|$$

$$\begin{aligned}
&= \frac{1}{\|u^*\|\|v^*\|} |\, \langle v_\epsilon, u_\epsilon\rangle - \langle r, u_\epsilon\rangle - \langle v_\epsilon, s\rangle + \langle r, s\rangle \,| \\
&= \frac{1}{\|u^*\|\|v^*\|} |\, \langle r, s\rangle - \langle v_\epsilon, u_\epsilon\rangle \,| && \text{by (2.17), (2.18)} \\
&\geq \frac{\alpha(1-\epsilon) - \|r\|\|s\|| \, \langle r/\|r\|, s/\|s\|\rangle \,|}{\sqrt{(1-\|r\|^2)(1-\|s\|^2)}} && \text{by (2.6), (2.10), (2.15)} \\
&\geq \frac{\alpha(1-\epsilon) - \alpha^3}{1-\alpha^2(1-\epsilon)^2} && \text{by (2.5), (2.9), (2.11),} \\
& && \text{(2.12), (2.14), (2.16)} \\
&= \alpha\left[\frac{1-\alpha^2-\epsilon}{1-\alpha^2(1-\epsilon)^2}\right] \to \alpha \quad \text{as } \epsilon \downarrow 0
\end{aligned}$$

$\square$

CHAPTER III

Angles between null spaces of the general $(N-k)$-plane transforms on $\mathbf{R}^N$

In this chapter I study the problem of determining the angle between null spaces of the general $(N-k)$-plane transforms. I prove an original result which reduces the problem to one of finding eigenvalues for explicitly given finite dimensional matrices. The Radon and X-ray transforms are special cases for which the eigenvalues can be given explicitly. These cases are developed in detail in the succeeding chapter.

3.1 Definitions

Let $e_1, e_2, \ldots, e_N$ be the usual orthonormal basis for $\mathbf{R}^N$, and let $z = (z_1, z_2, \ldots, z_N) \in \mathbf{R}^N$. Let $0 < k < N$ and decompose $\mathbf{R}^N$ into $\mathbf{R}^k \oplus \mathbf{R}^{N-k}$, with $z = x \oplus y$, $x = (x_1, x_2, \ldots, x_k) = (z_1, z_2, \ldots, z_k)$ and $y = (y_1, y_2, \ldots, y_{N-k}) = (z_{k+1}, z_{k+2}, \ldots, z_N)$. Now consider the integral operator

$$S : L^2(\Omega^N) \to L^2\left(\Omega^k, (1 - \|x\|^2)^{(k-N)/2}\right)$$

defined by

$$Sf(x) = \int_{\mathbf{R}^{N-k}} f(x \oplus y)\, d^{N-k}y \tag{3.1}$$

where $f(z) = f(x \oplus y)$ is extended from Ω^N to $\mathbf{R}^N$ by setting $f(z) = 0$ if $\|z\| > 1$. Thus S is the operator produced by integrating over affine subspaces of dimension $N - k$ parallel to the subspace $\{z \in \mathbf{R}^N \mid z_1 = z_2 = \cdots = z_k = 0\}$. Note that

$$\begin{aligned} \|Sf\|^2 &= \int_{\Omega^k} |Sf(x)|^2 (1 - \|x\|^2)^{(k-N)/2}\, d^k x \\ &= \int_{\Omega^k} \left| \int_{B_{N-k}(\sqrt{1-\|x\|^2})} f(x \oplus y)\, d^{N-k}y \right|^2 (1 - \|x\|^2)^{(k-N)/2}\, d^k x \\ &\le V_{N-k} \int_{\Omega^N} |f(z)|^2\, d^N z \\ &= V_{N-k} \|f\|^2, \end{aligned}$$

where $B_{N-k}(\sqrt{1 - \|x\|^2})$ denotes the ball about the origin of radius $\sqrt{1 - \|x\|^2}$ in $\mathbf{R}^{N-k}$, and V_{N-k} is the volume of the unit ball in $\mathbf{R}^{N-k}$ (see formula (A.4)). Thus $\|S\| \le V_{N-k}^{1/2}$. But the inequality above becomes an equality if f is a constant function, so in fact $\|S\| = V_{N-k}^{1/2}$.

Define

$$\mathcal{N} = \text{Null } S = \{f \in L^2(\Omega^N) \mid Sf = 0\},$$

and

$$A = \{f \in L^2(\Omega^N) \mid \exists \tilde{f}(z) = f(z) \text{ a.e. with } \tilde{f}(x \oplus y) = \tilde{f}(x \oplus 0)\ \forall x \oplus y \in \Omega^N\}.$$

Thus A consists of those functions which are constant on planes parallel to the plane $\{z \in \mathbf{R}^N \mid z_1 = z_2 = \ldots = z_k = 0\}$.

Lemma 1 *The set A is the orthogonal complement in $L^2(\Omega^N)$ to $\mathcal{N}$, the null space of the operator S.*

Proof: Clearly $A \subseteq \mathcal{N}^\perp$, so consider a fixed $f \in \mathcal{N}^\perp$ and we will prove that f is also in A. Define

$$f_0(z) = f_0(x \oplus 0) = V_{N-k}^{-1} Sf(x)(1 - \|x\|^2)^{(k-N)/2}.$$

Then since $Sf \in L^2\left(\Omega^k, (1 - \|x\|^2)^{(k-N)/2}\right)$ we have

$$\begin{aligned}
\|f_0\|^2 &= \int_{\Omega^N} |f_0(z)|^2 \, d^N z \\
&= V_{N-k}^{-1} \int_{\Omega^k} |Sf(x)|^2 (1 - \|x\|^2)^{(k-N)/2} d^k x \\
&= V_{N-k}^{-1} \|Sf\|^2.
\end{aligned}$$

Thus $f_0 \in L^2(\Omega^N)$, and in particular $f_0 \in A$. Notice that $Sf_0 = Sf$, so $f - f_0 \in \mathcal{N}$. Also $f \in \mathcal{N}^\perp$ by assumption, and $f_0 \in A \subseteq \mathcal{N}^\perp$, so $f - f_0 \in \mathcal{N}^\perp$. But $\mathcal{N} \cap \mathcal{N}^\perp = \{0\}$, so it must be that $f = f_0 \in A$. Since f is an arbitrary element of $\mathcal{N}^\perp$, we have $\mathcal{N}^\perp = A$. $\square$

Let $U \in SO_N$ (an $N \times N$ orthogonal matrix with determinant $= +1$), and define $S_U : L^2(\Omega^N) \to L^2\left(\Omega^k, (1 - \|x\|^2)^{(k-N)/2}\right)$ by

$$S_U f(x) = S(f \circ U^{-1})(x). \tag{3.2}$$

(Extending this definition to O_N does not yield any new transformations. If $U \in O_N$ with $\det(U) = -1$, then we can multiply the last row of U by -1 to get say $\tilde{U}$

with $\tilde{U} \in SO_N$ and $S_{\tilde{U}} = S_U$.) Also, let $\mathcal{N}_U$ be the null space for the operator S_U and define

$$A_U = \{f \in L^2(\Omega^N) \mid f \circ U^{-1} \in A\}, \tag{3.3}$$

so A_U are those functions which are constant on subspaces parallel to the subspace resulting from applying U to $\{z \in \mathbf{R}^N \mid z_1 = z_2 = \cdots = z_k = 0\}$. In particular, A_U is the orthogonal complement in $L^2(\Omega^N)$ to the null space $\mathcal{N}_U$ of the operator S_U. Also note that $f \in A$ if and only if $f \circ U \in A_U$.

The angle between the null spaces $\mathcal{N}$ and $\mathcal{N}_U$ of the operators S and S_U is the same as the angle between the subspaces A and A_U, defined by

$$\begin{aligned} \cos(\gamma(A, A_U)) &= \sup_{\substack{\|f_1\|=\|f_2\|=1 \\ f_1 \in A \cap (A \cap A_U)^\perp \\ f_2 \in A_U \cap (A \cap A_U)^\perp}} |\langle f_1, f_2 \rangle| \\ &= \sup_{\substack{\|f_1\|=\|f_2\|=1 \\ f_i \in A \cap (A \cap A_U)^\perp}} |\langle f_1, f_2 \circ U \rangle| \end{aligned} \tag{3.4}$$

3.2 An equivalence relation on orthogonal matrices

For N and k fixed, the collection $\{S_U\}$ forms a family of operators indexed by $U \in SO_N$. We want to study the angle between the null spaces of operators from this family. Since $\gamma(A_{U_1}, A_{U_2}) = \gamma(A, A_{U_1 \circ U_2^{-1}})$, it suffices to study the angle between the null spaces of S and S_U. If $A_{U_1} = A_{U_2}$, then $\gamma(A, A_{U_1}) = \gamma(A, A_{U_2})$, so it is natural to define an equivalence relation $\sim$ on SO_N by

$$U_1 \sim U_2 \text{ if } A_{U_1} = A_{U_2}. \tag{3.5}$$

Let $X, Y \subset \mathbf{R}^N$ be defined by $X = \{z \in \mathbf{R}^N \mid z_{k+1} = z_{k+2} = \cdots = z_N = 0\}$ and $Y = \{z \in \mathbf{R}^N \mid z_1 = z_2 = \cdots = z_k = 0\}$. Let $B \subset SO_N$ be defined by $B = \{T \in SO_N \mid T : X \to X, T|_X \in SO_k\}$. (Note that $T \in B$ forces T to act on Y as an element of SO_{N-k}.) Then $T \in B$ has the form

$$T = \begin{array}{cc} & \begin{array}{cc} \scriptstyle k & \scriptstyle N-k \end{array} \\ \begin{array}{c} \scriptstyle k \\ \\ \scriptstyle N-k \end{array} & \begin{pmatrix} V & 0 \\ & \\ 0 & W \end{pmatrix} \end{array} \tag{3.6}$$

where $V \in SO_k$ and $W \in SO_{N-k}$.

Lemma 2 *If $T \in B$ and $U \in SO_N$, then $U \sim TU$.*

Proof: By definition, $f \in A_{TU}$ means (for proper choice of representative f)

$$f \circ U^{-1} \circ T^{-1}(x \oplus y) = f \circ U^{-1} \circ T^{-1}(x \oplus 0) \quad \text{for a.e. } x \oplus y.$$

This can be rewritten as

$$f \circ U^{-1}(V^{-1}x \oplus W^{-1}y) = f \circ U^{-1}(V^{-1}x \oplus 0) \quad \text{for a.e. } x \oplus y,$$

which is equivalent to

$$f \circ U^{-1}(x \oplus y) = f \circ U^{-1}(x \oplus 0) \quad \text{for a.e. } x \oplus y.$$

But this is the defining condition for $f \in A_U$, so $A_U = A_{TU}$, hence $U \sim TU$. $\square$

The following lemma provides a canonical representation for orthogonal matrices which I shall use throughout the remainder of this paper. Related (and more general) results can be found in [7].

Lemma 3 *For each $U \in SO_N$, there exist $\tilde{U} \sim U$ and orthonormal basis $\tilde{e}_1$, $\tilde{e}_2$, $\ldots$, $\tilde{e}_N$ with*

$$\text{lin span}\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k\} = \text{lin span}\{e_1, e_2, \ldots, e_k\}$$

(and consequently $\text{lin span}\{\tilde{e}_{k+1}, \tilde{e}_{k+2}, \ldots, \tilde{e}_N\} = \text{lin span}\{e_{k+1}, e_{k+2}, \ldots, e_N\}$*), such that with respect to the basis $\tilde{e}_1$, $\tilde{e}_2$, $\ldots$, $\tilde{e}_N$, $\tilde{U}$ has the form*

$$U = \begin{array}{cc} & \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{array}{c} m \\ k-m \\ m \\ N-k-m \end{array} & \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & I & 0 & 0 \\ \hline B' & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{C} \end{array}\right) \end{array}. \tag{3.7}$$

where A, B, and B' are $m \times m$ diagonal matrices with $m \leq \min\{k, N-k\}$, and C is an $(N-k)\times(N-k)$ matrix. Let $A = \operatorname{diag}(a_1, a_2, \ldots, a_m)$, $B = \operatorname{diag}(b_1, b_2, \ldots, b_m)$, and $B' = \operatorname{diag}(b'_1, b'_2, \ldots, b'_m)$. Then A, B, and B' can be chosen above so that $0 \leq a_i < 1$, $|b_i| > 0$, and $b'_i = \pm b_i$ for $i = 1, 2, \ldots, m$.

Proof: Let S_1, $S_2 \in B$, and let the basis $\tilde{e}_1$, $\tilde{e}_2$, $\ldots$, $\tilde{e}_N$, be defined by $\tilde{e_j} = S_1(e_j)$. If we identify the operator U with its matrix representation with respect to the basis e_1, e_2, $\ldots$, e_N, then with respect to the new basis the matrix is written

$S_1US_1^{-1}$. Now let $T = S_1^{-1}S_2 \in B$, and let $\tilde{U} = S_1US_1^{-1} \sim S_1US_1^{-1}T = S_1US_2$. Therefore, the lemma is equivalent to showing that for any matrix $U \in SO_N$, there exist S_1, $S_2 \in B$ such that S_1US_2 has the form indicated in (3.7).

Let us write

$$S_i = \begin{array}{c c} & \begin{array}{cc} k & N-k \end{array} \\ \begin{array}{c} k \\ \\ N-k \end{array} & \begin{pmatrix} V_i & 0 \\ & \\ 0 & W_i \end{pmatrix} \end{array}$$

for $i = 1, 2$, and

$$U = \begin{array}{c c} & \begin{array}{cc} k & N-k \end{array} \\ \begin{array}{c} k \\ \\ N-k \end{array} & \begin{pmatrix} A_0 & B_0 \\ & \\ B_0' & C_0 \end{pmatrix} \end{array}.$$

Now use the singular value decomposition to choose V_1 and V_2 so that $V_1A_0V_2$ is a diagonal matrix. Moreover, choose V_1 and V_2 so that any 1's in $V_1A_0V_2$ appear at the bottom. Thus

$$\left(\begin{array}{c|c} V_1 & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} A_0 & B_0 \\ \hline B_0' & C_0 \end{array}\right)\left(\begin{array}{c|c} V_2 & 0 \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{ccc|c} a_1 & & & \\ & \ddots & & B_1 \\ & & a_k & \\ \hline & B_1' & & C_1 \end{array}\right),$$

where I denotes the appropriate identity matrix.

Denote the j^{th} row of B_1 by $(B_1)_j$, and note that the rows of B_1 are orthogonal, i.e., $\left\langle (B_1)_i^t, (B_1)_j^t \right\rangle = 0$ if $i \neq j$. Let $(W_2)_j$ denote the j^{th} **column** of W_2. For each j with $\|(B_1)_j\| \neq 0$ define

$$(W_2)_j = \left((B_1)_j / \|(B_1)_j\|\right)^t.$$

For each remaining j with $\|(B_1)_j\| = 0$, arbitrarily choose $(W_2)_j$ to make the matrix W_2 orthogonal. We can force W_2 into SO_{N-k} by multiplying one column by -1 if necessary. With this choice of W_2 we get $V_1 B W_2 = B_1 W_2$ diagonal.

Choose W_1 similarly, based on B_1', which has orthogonal columns. With V_1, W_1, V_2, and W_2 chosen in this manner we achieve

$$\tilde{U} = \left(\begin{array}{c|c} V_1 & 0 \\ \hline 0 & W_1 \end{array}\right) \left(\begin{array}{c|c} A_0 & B_0 \\ \hline B_0' & C_0 \end{array}\right) \left(\begin{array}{c|c} V_2 & 0 \\ \hline 0 & W_2 \end{array}\right) = \left(\begin{array}{cccc|ccc} a_1 & & & & b_1 & & \\ & \ddots & & & & \ddots & \\ & & \ddots & & & & b_m \\ & & & a_k & & & \\ \hline b_1' & & & & & & \\ & \ddots & & & & C & \\ & & b_m' & & & & \end{array}\right).$$

Also, since $\tilde{U} \in SO_N$, we have $a_j^2 + b_j^2 = 1 = a_j^2 + (b_j')^2$ for $j = 1, 2, \ldots, m$, which implies $b_j' = \pm b_j$ for each j, proving the lemma. $\square$

If $m = 0$ in the above lemma then $S_U = S$, so there is nothing to prove. We shall therefore assume that $m > 0$ for the remainder of this paper.

Lemma 4 *If U is in the canonical form (3.7) with $|a_i| < 1$ for $i = 1, 2, \ldots, m$, then $A \cap A_U$ is the set of all functions in $L^2(\Omega^N)$ that are functions of the coordinates $m+1$ through k alone.*

Proof: Let $f \in A \cap A_U$, so there exist representatives, say f_I and f_U, with $f(z) = f_I(z) = f_U(z)$ a.e., such that

$$f_I(x \oplus y) = f_I(x \oplus 0)$$

and

$$f_U \circ U^{-1}(x \oplus y) = f_U \circ U^{-1}(x \oplus 0)$$

for all $x \oplus y \in \Omega^N$. Let

$$U = \begin{array}{c} \\ m \\ k-m \\ m \\ N-k-m \end{array} \begin{array}{c} \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & I & 0 & 0 \\ \hline B' & 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ 0 & 0 & & \end{array}\right) \end{array},$$

and define

$$T = \begin{array}{c} \\ m \\ k-m \\ m \\ N-k-m \end{array} \begin{array}{c} \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ A & 0 & B & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{array}.$$

Since $A = \mathrm{diag}(a_1, \ldots, a_m)$ satisfies $|a_i| < 1$ for $i = 1, 2, \ldots, m$, it follows that $B = \mathrm{diag}(b_1, \ldots, b_m)$ satisfies $|b_i| > 0$ for $i = 1, 2, \ldots, m$. In particular, B is invertible, so

$$T^{-1} = \begin{array}{c} \\ m \\ k-m \\ m \\ N-k-m \end{array} \begin{array}{c} \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -B^{-1}A & 0 & B^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{array},$$

Next, define P_1 to be the projection onto the first k coordinates, i.e.,

$$P_1 = \begin{array}{c} \\ m \\ k-m \\ m \\ N-k-m \end{array} \begin{array}{c} \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

and define P_2 to be the projection onto coordinates e_{m+1} through e_{m+k},

$$P_2 = \begin{array}{c} \\ m \\ k-m \\ m \\ N-k-m \end{array} \begin{array}{c} \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Note that

$$P_1 \circ T^{-1} \circ P_1 = P_1 \circ T^{-1} \tag{3.8}$$

and

$$P_1 \circ U \circ T^{-1} \circ P_2 = P_1 \circ U \circ T^{-1}. \tag{3.9}$$

Set $g(z) = f(az+z_0)$ where $a \in \mathbf{R}^+$ is a scaling factor and $z_0 \in \Omega^N$. The domain of g is all z such that $az + z_0 \in \Omega^N$. In particular, the domain of g contains a neighborhood of the origin. Adjust the scaling factor a so that the domain of g contains $T^{-1}\left([-1,1]^N\right)$. Define $g_I(z) = f_I(az + z_0)$, $g_U(z) = f_U(az + z_0)$. By the defining properties of f_I and f_U we have

$$g_I \circ P_1(z) = g_I(z)$$

and

$$g_U \circ U^{-1} \circ P_1(z) = g_U \circ U^{-1}(z),$$

for all $z \in T^{-1}\left([-1,1]^N\right)$. Invoking the equalities (3.8) and (3.9) yields

$$\begin{aligned} g_I \circ T^{-1}(w) &= g_I \circ P_1 \circ T^{-1}(w) \\ &= g_I \circ P_1 \circ T^{-1} \circ P_1(w) \\ &= g_I \circ T^{-1} \circ P_1(w) \end{aligned}$$

and

$$\begin{aligned} g_U \circ T^{-1}(w) &= g_U \circ U^{-1} \circ U \circ T^{-1}(w) \\ &= g_U \circ U^{-1} \circ P_1 \circ U \circ T^{-1}(w) \\ &= g_U \circ U^{-1} \circ P_1 \circ U \circ T^{-1} \circ P_2(w) \\ &= g_U \circ T^{-1} \circ P_2(w), \end{aligned}$$

for all $w \in [-1,1]^N$. An application of Corollary 2 in Appendix B shows the existence of g_3, $g_3(z) = g(z)$ a.e., with

$$g_3 \circ T^{-1} \circ P_3 \circ T(z) = g_3(z)$$

for all $z \in T^{-1}([-1,1])$. But $T^{-1} \circ P_3 \circ T = P_3$, so

$$g_3(z) = g_3 \circ P_3(z)$$

for all $z \in T^{-1}([-1,1])$, i.e., g_3 is a function of the coordinates $m+1$ through k alone. Relating this to the original function f shows that there is a ball centered at z_0, call it M_0, for which there exists a representative f_0, $f_0(z) = f(z)$ a.e., with

$$f_0(w_1) = f_z(w_2) \qquad \forall\ w_1,\, w_2 \in M_0 \text{ satisfying } P_3(w_1) = P_3(w_2).$$

Let us extend f_0 from M_0 to $P_1^{-1}P_1(M_0)$ by defining (for $z \in P_1^{-1}P_1(M_0)$) $f_0(z)$ to equal $f_0(z')$ where z' is any point in M_0 such that $P_1(z') = P_1(z)$. Since f_I differs from f_0 on M_0 by at most a set of measure zero, it follows that f_I and f_0 differ on $P_1^{-1}P_1(M_0)$ by at most a set of measure zero. Therefore f_0 extended to $P_1^{-1}P_1(M_0)$ agrees with f up to a set of measure zero. The same argument can be used with P_2 replacing P_1 to show that f_0 can be extended to the set $P_2^{-1}P_2P_1^{-1}P_1(M_0)$. But P_1 and P_2 commute, so in this extension is to the set $P_3^{-1}P_3(M_0)$. Let us denote the extension of f_0 to $P_3^{-1}P_3(M_0)$ by h_0.

Select a (countable) sequence of points $\{z_n\}$ with the corresponding collection of balls $\{M_n\}$ such that

$$\bigcup_{n=0}^{\infty} P_3^{-1}P_3(M_n) \supset \Omega^N.$$

For $z \in \Omega^N$ define $\phi(z) = \min\{n \in \mathbf{N} \mid z \in P_3^{-1}P_3(M_n)\}$. Then define

$$h(z) = h_{\phi(z)}(z).$$

Then h has domain Ω^N and the property

$$h \circ P_3(z) = h(z) \qquad \text{for all } z \in \Omega^N.$$

Moreover, $h(z) = f(z)$ for almost every $z \in \Omega^N$. $\square$

3.3 An equivalent problem

Recall from (3.4) that the angle between A and A_U depends on the inner product $\langle f_1, f_2 \circ U \rangle$, where $f_1, f_2 \in A$. Since

$$|\langle f_1, f_2 \circ U \rangle| \leq \|f_1\| \|f_2\|, \tag{3.10}$$

U introduces a bounded bilinear form on A. It follows from the Riesz Representation Theorem that there exists a bounded linear operator L_A on A such that

$$\langle f_1, L_A f_2 \rangle = \langle f_1, f_2 \circ U \rangle \qquad \text{for all } f_1, f_2 \in A. \tag{3.11}$$

Lemma 5 *The set A and the space $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$ are isomorphic as Hilbert spaces.*

Proof: If $f \in A$, then there exists a representative $\tilde{f} = f$ with $\tilde{f}(x \oplus y) = \tilde{f}(x \oplus 0)$ for every $x \in \Omega^k$. Define the operator H by

$$Hf(x) = \tilde{f}(x \oplus 0).$$

For $f \in A$, $g \in A$ we have

$$\begin{aligned}
\langle f, g \rangle &= \int_{\Omega^N} f(z) g(z)\, d^N z \\
&= \int_{\Omega^k} \int_{B_{N-k}(\sqrt{1-\|x\|^2}} f(x \oplus y) g(x \oplus y)\, d^{N-k} y\, d^k x \\
&= V_{N-k} \int_{\Omega^k} Hf(x)\, Hg(x)\, (1-\|x\|^2)^{(N-k)/2}\, d^k x \\
&= \langle Hf, Hg \rangle ,
\end{aligned}$$

where the last inner product is in the space $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$. It is clear that H is bijective, so in fact

$$H : A \to L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$$

is a Hilbert space isomorphism. $\square$

Since A is isomorphic to $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$, the operator L_A induces an operator L on $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$ via

$$L = HL_AH^{-1}, \tag{3.12}$$

where H is the isomorphism from A to $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$. Let $\Pi_k : \mathbf{R}^N \to \mathbf{R}^k$ be the projection onto the first k-coordinates, i.e., $\Pi_k(x \oplus y) = x$. Then for $g_1, g_2 \in L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$, we have

$$\begin{aligned} \langle g_1, Lg_2 \rangle &= \left\langle H^{-1}g_1, L_AH^{-1}g_2 \right\rangle \\ &= \int_{\Omega^N} H^{-1}g_1(z)H^{-1}g_2 \circ U(z)\, d^Nz \\ &= \int_{\Omega^k} g_1(x) \int_{B_{N-k}(\sqrt{1-\|x\|^2})} g_2 \circ \Pi_k \circ U(x \oplus y)\, d^{N-k}y\, d^kx \end{aligned} \tag{3.13}$$

Since the angle between the null spaces $\mathcal{N}$ and $\mathcal{N}_U$ of the operators S and S_U can be determined from the norm of the operator L_A (compare (3.4) and (3.11)), it follows that

$$\cos\left(\gamma(\mathcal{N}, \mathcal{N}_U)\right) = \sup_{\substack{\|g_1\|=\|g_2\|=1 \\ g_i \in D}} |\langle g_1, Lg_2 \rangle|, \tag{3.14}$$

where the set $D = H(A \cap (A \cap A_U)^\perp$. From Lemma 4 we see that $D^\perp$ is the set of functions in $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$ which are functions of the coordinates $m+1$ through k alone. In particular, if $m = k$, then $D^\perp$ is the set of constant functions.

Properties of the operator L

We now study the operator L more closely.

Lemma 6 $\|L\| = 1$.

Proof: The fact that $\|L\| \leq 1$ follows immediately from (3.10) and (3.12). If f_2 is taken to be a constant function, then one sees that in fact $\|L\| = 1$. □

Lemma 7 *The operator L is self-adjoint.*

Proof: Without loss of generality, assume U is in the canonical form of (3.7). The Lebesgue measure on $\mathbf{R}^N$ is rotation invariant, so we may rotate the coordinate system by U to achieve (for $g_1, g_2 \in L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$)

$$\begin{aligned} \langle g_1, Lg_2\rangle &= \int_{\Omega^N} g_1 \circ \Pi_k \circ U^{-1}(z) g \circ \Pi_k(z)\, d^N z \\ &= \int_{\Omega^k} g_2(x) \int_{B_{N-k}(\sqrt{1-\|x\|^2})} g_1 \circ \Pi_k \circ U^{-1}(x \oplus y)\, d^{N-k}y\, d^k x. \end{aligned}$$

Since U is orthogonal, $U^{-1} = U^t$, and from (3.7) we note that $\Pi_k \circ U$ and $\Pi_k \circ U^t$ are identical with the possible exception of some ± 1's on the y variables. But the inner integral over y's in the last equation is symmetric with respect to the origin, so we can introduce the change of variables $y'_j = -y_j$ as necessary without changing the value of the integral. Thus

$$\begin{aligned}
\langle g_1, Lg_2 \rangle &= \int_{\Omega^k} g_2(x) \int_{B_{N-k}(\sqrt{1-\|x\|^2})} g_1 \circ \Pi_k \circ U(x \oplus y')\, d^{N-k}y'\, d^k x \\
&= \langle g_2, Lg_1 \rangle \\
&= \langle Lg_1, g_2 \rangle ,
\end{aligned}$$

which shows that L is self-adjoint. $\square$

Continuing our study of L, let us dilate the inner integral in (3.13) by $\sqrt{(1-\|x\|^2)}$ to get

$$\langle g_1, Lg_2 \rangle = \int_{\Omega^k} g_1(x)(1-\|x\|^2)^{(N-k)/2} \int_{\Omega^{N-k}} g_2 \circ \Pi_k \circ U(x, y\sqrt{1-\|x\|^2})\, d^{N-k}y\, d^k x.$$

In particular, this reveals an explicit representation for the operator L. If we make use of the canonical form (3.7) of the orthogonal operator U, we get

$$\begin{aligned}
Lg(x) &= \int_{\Omega^m} g(a_1 x_1 + b_1 y_1 \sqrt{1-\|x\|^2}, \ldots, a_m x_m + b_m y_m \sqrt{1-\|x\|^2}, x_{m+1}, \ldots, x_k) \\
&\quad \times \frac{V_{N-k-m}}{V_{N-k}} (1-\|y\|^2)^{(N-k-m)/2}\, d^m y, \qquad (3.15)
\end{aligned}$$

where $m \leq \min(k, N-k)$ and depends on U.

The action of L on polynomials

Let us first consider the action of L on monomials. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be a multi-index of length k, and define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$. Then

$$\begin{aligned} Lx^\alpha &= \frac{V_{N-k-m}}{V_{N-k}} x_{m+1}^{\alpha_{m+1}} \dots x_k^{\alpha_k} \\ &\quad \times \int_{\Omega^m} \prod_{j=1}^{m} \left(a_j x_j + b_j y_j \sqrt{1 - \|x\|^2} \right)^{\alpha_j} (1 - \|y\|^2)^{(N-k-m)/2} \, d^m y. \end{aligned} \tag{3.16}$$

This shows that the space $D^\perp$ (and hence D since L is self-adjoint) is an invariant subspace for the operator L.

It is convenient here to divide the k variables into two sets. Let $w \in \mathbf{R}^m$ and $z \in \mathbf{R}^{k-m}$ with $(w_1, w_2, \dots, w_m) = (x_1, x_2, \dots, x_m)$ and $(z_1, z_2, \dots, z_{k-m}) = (x_{m+1}, x_{m+2}, \dots, x_k)$. In a similar fashion, divide the multi-index α into multi-indices β of length m and η of length $k - m$. Using this notation (3.16) takes the form

$$Lw^\beta z^\eta = \frac{V_{N-k-m}}{V_{N-k}} z^\eta \int_{\Omega^m} \left(aw + by \sqrt{1 - \|w\|^2 - \|z\|^2} \right)^\beta (1 - \|y\|^2)^{(N-k-m)/2} \, d^m y, \tag{3.17}$$

where the product of vectors is defined coordinatewise, e.g.,

$$aw = (a_1 w_1, a_2 w_2, \dots, a_m w_m).$$

We need now to introduce some notation for multi-indices. For multi-index i of length n define

$$i! = \prod_{j=1}^{n} i_j!, \tag{3.18}$$

and similarly,

$$\Gamma(i) = \prod_{j=1}^{n} \Gamma(i_j). \tag{3.19}$$

We say multi-index $i \leq \sigma$ if $i_j \leq \sigma_j$ for $j = 1, 2, \ldots, n$. In conjunction with multi-indices of length n, let 0 denote the multi-index $(0, 0, \ldots, 0)$ and 1 the multi-index $(1, 1, \ldots, 1)$. For multi-index i with $0 \leq i \leq \sigma$ define

$$\binom{\sigma}{i} = \prod_{j=1}^{n} \binom{\sigma_j}{i_j}. \tag{3.20}$$

Finally, we say that multi-index i is **even** if i_j is even for each j, $j = 0, 1, \ldots, n$.

The binomial theorem allows (3.17) to be written in the form

$$\begin{aligned} Lw^\beta z^\eta &= \frac{V_{N-k-m}}{V_{N-k}} z^\eta \sum_{0 \leq i \leq \beta} \binom{\beta}{i} a^{\beta-i} w^{\beta-i} b^i (1 - \|w\|^2 - \|z\|^2)^{|i|/2} \\ &\quad \times \int_{\Omega^m} y^i (1 - \|y\|^2)^{(N-k-m)/2} \, d^m y. \end{aligned} \tag{3.21}$$

Note that if any of i_j are odd, then the integrand is odd with respect to the j^{th} variable, so the integral evaluates to zero. Otherwise, the integral can be evaluated sequentially as a product of iterated integrals via (A.3). This produces

$$\begin{aligned} J(i) &= \int_{\Omega^m} y^i (1 - \|y\|^2)^{(N-k-m)/2} \, d^m y \\ &= \begin{cases} \displaystyle\prod_{j=1}^{m} B\left(\frac{i_j + 1}{2}, \frac{i_{j+1} + i_{j+2} + \cdots + i_m + N - k - j + 2}{2}\right) & \text{for } i \text{ even} \\ 0 \text{ otherwise} \end{cases} \end{aligned} \tag{3.22}$$

where $B(\,\cdot\,, \,\cdot\,)$ denotes the Beta function. If one expands the Beta function in terms of the Gamma function and cancels like terms one achieves

$$J(i) = \begin{cases} \dfrac{\Gamma\left((N - k - m + 2)/2\right)}{\Gamma\left((N - k + 2 + |i|)/2\right)} \Gamma\left((i + 1)/2\right) & \text{for } i \text{ even} \\ 0 \text{ otherwise.} \end{cases} \tag{3.23}$$

In particular, note that $J(0) = V_{N-k}/V_{N-k-m}$.

Thus (3.17) can be explicitly written as

$$Lw^\beta z^\eta = \frac{V_{N-k-m}}{V_{N-k}}\, z^\eta \sum_{\substack{0 \le i \le \beta \\ i \text{ even}}} \binom{\beta}{i} J(i) a^{\beta-i} w^{\beta-i} b^i (1 - \|w\|^2 - \|z\|^2)^{|i|/2}. \quad (3.24)$$

Invariant subspaces of the operator L

We can use the expression 3.24 to reveal some invariant subspaces of the operator L. Note that the restriction that the multi-index i be even forces $|i|/2$ to be integral, thus L maps polynomials back to polynomials. Moreover, notice that L preserves the total degree. Thus, if we set

$$E_d = \text{lin span } \{x^\alpha \mid |\alpha| \le d\},$$

then L maps E_d back into E_d.

Another consequence of the evenness of i is a bit more subtle. To ease the discussion, let us introduce the concept of **parity** for multi-indices. We say that two multi-indices α and σ (of the same length) have the same parity (written $\alpha \sim \sigma$) if $\alpha_j - \sigma_j$ is even for each j, i.e., $\alpha - \sigma$ is even. For example, $i \sim 0$ if i is even. Let ϵ be a multi-index of length k such that $\epsilon_j \in \{0, 1\}$ for each j. There are 2^k distinct ϵ of this type. Each such ϵ defines a parity class of polynomials. Specifically, define

$$F_\epsilon = \text{lin span } \{x^\alpha \mid \alpha \sim \epsilon\}. \quad (3.25)$$

Then we see from (3.24) that the evenness of i causes L to map F_ϵ onto itself. Moreover, in $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$ note that

$$\begin{aligned} \langle x^\alpha, x^\sigma\rangle &\geq 0 \text{ if } \alpha\sim\sigma \\ \langle x^\alpha, x^\sigma\rangle &= 0 \text{ otherwise,} \end{aligned} \tag{3.26}$$

which shows

Lemma 8 *The sets F_ϵ form an orthogonal decomposition of the Hilbert space $L^2\left(\Omega^k, (1-\|x\|^2)^{(N-k)/2}\right)$, i.e.,*

$$L^2\left(\Omega^k, (1-\|x\|^2)^{(N-k)/2}\right) = \oplus_\epsilon F_\epsilon^{\mathrm{cl}},$$

where the sum is over distinct ϵ. The number of distinct ϵ (and hence F_ϵ) is 2^k.

Let us now develop a convenient basis for $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$. Define

$$P_\alpha(x) = x^\alpha - \sum_{|\sigma|<|\alpha|} \frac{\langle x^\alpha, x^\sigma\rangle}{\langle x^\sigma, x^\sigma\rangle} x^\sigma. \tag{3.27}$$

Then the set $\{P_\alpha\}$, where α runs over all multi-indices of length k, is a basis for $L^2\left(\Omega^k, V_{N-k}(1-\|x\|^2)^{(N-k)/2}\right)$, with the property

$$\langle P_\alpha(x), x^\sigma\rangle = 0 \qquad \text{for all } \sigma \text{ with } |\sigma|<|\alpha|. \tag{3.28}$$

Notice that no claim is made for $\langle P_\alpha, x^\sigma\rangle$ for $|\sigma| = |\alpha|$. In particular, this is **not** an orthogonal basis. From (3.26) and (3.27), it follows that $P_\alpha \in F_\epsilon$ if $\alpha\sim\epsilon$, so the P_α's respect the decomposition of Lemma 8.

Next let

$$G_d = \text{lin span } \{P_\alpha \mid |\alpha| = d\}. \tag{3.29}$$

Note that $G_d = E_d \cap E_{d-1}^\perp$. A simple counting argument (see, for example, page 38 of [8]) shows that for each d,

$$\dim G_d = \binom{d+k-1}{k-1}. \tag{3.30}$$

The space E_{d-1} (the space of all polynomials of degree less than d) is an invariant subspace for L, so $E_{d-1}^\perp$ is also an invariant subspace since L is self-adjoint. Therefore $G_d = E_d \cap E_{d-1}^\perp$ is also an invariant subspace for L. Moreover, we can use the F_ϵ decomposition to decompose G_d into smaller invariant subspaces. If F_ϵ intersects G_d nontrivially, then there must exist α with $|\alpha| = d$ such that $\alpha \sim \epsilon$. But then $d - |\epsilon|$ must be even, so only half of the F_ϵ's intersect G_d nontrivially. To be precise, if $d - |\epsilon|$ is even and the ϵ_j are given for $j = 1, 2, \ldots, k-1$, then the parity of d determines ϵ_k. Thus G_d decomposes (with respect to the basis $\{P_\alpha\}$) into 2^{k-1} smaller invariant subspaces, say

$$G_{d,\epsilon} = G_d \cap F_\epsilon, \tag{3.31}$$

where ϵ is restricted to those ϵ with $d - |\epsilon|$ even. The count 2^{k-1} is actually only accurate for $d \geq k-1$. In particular, for F_ϵ to intersect G_d nontrivially, it is necessary that $|\epsilon|$ be not larger than d. If $j \leq d$ then there are $\binom{d}{j}$ different ϵ

with $|\epsilon| = j$. Thus for each d

$$|\{\epsilon \mid G_{d,\epsilon} \neq \{0\}\}| = \sum_{j=0}^{d} \binom{k-1}{j} \qquad \text{if } d \leq k-1.$$

Note that for $d = k-1$ this sum evaluates to 2^{k-1}. Also, $|\epsilon| \leq k$ by definition of ϵ, so if $d \geq k$ then $|\epsilon| \leq d$ automatically.

An argument similar to that used in (3.30) can be used to explicitly calculate the dimension of the subspace $G_{d,\epsilon}$. If $d \geq |\epsilon|$ and $d - |\epsilon|$ is even then

$$\dim G_{d,\epsilon} = \binom{(d-|\epsilon|)/2 + k - 1}{k-1}. \tag{3.32}$$

Note that for fixed d the dimension is largest with $\epsilon = 0$, and decreases as $|\epsilon|$ increases.

These results are accumulated in Theorem 2.

Theorem 2 *Let L be the operator on $L^2\left(\Omega^k, (1-\|x\|^2)^{(N-k)/2}\right)$ defined by (3.15), and let F_ϵ, G_d, and $G_{d,\epsilon}$ be the subspaces described in (3.25), (3.29), and (3.31) respectively. Then the following hold:*

1. *Each of the sets $\{F_\epsilon\}$, $\{G_d\}$, and $\{G_{d,\epsilon}\}$ produce an orthogonal decomposition of $L^2\left(\Omega^k, (1-\|x\|^2)^{(N-k)/2}\right)$.*

2. *$|\{F_\epsilon\}| = 2^k$, and $\dim F_\epsilon = \infty$ for each ϵ.*

3. *$|\{G_d\}| = \infty$, and*

$$\dim G_d = \binom{d+k-1}{k-1}.$$

4. For each d, if $d < \epsilon$ or $d - |\epsilon|$ is odd then $G_{d,\epsilon} = 0$. Otherwise,

$$|\{\epsilon \mid G_{d,\epsilon} \neq \{0\}\}| = \begin{cases} \sum_{j=0}^{d} \binom{k-1}{j} & \text{if } d < k-1 \\ 2^{k-1} & \text{if } d \geq k-1 \end{cases}$$

and

$$\dim G_{d,\epsilon} = \binom{(d-|\epsilon|)/2 + k - 1}{k-1}$$

Matrix representation for the operator L

Let M be the representation of the operator L with respect to the basis $\{P_\alpha\}$. Theorem 2 shows that M is in block diagonal form, with a (finite) block corresponding to each $G_{d,\epsilon}$. Since L is self-adjoint, it follows that each block is diagonalizable, even though M is not (in general) symmetric since the basis $\{P_\alpha\}$ is not orthogonal. But M is diagonalizable, so from (3.14) we have that the angle between the null spaces of S and S_U is just the largest eigenvalue of L restricted to D.

We now develop an explicit formula for the entries of M. Label the entries of M with the indices α of the basis $\{P_\alpha\}$, i.e., define $M_{\sigma\alpha}$ by

$$LP_\alpha = \sum_{\substack{|\sigma| = |\alpha| \\ \sigma \sim \alpha}} M_{\sigma\alpha} P_\sigma.$$

($M_{\sigma\alpha} = 0$ if $|\sigma| \neq |\alpha|$ or $\sigma \not\sim \alpha$.) From the definition of P_α (refer to (3.27)) it follows that this statement is equivalent to

$$Lx^\alpha = \sum_{\substack{|\sigma| = |\alpha| \\ \sigma \sim \alpha}} M_{\sigma\alpha} x^\sigma \quad + \text{lower order terms}.$$

Writing (3.24) in this form yields

$$\begin{aligned} Lw^\beta z^\eta &= \frac{V_{N-k-m}}{V_{N-k}} z^\eta \sum_{\substack{0 \le i \le \beta \\ i \text{ even}}} (-1)^{|i|/2} \binom{\beta}{i} J(i) a^{\beta-i} w^{\beta-i} b^i (\|w\|^2 + \|z\|^2)^{|i|/2} \\ &\quad + \text{lower order terms.} \end{aligned}$$

Use the multinomial expansion

$$\left(x_1^2 + x_2^2 + \ldots + x_k^2\right)^{|i|/2} = \sum_{|\sigma|=|i|/2} \frac{(|i|/2)!}{\sigma!} x^{2\sigma}$$

(see (3.18)), and collect terms to get (where $\alpha = \beta \oplus \eta$)

$$M_{\sigma\alpha} = \frac{V_{N-k-m}}{V_{N-k}} \sum_{\substack{0 \le i \le \beta \\ i \text{ even} \\ (\beta - i) \oplus \eta \le \sigma}} (-1)^{|i|/2} \binom{\beta}{i} J(i) a^{\beta-i} b^i \frac{(|i|/2)!}{[[\sigma - ((\beta - i) \oplus \eta)]/2]!} \tag{3.33}$$

3.4 Results

In this section I collect the results from the preceding sections, which are given with respect to the "equivalent" problem, and reinterpret them in terms of the original $(N-k)$-plane transform question.

Since the operator L of (3.12) is self-adjoint, for each set $G_{d,\epsilon}$ of (3.31) the corresponding (finite) block in the matrix representation M of L is diagonalizable. Since the dimension of the space $G_{d,\epsilon}$ is the same as the number of indices α with $P\alpha \in G_{d,\epsilon}$, we can label the eigenvalues and eigenvectors by the indices α. Let $\{\lambda_\alpha\}$ denote the set of eigenvalues of L restricted to $G_{d,\epsilon}$, where there is in general no direct relationship between λ_α and P_α.

Note, however, if $\alpha = \beta \oplus \eta$ with $\beta = 0$, then $M_{\sigma\alpha} \neq 0$ implies that $\sigma = \alpha$ and $M_{\alpha\alpha} = 1$. (See (3.33)). In particular, $P_{0\oplus\eta}$ is an eigenvector for L with eigenvalue, say $\lambda_{0\oplus\eta}$, equal to 1. Furthermore,

$$\text{lin span } \{P_{0\oplus\eta}\}^{\text{cl}} = D^{\perp},$$

where $D = H(A \cap (A \cap A_U)^{\perp})$. (Refer to (3.14)). Recall that the angle between $\mathcal{N}$ and $\mathcal{N}_U$ (the null spaces for the operators S and S_U respectively), written $\gamma(\mathcal{N}, \mathcal{N}_U)$, is equal to the arccos of the norm of L restricted to D. Thus we have

$$\begin{aligned} \cos\left(\gamma(\mathcal{N}, \mathcal{N}_U)\right) &= \|L|_D\| \\ &= \sup_{\beta\oplus\eta,\, |\beta|>0} |\lambda_{\beta\oplus\eta}| \end{aligned} \tag{3.34}$$

In some special cases one can explicitly calculate the eigenvalues $\lambda_{\beta\oplus\eta}$. For example, if $d = 0$ then $G_{d,\epsilon}$ contains only the constant functions, which are always contained in $D^{\perp}$. If $d = 1$, then the index i in (3.33) can take only the value 0, so $M_{\sigma\alpha} \neq 0$ implies $\sigma = \alpha$ and $M_{\alpha\alpha} = a^{\beta}$. So for $|\alpha| = 1$ we can identify the eigenvalue $\lambda_{\alpha} = a^{\beta}$ and eigenvector $P_{\alpha} = x^{\alpha}$. (Of course, if $\beta = 0$ then $\lambda_{\alpha} = a^0 = 1$, and the eigenvector $x^{\alpha} \in D^{\perp}$.) On the other hand, for $d = 2$ the subspace $G_{d,0}$ has dimension k, and already the eigenvalue problem in the general case is intractable.

However, if $m = 1$ then the eigenvalue problem is completely solvable. Let us order the indices of each degree class in lexicographical order, i.e., for $|\alpha| = |\sigma|$, define

$$\alpha \ll \sigma \quad \text{ if } \alpha_i = \sigma_i \text{ for } i = k, k-1, \ldots, j > 1 \text{ and } \alpha_{j-1} < \sigma_{j-1}.$$

Then it follows from (3.33) that

$$M_{\sigma\alpha} = 0 \qquad \text{if } \sigma \ll \alpha,\ m = 1. \tag{3.35}$$

Therefore, with respect to the order $\ll$, the matrix M is lower triangular. Hence the eigenvalues are just the diagonal entries, $M_{\alpha\alpha}$.

Theorem 3 *If U has the form given in (3.7) with $m = 1$, then the angle between the null spaces $\mathcal{N}$ and $\mathcal{N}_U$ of S and S_U satisfies*

$$\cos\left(\gamma(\mathcal{N}, \mathcal{N}_U)\right) = \sup_{\substack{\alpha=\beta\oplus\eta \\ \beta>0}} |M_{\alpha\alpha}|, \tag{3.36}$$

where $M_{\alpha\alpha}$ is given in (3.33).

The Radon ($k = 1$) and X-ray ($k = N - 1$) transforms are particular examples of the situation $m = 1$. They are studied in detail in the following chapter.

CHAPTER IV

Angle between null spaces of the Radon and X-ray transforms

In this chapter I extend the results of Chapter III to the special cases of the Radon and X-ray transform. The expression for the angle between the null spaces given in (4.8) is a known result for $\mathbf{R}^2$ (see [4] and [5]), but the result is new for $\mathbf{R}^N$. Moreover, the explicit evaluation of this expression, given in Theorem 4, is a new result for all $N \geq 2$.

4.1 The Radon transform

The Radon transform is the special case of the general $(N-k)$-plane transform with $N-k = N-1$. For $f \in L^2(\Omega^N)$, $\sigma \in S^{N-1}$, $t \in \mathbf{R}$, denote by $Rf(\sigma, t)$ the Radon transform of f, i.e.,

$$Rf(\sigma, t) = \int_{\langle x, \sigma \rangle = t} f(x)\, dx = \int_{\sigma^\perp} f(t\sigma + y)\, dy, \tag{4.1}$$

where f is extended from Ω^N to $\mathbf{R}^N$ by $f(x) = 0$ if $\|x\| > 1$. A straightforward application of Schwarz's inequality ([3], page 17) shows that

$$R : L^2(\Omega^N) \to L^2\left(S^{N-1} \times \mathbf{R}, (1-t^2)^{(1-N)/2}\right) \tag{4.2}$$

is continuous. Fix $\sigma \in S^{N-1}$, and consider $R_\sigma f(t)$ ($\equiv Rf(\sigma, t)$), where

$$R_\sigma : L^2(\Omega^N) \to L^2\left([-1,1], (1-t^2)^{(1-N)/2}\right). \tag{4.3}$$

This operator is also continous, which follows from the general discussion in Chapter III.

In the language of Chapter III, $R_\sigma = S_U$ where U is any element of SO_N that sends the first coordinate vector e_1 to σ. The canonical form (refer to (3.7) with $k = m = 1$) for U is

$$U = \left(\begin{array}{c|cccc} \cos\theta & \sin\theta & 0 & \dots & 0 \\ \hline -\sin\theta & \cos\theta & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{array}\right),$$

where θ is the angle between e_1 and σ.

Let $\mathcal{N}_{\sigma_i}$ be the null space for R_{σ_i}, $i = 1, 2$, and let θ be the angle between σ_1 and σ_2. Assuming $\theta \neq 0$, by (3.36) we have

$$\cos\left(\gamma(\mathcal{N}_{\sigma_1}, \mathcal{N}_{\sigma_2})\right) = \sup_{\alpha=\beta\oplus\eta,\, \beta>0} |M_{\alpha\alpha}|. \tag{4.4}$$

Using (3.22), (3.33), and (A.4) we have

$$M_{\alpha\alpha} = \frac{\Gamma(N)}{2^{N-1}(\Gamma(N/2))^2} \sum_{i=0}^{[\beta/2]} (-1)^i \binom{\beta}{2i} B(i+1/2, N/2)(\cos\theta)^{\beta-2i}(\sin\theta)^{2i}. \tag{4.5}$$

Comparing to (A.12) and (A.13) shows that

$$M_{\alpha\alpha} = C_\beta^{(N/2)}(\cos\theta)/C_\beta^{(N/2)}(1), \tag{4.6}$$

where $C_\beta^{(N/2)}$ denotes the Gegenbauer (ultraspherical) polynomial of degree β with parameter $N/2$.

It follows that the angle between the null spaces of R_{σ_1} and R_{σ_2} reduces from (4.4) to

$$\cos\left(\gamma(\mathcal{N}_{\sigma_1}, \mathcal{N}_{\sigma_2})\right) = \sup_{n\in\mathbf{N}} |C_n^{(N/2)}(\cos\theta)/C_n^{(N/2)}(1)|. \tag{4.7}$$

I explicitly evalutate this supremum in Theorem 4.

An important case is $N = 2$. The Gegenbauer polynomial $C_n^{(1)}$ is the Chebyshev polynomial of the second kind, U_n (see (A.9)). In this case

$$\cos\left(\gamma(\mathcal{N}_{\sigma_1}, \mathcal{N}_{\sigma_2})\right) = \sup_{n\in\mathbf{N}} \left|\frac{\sin(n+1)\theta)}{(n+1)\sin\theta}\right|. \tag{4.8}$$

This result was obtained by Hamaker and Solmon [4], though they did not provide the explicit evaluation that I give in Theorem 4.

4.2 The X-ray transform

The X-ray transform is the special case of the general $(N-k)$-plane transform with $N - k = 1$. For $f \in L^2(\Omega^N)$, $\sigma \in S^{N-1}$, $x \in \mathbf{R}^N$, let $Pf(\sigma, x)$ denote the X-ray transform of f, defined by

$$Pf(\sigma, x) = \int_{-\infty}^{\infty} f(x + \sigma t)\, dt, \tag{4.9}$$

where f is extended from Ω^N to $\mathbf{R}^N$ by $f(x) = 0$ if $\|x\| > 1$. As with the Radon transform, it is straightforward to show that

$$P : L^2(\Omega^N) \to L^2(S^{N-1} \times \mathbf{R}^N, (1 - \|x\|^2)^{-1/2}$$

is continuous.

Fix $\sigma \in S^{N-1}$, restrict x to $\sigma^\perp$, and let $P_\sigma(x)$ denote $P(\sigma, x)$. Then

$$P_\sigma : L^2(\Omega^N) \to L^2\left(\sigma^\perp \cap \Omega^N, (1 - \|x\|^2)^{-1/2}\right)$$

is also continuous, as follows from the general discussion in Chapter III.

In terms of the development in Chapter III, P_σ is identified with S_U, where $U \in SO_N$ maps the first coordinate vector e_1 to σ, and the range space $L^2(\sigma^\perp \cap \Omega^N, (1-\|x\|^2)^{-1/2})$ of P_σ is identified with $L^2(\Omega^{N-1}, (1-\|x\|^2)^{-1/2})$ in the obvious way (through U).

The matrix U has the canonical form (see (3.7), $m = 1$, $k = N - 1$)

$$U = \left(\begin{array}{cccc|c} \cos\theta & & & & \sin\theta \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ \hline -\sin\theta & 0 & \cdots & 0 & \cos\theta \end{array}\right), \tag{4.10}$$

where θ is the angle between e_1 and σ.

Let $\mathcal{N}_{\sigma_i}$ be the null space for P_{σ_i}, $i = 1, 2$, and let θ be the angle between σ_1 and σ_2. Then for $\theta \neq 0$, we have from (3.36)

$$\cos\left(\gamma(\mathcal{N}_{\sigma_1}, \mathcal{N}_{\sigma_2})\right) = \sup_{\alpha=\beta\oplus\eta,\, \beta>0} |M_{\alpha\alpha}|.$$

Using (3.22), (3.33), and (A.4) we have

$$M_{\alpha\alpha} = \frac{1}{2}\sum_{i=0}^{[\beta/2]}(-1)^i\binom{\beta}{2i}B(i+1/2,1)(\cos\theta)^{\beta-i}(\sin\theta)^i. \tag{4.11}$$

Comparison to (A.12) and (A.13) shows

$$M_{\alpha\alpha} = C_\beta^{(1)}(\cos\theta)/C_\beta^{(1)}(1).$$

Therefore

$$\cos\left(\gamma(\mathcal{N}_{\sigma_1},\mathcal{N}_{\sigma_2})\right) = \sup_{n\in\mathbf{N}}|C_n^{(1)}(\cos\theta)/C_n^{(1)}(1)|. \tag{4.12}$$

This is the same as (4.7) with $N = 2$. As pointed out in that section, $C_n^{(1)}$ is the Chebyshev polynomial of the second kind, U_n, and so (4.12) can be rewritten as in (4.8), a result achieved by Hamaker and Solmon [4].

4.3 Supremum of normalized Gegenbauer polynomials

Theorem 4 *Let $\alpha \geq 1$ be fixed, and define $u_m(\theta) \equiv C_m^{(\alpha)}(\cos\theta)/C_m^{(\alpha)}(1)$. Then*

$$|u_m(\theta)| \leq \max\left(|u_1(\theta)|,|u_2(\theta)|\right) \qquad \text{for all } m\in\mathbf{N}. \tag{4.13}$$

In particular,

$$\max\left(|u_1(\theta)|,|u_2(\theta)|\right) = \begin{cases} u_1(\theta) = \cos\theta & \text{for } 0\leq\theta\leq\arccos\left(\frac{1}{2\alpha+1}\right) \\ -u_2(\theta) = \dfrac{1-(2\alpha+2)(\cos\theta)^2}{2\alpha+1} & \text{for } \arccos\left(\frac{1}{2\alpha+1}\right)\leq\theta\leq\frac{\pi}{2}. \end{cases}$$

Proof: Note that

$$u_m(\theta+\pi) = (-1)^m u_m(\theta) u_m(-\theta) = u_m(\theta) \tag{4.14}$$

as can be verified by means of (A.12) in Appendix A. This shows that the function $|u_m|$ is $\pi/2$ periodic, so it suffices to show (4.13) holds for θ in the interval $[0, \pi/2]$.

Using formulae (A.11) and (A.13) of Appendix A yields

$$u_m(\theta) = \left(B(\alpha, \tfrac{1}{2})\right)^{-1} \int_0^\pi (\cos\theta + i \sin\theta \cos\phi)^m (\sin\phi)^{2\alpha-1}\, d\phi, \tag{4.15}$$

which holds for $\alpha > 0$. Therefore,

$$\begin{aligned} |u_m(\theta)| &\leq \left(B(\alpha, \tfrac{1}{2})\right)^{-1} \int_0^\pi |\cos\theta + i\sin\theta\cos\phi|^m (\sin\phi)^{2\alpha-1}\, d\phi \\ &= 2\left(B(\alpha, \tfrac{1}{2})\right)^{-1} \int_0^{\pi/2} \left((\cos\theta)^2 + (\sin\theta\cos\phi)^2\right)^{m/2} (\sin\phi)^{2\alpha-1}\, d\phi \\ &\stackrel{\text{def}}{=} F_m(\theta). \end{aligned} \tag{4.16}$$

Notice that $F_m(\theta)$ is decreasing as a function of m, thus

$$|u_k(\theta)| \leq F_m(\theta) \qquad \text{for all } k \geq m. \tag{4.17}$$

The reader may also readily verify that $F_m(\theta)$ is also decreasing as a function of θ for $0 \leq \theta \leq \pi/2$.

Fig. 4.1 graphs the first few $|u_m(\theta)|$ and $F_m(\theta)$. Notice that $u_m(0) = F_m(0)$ for every m, and $|u_m(\pi/2)| = F_m(\pi/2)$ for all even m. These equalities follow easily from (4.15) and (A.2). I now prove Theorem 4 in three steps, using Fig. 4.1 as a guide:

Step 1 $\max\limits_{j=1,2} |u_j(\theta)| = \begin{cases} u_1(\theta) & \text{for } 0 \leq \theta \leq \arccos\left(\frac{1}{2\alpha+1}\right) \\ -u_2(\theta) & \text{for } \arccos\left(\frac{1}{2\alpha+1}\right) \leq \theta \leq \frac{\pi}{2}. \end{cases}$

Step 2 $|u_3(\theta)| \leq \max\limits_{j=1,2} |u_j(\theta)|$

Step 3 $|u_k(\theta)| \leq F_4(\theta) \leq \max\limits_{j=1,2} |u_j(\theta)|$ for all $k \geq 4$.

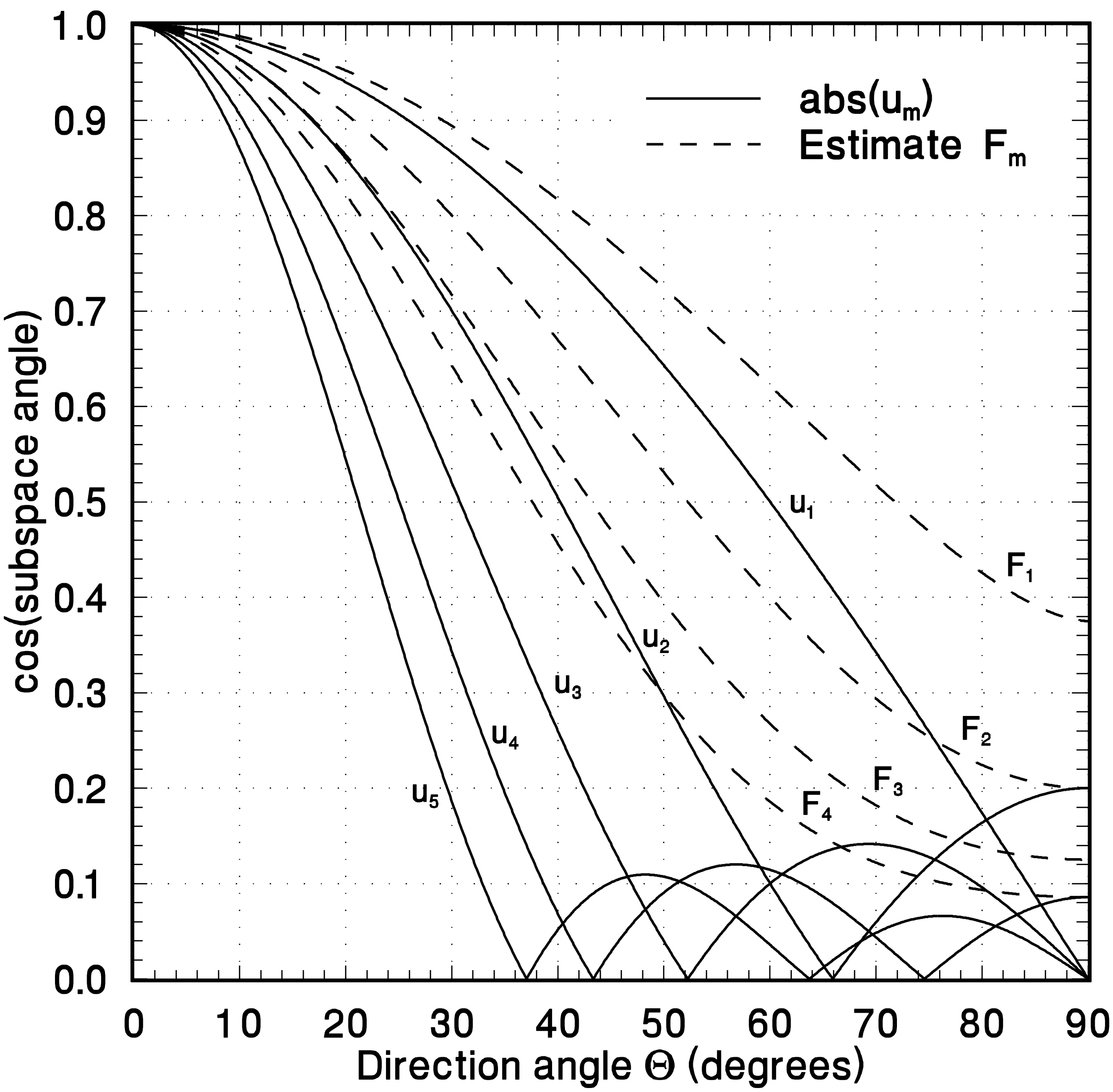

Figure 4.1: Comparison of normalized Gegenbauer polynomials $|u_m|$, m = 1–5, $\alpha = 2$, and estimate functions F_m, $m = 1$–4.

Step 1: Referring to formulae (A.16) and (A.17) in Appendix A, we have

$$\begin{aligned} u_1(\theta) &= \cos\theta \\ u_2(\theta) &= \Big[(2\alpha+2)(\cos\theta)^2 - 1\Big]/(2\alpha+1) \end{aligned}$$

Solving for intersections between $|u_1(\theta)|$ and $|u_2(\theta)|$ in the interval $0 \le \theta \le \pi/2$ yields the solutions

$$\theta \in \left\{0, \arccos\left(\tfrac{1}{2\alpha+2}\right)\right\}. \tag{4.18}$$

A simple check at $\tilde{\theta} = \arccos(\frac{1}{2})$ shows that

$$|u_2(\tilde{\theta})| = \left|\frac{\alpha-1}{4\alpha+2}\right| \le \frac{1}{4} < \tfrac{1}{2} = u_1(\tilde{\theta}) \qquad \text{for } \alpha \ge 1. \tag{4.19}$$

Since (for $\alpha \ge 1$) $0 < \arccos(\frac{1}{2}) < \arccos(1/(2\alpha+2))$, it follows that

$$u_1(\theta) \ge |u_2(\theta)| \qquad \text{for } 0 \le \theta \le \arccos\left(\tfrac{1}{2\alpha+2}\right). \tag{4.20}$$

Next notice that on the interval $\theta \in [\arccos(1/(2\alpha+2)), \pi/2]$,

$$(\cos\theta)^2 \le (2\alpha+2)^{-2},$$

so $u_2(\theta) < 0$, implying that $|u_2(\theta)| = -u_2(\theta)$. Also,

$$-\, u_2(\pi/2) = \frac{1}{2\alpha+1} > 0 = u_1(\pi/2), \tag{4.21}$$

so $-u_2(\theta) \ge |u_1(\theta)|$ for $\theta \in [\arccos(1/(2\alpha+2)), \pi/2]$. This shows that

$$\max_{j=1,2} |u_j(\theta)| = \begin{cases} u_1(\theta) = \cos\theta & \text{for } 0 \le \theta \le \arccos\left(\frac{1}{2\alpha+1}\right) \\ -u_2(\theta) = \frac{1-(2\alpha-2)(\cos\theta)^2}{2\alpha+1} & \text{for } \arccos\left(\frac{1}{2\alpha+1}\right) \le \theta \le \frac{\pi}{2}. \end{cases} \tag{4.22}$$

Step 2: Referring now to formula (A.19) in Appendix A yields

$$u_3(\theta) = \frac{2(\alpha+2)(\cos\theta)^3 - 3\cos\theta}{2\alpha+1} \tag{4.23}$$

The intersections between the graphs of $|u_1(\theta)|$ and $|u_3(\theta)|$ for $0 \leq \theta \leq \pi/2$ occur at θ satisfying

$$\pm (2\alpha+1)\cos\theta = 2(\alpha+2)(\cos\theta)^3 - 3\cos\theta, \tag{4.24}$$

which implies

$$(\cos\theta)^2 = \frac{3 \pm (2\alpha+1)}{2\alpha+4} \qquad \text{or} \qquad \cos\theta = 0. \tag{4.25}$$

Since $\alpha \geq 1$, the $\pm$ above must be $+$. Moreover, $0 \leq \theta \leq \pi/2 \Rightarrow \cos\theta \geq 0$, so $\cos\theta$ must be either 0 or 1, i.e.,

$$\theta \in \{0, \pi/2\} \tag{4.26}$$

Comparing $|u_3(\tilde{\theta})|$ and $u_1(\tilde{\theta})$ at $\tilde{\theta} = \arccos(\frac{1}{2})$ shows that $|u_3(\tilde{\theta})| \leq u_1(\tilde{\theta})$ for $\alpha \geq 1$. Therefore

$$|u_3(\theta)| \leq u_1(\theta) \qquad \text{for } 0 \leq \theta \leq \pi/2,\ \alpha \geq 1). \tag{4.27}$$

Step 3: Recall the definition of $F_m(\theta)$ given in (4.16). Thus

$$\begin{aligned} F_4(\theta) &= \frac{2}{B(\alpha,\frac{1}{2})} \int_0^{\pi/2} \left[(\cos\theta)^2 + (\sin\theta\cos\phi)^2\right]^2 (\sin\phi)^{2\alpha-1}\, d\phi \\ &= \frac{1}{B(\alpha,\frac{1}{2})} \left[B(\alpha,\tfrac{1}{2})(\cos\theta)^4 + 2B(\alpha,\tfrac{3}{2})(\cos\theta\sin\theta)^2 + B(\alpha,\tfrac{5}{2})(\sin\theta)^4\right] \end{aligned}$$

Using the fact that $B(w,z) = \Gamma(w)\Gamma(z)/\Gamma(w+z)$ and $\Gamma(z+1) = z\Gamma(z)$ reduces the above to

$$F_4(\theta) = (\cos\theta)^4 + \frac{2(\cos\theta\sin\theta)^2}{2\alpha+1} + \frac{3(\sin\theta)^4}{(2\alpha+3)(2\alpha+1)} \tag{4.28}$$

The work is now easier if we replace $\cos\theta$ with x, i.e., let $G(x) = F_4(\arccos x)$, for $0 \le x \le 1$. Then

$$\begin{aligned} G(x) &= x^4 + \frac{2x^2(1-x^2)}{2\alpha+1} + \frac{3(1-x^2)^2}{(2\alpha+3)(2\alpha+1)} && (4.29) \\ G'(x) &= 4x^3 + \frac{4x-8x^3}{2\alpha+1} - \frac{12x(1-x^2)}{(2\alpha+3)(2\alpha+1)} && (4.30) \\ G''(x) &= \frac{8\alpha}{(2\alpha+3)(2\alpha+1)}\left[6(\alpha+1)x^2+1\right] && (4.31) \end{aligned}$$

For $\alpha \ge 1$, $G''(x)$ is clearly positive, so $G(x)$ is concave.

Let us compare first $F_4(\theta)$ to $u_1(\theta)$ for $0 \le \theta \le \arccos(1/(2\alpha+2))$. This is equivalent to comparing (under $\cos\theta \to x$) $G(x)$ to x for $\frac{1}{2\alpha+2} \le x \le 1$. Refer to Fig. 4.2, which is a representative sketch of $G(x)$ (for $\alpha = 2$).

Note that $G(1) = 1$ and $G'(1) = 4 - 4/(2\alpha+1) > 1$ (since $\alpha \ge 1$), so there is some interval $\xi \le x \le 1$ for which $G(x) \le x$. Also, since G is concave, $G(x)$ and x intersect at no more than 2 points. One such point is $x = 1$. Since $G(0) > 0$, the second point, $x = \xi$, lies in the interval $(0, 1)$. I need to show that $\xi < 1/(2\alpha+2)$. To do this it suffices to show that $G(1/(2\alpha+2)) < 1/(2\alpha+2)$. But

$$G\left(\frac{1}{2\alpha+2}\right) = \left[1 + (2\alpha+3)(6\alpha+5)\right]/(2\alpha+2)^4. \tag{4.32}$$

Now use the fact that for $\alpha \ge 1$, we have

$$\begin{aligned} (2\alpha+3)/(2\alpha+2) &\le 5/4 \\ (6\alpha+5)/(2\alpha+2) &\le 3 \\ 1/(2\alpha+2) &\le 1/4, \end{aligned}$$

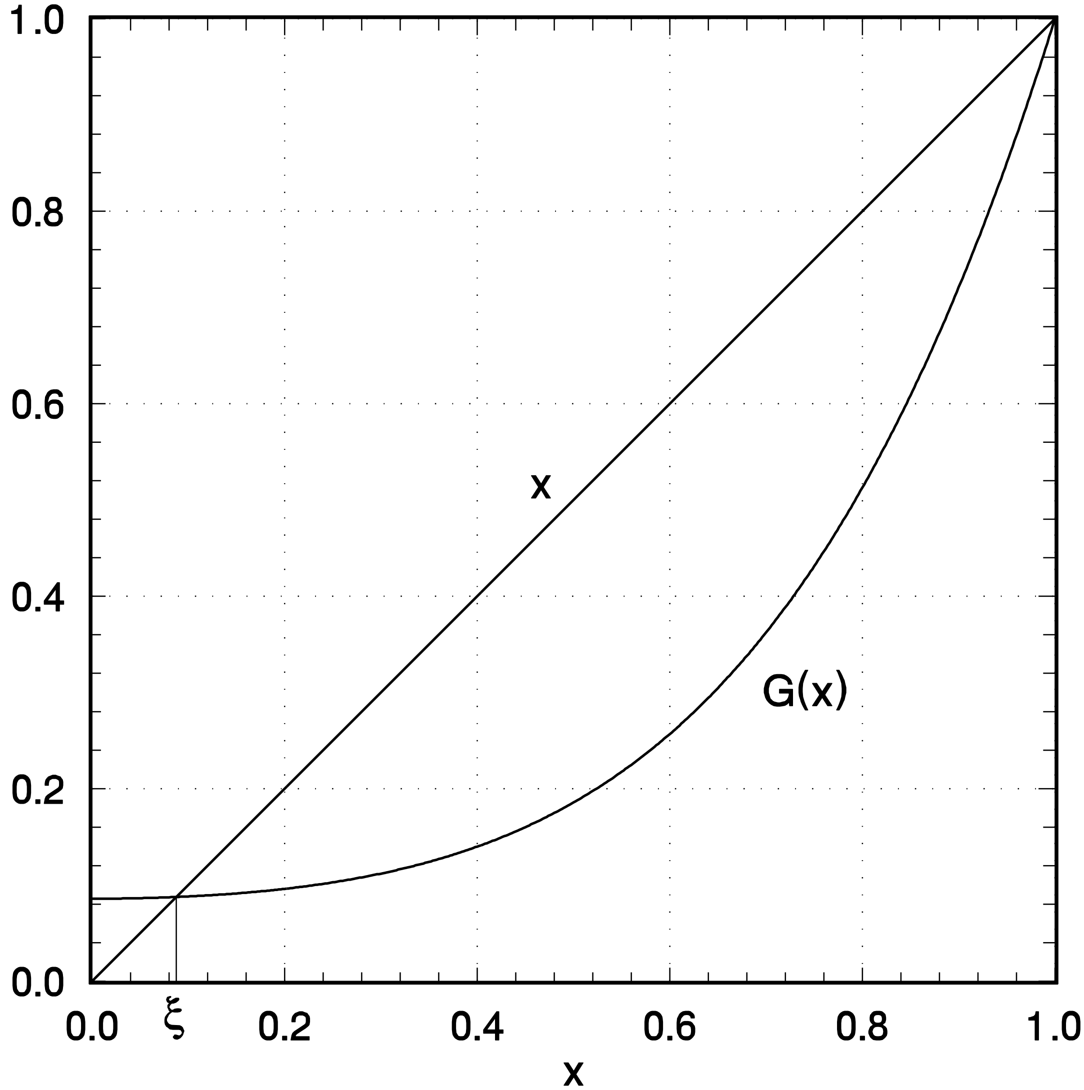

Figure 4.2: Comparison of $G(x) = F_4(\arccos x)$ and $u_1(\arccos x)$ $(= x)$ for $\alpha = 2$.

so

$$G\left(\frac{1}{2\alpha+2}\right) \le \frac{61}{64}\left(\frac{1}{2\alpha+2}\right) < \frac{1}{2\alpha+2},$$

which proves

$$F_4(\theta) \le u_1(\theta) \qquad \text{for } 0 \le \theta \le \arccos(1/4). \tag{4.33}$$

Now let $\theta_0 = \arccos(1/4)$. As noted previously, $F_m(\theta)$ is decreasing on the interval $[0, \pi/2]$. Thus

$$F_4(\theta) \le F_4(\theta_0) \qquad \text{for } \theta_0 \le \theta \le \pi/2.$$

A simple check of (4.22) shows that $-u_2(\theta) = \max\{|u_1(\theta)|, |u_2(\theta)|\}$ is increasing on the interval $\theta \in [\theta_0, \pi/2]$. Therefore,

$$F_4(\theta) \le F_4(\theta_0) < \frac{1}{2\alpha+2} = -u_2(\theta_0) \le -u_2(\theta) \qquad \text{for } \theta_0 \le \theta \le \pi/2.$$

This combined with the preceding discussion shows that

$$F_4(\theta) \le \max\{|u_1(\theta)|, |u_2(\theta)|\} \qquad \text{for } 0 \le \theta \le \pi/2. \tag{4.34}$$

Combining (4.22), (4.27) and (4.34) with (4.17) completes the proof of Theorem 4.

□

4.4 Radon and X-ray transform results

Combining the work in the preceding sections proves the following two theorems:

Theorem 5 *Let $\sigma_1 \in S^{N-1}$, $\sigma_2 \in S^{N-1}$, let $\theta \in [0, \pi/2]$ be the angle between σ_1 and σ_2, and let $\mathcal{N}_{\sigma_1}$, $\mathcal{N}_{\sigma_2}$ be the null spaces of the Radon transforms R_{σ_1}, R_{σ_2}. Then*

$$\gamma(\mathcal{N}_{\sigma_1}, \mathcal{N}_{\sigma_2}) = \begin{cases} \theta & \text{if } 0 \leq \theta \leq \arccos\left(\frac{1}{N+2}\right) \\ \arccos\left(\dfrac{1-(N+2)(\cos\theta)^2}{N+1}\right) & \text{if } \arccos\left(\frac{1}{N+2}\right) \leq \theta \leq \frac{\pi}{2} \end{cases}$$

Theorem 6 *Let $\sigma_1 \in S^{N-1}$, $\sigma_2 \in S^{N-1}$, let $\theta \in [0, \pi/2]$ be the angle between σ_1 and σ_2, and let $\mathcal{N}_{\sigma_1}$, $\mathcal{N}_{\sigma_2}$ be the null spaces of the X-ray transforms P_{σ_1}, P_{σ_2}. Then*

$$\gamma(\mathcal{N}_{\sigma_1}, \mathcal{N}_{\sigma_2}) = \begin{cases} \theta & \text{if } 0 \leq \theta \leq \arccos\left(\frac{1}{4}\right) \\ \arccos\left(\dfrac{\sin 3\theta}{3\sin\theta}\right) & \text{if } \arccos\left(\frac{1}{4}\right) \leq \theta \leq \frac{\pi}{2} \end{cases}$$

CHAPTER V

Angles between null spaces of the general k-plane transforms on $\mathbf{R}^N$ with Gaussian measure

We now consider the general integral transform of Chapter III, but this time we work on $L^2\left(\mathbf{R}^N, (2\pi)^{-N/2}e^{-\|z\|^2/2}\right)$ instead of $L^2(\Omega^N)$. The development is parallel, but the end result is simpler.

Let $0 < k < N$ as before, and decompose $\mathbf{R}^N$ into $\mathbf{R}^k \oplus \mathbf{R}^{N-k}$ with $z = x \oplus y$. Define $S : L^2\left(\mathbf{R}^N, (2\pi)^{-N/2}e^{-\|z\|^2/2}\right) \to L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ by

$$Sf(x) = (2\pi)^{(k-N)/2} \int_{\mathbf{R}^{N-k}} f(x \oplus y)e^{-\|y\|^2/2}\, d^{N-k}y. \tag{5.1}$$

The norm of the operator is easily computed:

$$\begin{aligned}
\|Sf\|^2 &= (2\pi)^{-k/2} \int_{\mathbf{R}^k} |Sf(x)|^2 e^{-\|x\|^2/2} d^k x \\
&= (2\pi)^{-k/2} \int_{\mathbf{R}^k} (2\pi)^{(k-N)/2} \left| \int_{\mathbf{R}^{N-k}} f(x \oplus y) e^{-\|y\|^2/2}\, d^{N-k}y \right|^2 e^{-\|x\|^2/2} d^k x \\
&\leq (2\pi)^{-N/2} \int_{\mathbf{R}^N} |f(z)|^2 e^{-\|z\|^2/2}\, d^N z \\
&= \|f\|^2,
\end{aligned}$$

so $\|S\| \leq 1$. An easy check with f equal to a constant shows the $\|S\| = 1$. Continuing as in Chapter III, define

$$\mathcal{N} = \text{Null } S = \{f \in L^2\left(\mathbf{R}^N, (2\pi)^{-N/2}e^{-\|z\|^2/2}\right) \mid Sf = 0\},$$

and let A be the subset of $L^2\left(\mathbf{R}^N, (2\pi)^{-N/2}e^{-\|z\|^2/2}\right)$ defined by

$$A = \{f \mid \exists \tilde{f}(z) = f(z) \text{ a.e. with } \tilde{f}(x \oplus y) = \tilde{f}(x \oplus 0)\ \forall x \in \mathbf{R}^k\}.$$

Then we have

Lemma 9 *The set A is the orthogonal complement in $L^2\left(\mathbf{R}^N, (2\pi)^{-N/2}e^{-\|z\|^2/2}\right)$ to the null space $\mathcal{N}$ of the operator S.*

Proof: The proof is essentially the same as in Lemma 1 in Chapter III, with allowances for the difference in measures.

Clearly $A \subseteq \mathcal{N}^\perp$, so consider a fixed $f \in \mathcal{N}^\perp$. We shall show that f is in A. Define

$$f_0(z) = f_0(x \oplus y) = Sf(x).$$

Then

$$\begin{aligned}
\|f_0\|^2 &= (2\pi)^{-N/2} \int_{\mathbf{R}^N} |f_0(z)|^2 e^{-\|z\|^2/2}\, d^N z \\
&= (2\pi)^{-k/2} \int_{\mathbf{R}^k} \left|(2\pi)^{(k-N)/2} \int_{\mathbf{R}^{N-k}} f(x \oplus t) e^{-\|t\|^2/2}\, d^{N-k} t\right|^2 e^{-\|x\|^2/2}\, d^k x \\
&\leq (2\pi)^{-N/2} \int_{\mathbf{R}^k} \int_{\mathbf{R}^{N-k}} |f(x \oplus t)|^2 e^{-\|t\|^2/2}\, d^{N-k} t\, e^{-\|x\|^2/2}\, d^k x \\
&= \|f\|^2.
\end{aligned}$$

Therefore, $f_0 \in L^2\left(\mathbf{R}^N, (2\pi)^{-N/2}e^{-\|z\|^2/2}\right)$, and in particular $f_0 \in A$. Since $Sf_0 = Sf$, we have $f - f_0 \in \mathcal{N}$. Moreover, $f \in \mathcal{N}^\perp$ by assumption, and $f_0 \in A \subseteq \mathcal{N}^\perp$ by construction, so $f - f_0$ is also an element of $\mathcal{N}^\perp$. But $\mathcal{N} \cap \mathcal{N}^\perp = \{0\}$, so $f = f_0 \in A$. Since f is an arbitrary element of $\mathcal{N}^\perp$, it follows that $\mathcal{N}^\perp = A$. $\square$

Lemma 10 *The sets A and $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ are isomorphic as Hilbert spaces.*

Proof: Proceed as in Lemma 5. If $f \in A$, then there exists a representative $\tilde{f} = f$ with $\tilde{f}(x \oplus y) = \tilde{f}(x \oplus 0)$ for every $x \in \mathbf{R}^k$. Define the operator H by

$$Hf(x) = \tilde{f}(x \oplus 0).$$

For $f \in A$, $g \in A$ we have

$$\begin{aligned}
\langle f, g\rangle &= (2\pi)^{-N/2}\int_{\mathbf{R}^N} f(z)g(z)\, e^{-\|z\|^2/2}\, d^N z \\
&= (2\pi)^{-N/2}\int_{\mathbf{R}^k} e^{-\|x\|^2/2}\int_{\mathbf{R}^{N-k}} f(x\oplus y)g(x\oplus y)\, e^{-\|y\|^2/2}, d^{N-k}y\, d^k x \\
&= (2\pi)^{-k/2}\int_{\mathbf{R}^k} Hf(x)\, Hg(x)\, e^{-\|x\|^2/2}\, d^k x \\
&= \langle Hf, Hg\rangle\,,
\end{aligned}$$

where the last inner product is in the space $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$. It is clear that H is bijective, so in fact

$$H : A \to L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$$

is a Hilbert space isomorphism. $\square$

Now define S_U and A_U as in Chapter III, i.e.,

$$S_U f(x) = S(f \circ U^{-1})(x). \tag{5.2}$$

and

$$A_U = \{f \in L^2(\Omega^N) \mid f \circ U^{-1} \in A\}, \tag{5.3}$$

where $U \in SO_N$. Also, as before, let $\mathcal{N}_U$ be the null space for the operator S_U.

Lemma 11 *If U is in the canonical form (3.7) with $|a_i| < 1$ for $i = 1, 2, \ldots, m$, then $A \cap A_U$ is the set of all functions in $L^2\left(\mathbf{R}^N, (2\pi)^{-N/2} e^{-\|z\|^2/2}\right)$ that are functions of the coordinates $m+1$ through k alone.*

Proof: The proof is similar to that of Lemma 4. For $f \in A \cap A_U$ there exist representatives, say f_I and f_U, with $f(z) = f_I(z) = f_U(z)$ a.e., such that

$$f_I(x \oplus y) = f_I(x \oplus 0)$$

and

$$f_U \circ U^{-1}(x \oplus y) = f_U \circ U^{-1}(x \oplus 0)$$

for all $x \oplus y \in \mathbf{R}^N$. Assuming U is given by

$$U = \begin{array}{cc} & \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{array}{c} m \\ k-m \\ m \\ N-k-m \end{array} & \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & I & 0 & 0 \\ \hline B' & 0 & & \\ 0 & 0 & & C \end{array}\right) \end{array},$$

define

$$\mu T = \begin{array}{c c} & \begin{array}{cccc} m & k-m & m & N-k-m \end{array} \\ \begin{array}{c} m \\ k-m \\ m \\ N-k-m \end{array} & \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ A & 0 & B & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{array},$$

where $\mu > 0$ is a scaling factor. Since $A = \operatorname{diag}(a_1, \ldots, a_m)$ satisfies $|a_i| < 1$ for $i = 1, 2, \ldots, m$, it follows that $B = \operatorname{diag}(b_1, \ldots, b_m)$ satisfies $|b_i| > 0$ for $i = 1, 2, \ldots, m$. In particular, B is invertible, so therefore T is invertible as well.

Next, define P_1 to be the projection onto the first k coordinates and define P_2 to be the projection onto coordinates e_{m+1} through e_{m+k}. Notice that

$$P_1 \circ T^{-1} \circ P_1 = P_1 \circ T^{-1}$$

and

$$P_1 \circ U \circ T^{-1} \circ P_2 = P_1 \circ U \circ T^{-1}.$$

(Refer to the proof of Lemma 4 for more details.)

Apply Corollary 2 of Appendix B to f with mapping T and space $X = T^{-1}\left([-1,1]^N\right)$. This shows the existence of a function f_3 with $f_3(z) = f(z)$ a.e. $z \in T^{-1}\left([-1,1]^N\right)$ and $f_3 \circ T^{-1} \circ P_3 \circ T(z) = f_3(z)$ for all $z \in T^{-1}\left([-1,1]^N\right)$. But

$$T^{-1} \circ P_3 \circ T = P_3,$$

so $f_3(z) = f_3 \circ P_3(z)$ for all $z \in T^{-1}\left([-1,1]^N\right)$.

The set $T^{-1}\left([-1,1]^N\right)$ is a neighborhood of the origin with size depending on the scaling parameter μ. By making μ small enough, the set $T^{-1}\left([-1,1]^N\right)$ can be made to fill an arbitrarily large region of $\mathbf{R}^N$ (with respect to the measure $(2\pi)^{-k/2}e^{-\|x\|^2/2}$). This permits construction of a sequence of functions $\{h_n\}$ that agree with f a.e. and such that $h_n(z) = h_n \circ P_3(z)$ for all z with $|z| < n$. Since $h_{n+1}(z) = h_n(z)$ a.e., we can, if necessary, redefine h_{n+1} on a set of measure zero so that $h_{n+1}(z) = h_n(z)$ for all $|z| < n$ and still maintain $h_{n+1}(z) = h_{n+1} \circ P_3(z)$ for all z with $|z| < n+1$. Then the sequence of functions $\{h_n\}$ has a limit, say h, with $h(z) = f(z)$ a.e., and $h(z) = h \circ P_3(z)$ for all $z \in \mathbf{R}^N$.

□

The orthogonal operator U induces a bounded linear operator L_A on A by

$$\langle f_1, L_A f_2\rangle = \langle f_1, f_2 \circ U\rangle\,,$$

corresponding to (3.11). Via the isomorphism H we get the corresponding operator, L, on $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ by the relation

$$L = HL_AH^{-1}.$$

In terms of the inner product on $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ the operator L satisfies

$$\langle g_1, Lg_2\rangle = (2\pi)^{-N/2}\int_{\mathbf{R}^k} g_1(x)e^{-\|x\|^2/2}\int_{\mathbf{R}^{N-k}} g_2 \circ \Pi_k \circ U(x \oplus y)e^{-\|y\|^2/2}\, d^{N-k}y\, d^kx \tag{5.4}$$

(compare to (3.13)), where Π_k is the projection of R^N onto the first k coordinates. Working as before shows that this L is also self-adjoint. An explicit formula for L can written directly from (5.4), namely

$$Lg(x) = (2\pi)^{(k-N)/2} \int_{\mathbf{R}^{N-k}} g \circ P \circ U(x \oplus y) e^{-\|y\|^2/2}\, d^{N-k}y.$$

Using the canonical form for U given in (3.7), we have

$$\begin{aligned} Lg(x) &= (2\pi)^{(k-N)/2} \int_{\mathbf{R}^{N-k}} g(a_1x_1 + b_1y_1, \ldots, a_mx_m + b_my_m, x_{m+1}, \ldots, x_k) \\ &\quad \times e^{-\|y\|^2/2}\, d^{N-k}y. \end{aligned} \tag{5.5}$$

Let us study the effect of L on monomials. Given monomial x^α, we have

$$\begin{aligned} Lx^\alpha &= (2\pi)^{(k-N)/2} x_{m+1}^{\alpha_{m+1}} \ldots x_k^{\alpha_k} \int_{\mathbf{R}^{N-k}} \prod_{j=1}^{m} (a_jx_j + b_jy_j)^{\alpha_j} e^{-\|y\|^2/2}\, d^{N-k}y \\ &= (2\pi)^{-m/2} x_{m+1}^{\alpha_{m+1}} \ldots x_k^{\alpha_k} \prod_{j=1}^{m} \int_{\mathbf{R}} (a_jx_j + b_jy)^{\alpha_j} e^{-y^2/2}\, dy \\ &= a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_m^{\alpha_m} x^\alpha \ + \text{ lower order terms.} \end{aligned} \tag{5.6}$$

In particular, the set

$$E_d = \{\text{lin span } x^\alpha \mid |\alpha| \le d\}$$

is an invariant subspace for L. Also, since L is self-adjoint, it follows that $E_d^\perp$ is also an invariant subspace.

Next define the polynomials P_α as in (3.27), i.e.,

$$P_\alpha(x) = x^\alpha - \sum_{|\sigma|<|\alpha|} \frac{\langle x^\alpha, x^\sigma \rangle}{\langle x^\sigma, x^\sigma \rangle} x^\sigma, \tag{5.7}$$

except in this case the inner products are on the space $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$. Note that since $P_\alpha \in E^\perp_{|\alpha|-1}$, it follows that LP_α is a polynomial of degree $|\alpha|$ which is orthogonal to $E_{|\alpha|-1}$. But from (5.6) and (5.7) we see that LP_α must have the form

$$LP_\alpha(x) = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} x^\alpha + \sum_{|\sigma|<|\alpha|} c_\sigma x^\sigma.$$

The space of polynomials of this form in $E^\perp_{|\alpha|-1}$ is a one dimensional space that contains P_α. Therefore

$$LP_\alpha = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} P_\alpha. \tag{5.8}$$

This with the fact that polynomials are dense in $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ proves the following result:

Theorem 7 *The operator L on $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ is diagonalizable. The functions P_α of (5.7) are the eigenfunctions for L, with corresponding eigenvalues $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m}$, where the values m, a_1, a_2, $\ldots$, a_m are invariants of the transformation U, as given by the canonical representation in (3.7).*

An interesting consequence of this theorem involves the orthogonality of the polynomials P_α. In Chapter III, the blocks in the decomposition (3.33) of L are not symmetric because the basis $\{P_\alpha\}$ is not orthogonal. Theorem 7 shows that the situation is different in $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$.

Corollary 1 *The polynomials $\{P_\alpha\}$ as defined by (5.7) are pairwise orthogonal in $L^2\left(\mathbf{R}^k, (2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$.*

Proof: Theorem 7 states that the P_α are eigenfunctions for the self-adjoint operator L. Eigenfunctions corresponding to distinct eigenvalues of self-adjoint operators are orthogonal. Therefore, if

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m} \neq a_1^{\tilde{\alpha}_1} a_2^{\tilde{\alpha}_2} \cdots a_m^{\tilde{\alpha}_m}, \tag{5.9}$$

then P_α and $P_{\tilde{\alpha}}$ are orthogonal. The values $\{a_i\}$ depend only on the orthogonal matrix U of (3.7), whereas the P_α's are independent of U. Therefore, for any α and $\tilde{\alpha}$, $\alpha \neq \tilde{\alpha}$, we may choose U with $m = k$ and a_i's depending on α and $\tilde{\alpha}$ in such a way that (5.9) is satisfied. Thus $P_\alpha \perp P_{\tilde{\alpha}}$ for all $\alpha \neq \tilde{\alpha}$. $\square$

We return now to the main result of this chapter.

Theorem 8 *Let $U \in SO_N$, and let S and S_U be the operators defined by (5.1) and (5.2) on $L^2\left(\mathbf{R}^N, (2\pi)^{-N/2} e^{-\|z\|^2/2}\right)$. Then the angle between the null spaces of S and S_U is*

$$\gamma(\mathcal{N}, \mathcal{N}_U) = \arccos\left(\max_{1 \leq i \leq m} |a_i|\right), \tag{5.10}$$

where the constants m, a_1, a_2, $\ldots$, a_m are invariants of the matrix U, given by the size and entries of the diagonal matrix A in (3.7).

Proof: As in Chapter III, the angle between the null spaces $\mathcal{N}$ and $\mathcal{N}_U$ is given by

$$\begin{aligned}\cos\left(\gamma(\mathcal{N},\mathcal{N}_U)\right) &= \cos\left(\gamma(A,A_U)\right)\\ &= \sup_{\substack{\|f_1\|=\|f_2\|=1\\ f_i\in A\cap(A\cap A_U)^\perp}} |\langle f_1, f_2\circ U\rangle|.\end{aligned}$$

(See (3.4).) Under the isomorphism H between A and $L^2\left(\mathbf{R}^k,(2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$, this corresponds to

$$\cos\left(\gamma(\mathcal{N},\mathcal{N}_U)\right) = \sup_{\substack{\|g_1\|=\|g_2\|=1\\ g_i\in D}} |\langle g_1, Lg_2\rangle|,$$

where $D = H(A\cap(A\cap A_U)^\perp)$. (Compare to (3.14).) It follows from Lemma 11 that $D^\perp$ is the set of functions in $L^2\left(\mathbf{R}^k,(2\pi)^{-k/2}e^{-\|x\|^2/2}\right)$ which are functions of the coordinates $m+1$ through k alone. Thus

$$D^\perp = \text{ lin span } \{P_{0\oplus\gamma}\}^{\text{cl}}.$$

Since $\{P_\alpha\}$ is an orthogonal set, it follows that

$$D = \text{ lin span } \{P_{\beta\oplus\gamma}\}^{\text{cl}}_{|\beta|>0}.$$

Let $\lambda_\alpha = a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_m^{\alpha_m}$ denote the eigenvalue for eigenvector P_α. Then

$$\begin{aligned}\sup_{\substack{\|g_1\|=\|g_2\|=1\\ g_i\in D}} |\langle g_1, Lg_2\rangle| &= \sup_{\beta\oplus\gamma,\,|\beta|>0} |\lambda_{\beta\oplus\gamma}|\\ &= \sup_{|\beta|>0} |a_1^{\beta_1}a_2^{\beta_2}\cdots a_m^{\beta_m}|. \qquad (5.11)\end{aligned}$$

But $0\le|a_i|<1$ for $i=1,2,\ldots,m$, so clearly the last supremum equals $\max_{1\le i\le m}|a_i|$.

□

CHAPTER VI

Angle between null spaces in $L^2(S^2)$

Let us change our considerations from $\mathbf{R}^N$ to S^2, the unit sphere in $\mathbf{R}^3$. In 1916 Funk [1] studied the inversion of the transform produced by integration over great circles on S^2. In this chapter I explicitly evaluate the angle between null spaces of a related transform, that obtained by integrating over the "latitude" circles on S^2.

6.1 The "latitude" integral transforms on S^2

Consider the space $L^2(S^2)$, and let T be the operator

$$T : L^2(S^2) \longrightarrow L^2([-1,1],(1-z^2)^{-1})$$

defined via

$$Tf(z) = \int_0^{2\pi} f\left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)\sqrt{1-z^2}\,d\theta$$

where $(x,y,z) \in S^2$ is parameterized with respect to the standard basis on $\mathbf{R}^3$.

Note that

$$\|Tf\|^2 = \int_{-1}^{1}\left[\int_0^{2\pi} f\left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)\sqrt{1-z^2}\,d\theta\right]^2 (1-z^2)^{-1}\,dz$$
$$\leq \int_{-1}^{1}\left[\int_0^{2\pi}\left(f\left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)\right)^2 d\theta\right]\left[2\pi(1-z^2)\right](1-z^2)^{-1}\,dz.$$

Substituting $z = \cos\phi$ gives

$$\begin{aligned}\|Tf\|^2 &\leq 2\pi\int_0^{\pi}\int_0^{2\pi}\left(f\left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)\right)^2 \sin\phi\,d\theta\,d\phi \\ &= 2\pi\|f\|^2.\end{aligned}$$

Thus, $\|T\| \leq \sqrt{2\pi}$. Moreover, if f is a function of z alone (for example, $f \equiv 1$), then the above inequality becomes an equality, so

$$\|T\| = \sqrt{2\pi} \tag{6.1}$$

The value $Tf(z_0)$ is the integral of f on the circle $S^2 \cap \{(x,y,z) \mid z = z_0\}$. For each z_0 the integrating set is a different circle. The collection of such circles consists of those circles that are perpendicular to and have centers on the z-axis. These circles are "latitudes" on the unit sphere.

Next let $w \in S^2$, and define $T_w f(t)$ to be the integral of f on the circle $S^2 \cap \{x \in \mathbf{R}^3 \mid \langle x, w\rangle = t\}$. As we vary t we get a collection of "latitudes" on S^2 about the axis w. In particular, $T_{e_3} = T$ (where e_1, e_2, e_3 is the usual basis on $\mathbf{R}^3$).

Theorem 9 *Let $\mathcal{N}_w$ be the null space for T_w, i.e.,*

$$\mathcal{N}_w = \{f \in L^2(S^2) \mid T_w f = 0 \text{ a.e. } t \in [-1,1]\}.$$

Then the angle between $\mathcal{N}_{w_1}$ ($\equiv \mathcal{N}_1$) and $\mathcal{N}_{w_2}$ ($\equiv \mathcal{N}_2$), written $\gamma(\mathcal{N}_{w_1}, \mathcal{N}_{w_2})$, is given by

$$\cos\left(\gamma(\mathcal{N}_{w_1}, \mathcal{N}_{w_2})\right) = \begin{cases} P_1(\cos\psi) & \text{if } 0 \leq \psi < \arccos\left(\frac{1}{\sqrt{5}}\right) \\ -P_3(\cos\psi) & \text{if } \arccos\left(\frac{1}{\sqrt{5}}\right) \leq \psi < \arccos\left(\frac{\sqrt{6}-1}{5}\right) \\ -P_2(\cos\psi) & \text{if } \arccos\left(\frac{\sqrt{6}-1}{5}\right) \leq \psi \leq \frac{\pi}{2} \end{cases}$$

where ψ is the angle between w_1 and w_2, and P_k are the Legendre polynomials of degree k, i.e., $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$.

Proof: As in the case with the Radon transform, we use the fact that $\gamma(\mathcal{N}_{w_1}, \mathcal{N}_{w_2}) = \gamma(\mathcal{N}_{w_1}^{\perp}, \mathcal{N}_{w_2}^{\perp})$. The space $\mathcal{N}_w^{\perp}$ consists of all functions that are constant a.e. on circles on S^2 which are perpendicular to w, i.e.,

$$\mathcal{N}_w^{\perp} = \{f \in L^2(S^2 \mid f(x_1) = f(x_2) \text{ for a.e. } x_1 \in S^2,\ x_2 \in S^2 \text{ with } \langle x_1, w\rangle = \langle x_2, w\rangle\}.$$

The proof is analogous to the proof of Lemma 1.

Let Π be the projection of the hemisphere $\{x \in \mathbf{R}^3 \mid \langle x, w_1 \times w_2\rangle > 0\}$ onto the unit disk in $\mathbf{R}^2$, where $\times$ denotes the vector cross product. If $g \in L^2(S^2)$, then $g \circ \Pi \in L^2(\Omega^2, (1 - x^2)^{1/2})$. Consider $f \in \mathcal{N}_1^{\perp} \cap \mathcal{N}_2^{\perp}$. Applying Lemma 4 to the function $(1 - x^2)^{1/2} f \circ \Pi(x)$ shows that f is constant (a.e.) in the hemisphere. Then the rotational invariance of f (with respect to either axis) shows that f is constant throughout the sphere. Therefore $\mathcal{N}_1^{\perp} \cap \mathcal{N}_2^{\perp}$ is the set of (a.e.) constant functions.

Without loss of generality, take $w_1 = e_3$ and $w_2 = -e_2 \sin\psi + e_3 \cos\psi$, so that ψ is the angle between w_1 and w_2. Identify to each $f \in \mathcal{N}_i^{\perp}$ the function $\tilde{f} \in L^2([-1,1])$ given by $\tilde{f}(t) = f(x)$ for a.e. x satisfying $\langle x, w_i \rangle = t$. In particular, note that for $f \in \mathcal{N}_i^{\perp}$, $g \in \mathcal{N}_i^{\perp}$, we have

$$\begin{aligned}\int_{S^2} fg &= \int_0^{2\pi}\int_0^{\pi} \tilde{f}(\cos\phi)\tilde{g}(\cos\phi)\sin\phi \, d\phi \, d\theta \\ &= 2\pi \int_{-1}^{1} \tilde{f}(z)\tilde{g}(z)\, dz \qquad (6.2)\end{aligned}$$

i.e., $\langle f, g\rangle_{L^2(S^2)} = 2\pi \left\langle \tilde{f}, \tilde{g}\right\rangle_{L^2([-1,1])}$.

Next let $f_i \in \mathcal{N}_i^{\perp}$, and using the fact $f_i(x) = \tilde{f}_i(\langle x, w_i\rangle)$ for a.e. x yields

$$\begin{aligned}\int_{S^2} f_1 f_2 &= \int_0^{2\pi}\int_0^{\pi} \tilde{f}_1(\cos\phi)\tilde{f}_2(\cos\psi\cos\phi - \sin\psi\sin\phi\sin\theta)\sin\phi\, d\phi\, d\theta \\ &= \int_{-1}^{1} \tilde{f}_1(z) \int_0^{2\pi} \tilde{f}_2\left(z\cos\psi - \sqrt{1-z^2}\sin\psi\sin\theta\right) d\theta\, dz. \qquad (6.3)\end{aligned}$$

The problem has thus been reduced to a question about functions on $L^2([-1,1])$. In particular, the right hand side of (6.3) is a bounded bilinear form on $\tilde{f}_1$, $\tilde{f}_2$, so there exists a bounded linear operator $L : L^2([-1,1]) \to L^2([-1,1])$ such that

$$\left\langle \tilde{f}_1, L\tilde{f}_2 \right\rangle = \int_{-1}^{1} \tilde{f}_1(z) \int_0^{2\pi} \tilde{f}_2\left(z\cos\psi - \sqrt{1-z^2}\sin\psi\sin\theta\right) d\theta\, dz. \qquad (6.4)$$

In fact, the operator L is seen to be

$$L\tilde{f}(z) = \int_0^{2\pi} \tilde{f}\left(z\cos\psi - \sqrt{1-z^2}\sin\psi\sin\theta\right) d\theta. \qquad (6.5)$$

We now follow a process analogous to the one used in Chapter IV. To ease the notation, let us temporarily drop the tilde notation. Similarly, all inner products are henceforth in $L^2([-1,1])$ until specified differently.

Note that the expression in (6.3) was obtained by introducing a spherical coordinate system with w_1 as the central ("z") axis. If the coordinate system is built around w_2 instead, then we get

$$\langle f_1, Lf_2\rangle = \int_0^{2\pi}\int_0^{\pi} f_1(\cos\psi\cos\phi + \sin\psi\sin\phi\sin\theta) f_2(\cos\phi)\sin\phi\, d\phi\, d\theta,$$

which should be compared to (6.3). Replacing θ with $-\theta$ and using the 2π-periodicity of $\sin\theta$, $\cos\theta$, gives

$$\begin{aligned}\langle f_1, Lf_2\rangle &= \int_0^{2\pi}\int_0^{\pi} f_1(\cos\psi\cos\phi - \sin\psi\sin\phi\sin\theta) f_2(\cos\phi)\sin\phi\, d\phi\, d\theta \\ &= \langle f_2, Lf_1\rangle \\ &= \langle Lf_1, f_2\rangle\,.\end{aligned}$$

Thus L is self-adjoint.

Let $Q_m(x) = x^m$, and use Gram-Schmidt orthogonalization to produce a sequence of polynomials $u_m(x)$ satisfying

$$\langle u_m, u_n\rangle = \int_{-1}^{1} u_m(x)u_n(x)\, dx = \delta_{mn} \qquad \forall m, n \tag{6.6}$$

$$\langle u_m, p\rangle = \int_{-1}^{1} u_m(x)p(x)\, dx = 0 \qquad \forall \text{ polynomials } p \text{ with } \deg p < m \tag{6.7}$$

In particular, $u_m(x) = (m + \frac{1}{2})P_m(x)$, where P_m is the usual Legendre polynomial of degree m, normalized so that $P_m(1) = 1$.

We next show that with respect to the basis $\{u_m\}$, the operator L is diagonal, i.e.,

$$\langle u_m, Lu_n\rangle = 0 \qquad \text{if } m \neq n. \tag{6.8}$$

Let us use the explicit representation of L, (6.5) to calculate $LQ_m(x)$, where $Q_m(x) = x^m$:

$$\begin{aligned} LQ_m(x) &= \int_0^{2\pi} \left(x\cos\psi - \sqrt{1-x^2}\sin\psi\sin\theta\right)^m \, d\theta \\ &= \sum_{k=0}^{m} \binom{m}{k} x^{m-k}(\cos\psi)^{m-k}(-1)^k(1-x^2)^{k/2}(\sin\psi)^k \int_0^{2\pi} (\sin\theta)^k \, d\theta \\ &= 2\sum_{k=0}^{[m/2]} \binom{m}{2k} x^{m-2k}(1-x^2)^k B(k+\tfrac{1}{2},\tfrac{1}{2})(\cos\psi)^{m-2k}(\sin\psi)^{2k}. \quad (6.9) \end{aligned}$$

In particular, $LQ_m(x)$ is a polynomial in x of degree not greater than n. Therefore, via (6.7),

$$\langle u_m, Lu_n \rangle = 0 \qquad \text{if } n < m.$$

But L is self-adjoint, so the same result holds if $m < n$, which proves (6.8).

It remains to calculate the eigenvalues λ_m, which are equal to the coefficients of x^m in (6.9), i.e.,

$$\lambda_m = \langle u_m, Lu_m \rangle = 2\sum_{k=0}^{[m/2]} (-1)^k \binom{m}{2k} B(k+\tfrac{1}{2},\tfrac{1}{2})(\cos\psi)^{m-2k}(\sin\psi)^{2k} \qquad (6.10)$$

Comparing this to (A.12) in Appendix A shows that $\lambda_m = \lambda_m(\psi)$ is a Gegenbauer polynomial with $\alpha = 1/2$, i.e.,

$$\begin{aligned} \lambda_m &= 2C_m^{(1/2)}(\cos\psi)B(\tfrac{1}{2},\tfrac{1}{2})/C_m(1/2)(1) \\ &= 2\pi C_m^{(1/2)}(\cos\psi)/C_m^{(1/2)}(1). \end{aligned}$$

Using the fact that $C_m^{(1/2)}(x) = P_m(x)$ (the Legendre polynomial of degree m) gives

$$\lambda_m = 2\pi P_m(\cos\phi) \qquad (6.11)$$

since it follows from (A.13) that $C_m^{(1/2)}(1) = P_m(1) = 1$.

Let us now reintroduce the tilde notation to denote functions in $L^2([-1,1])$. Then note that (6.11) specifies the eigenvalues of the operator L as an operator on $L^2([-1,1])$. In the original problem, however, L should be viewed as an operator mapping $\mathcal{N}_1^\perp$ to $\mathcal{N}_2^\perp$ under the norm in $L^2(S^2)$. In particular, the orthogonal polynomials $\{\tilde{u}_m\}$ of (6.6) and (6.7) are unit vectors in $L^2([-1,1])$, but their source functions $\{u_m\} \subset \mathcal{N}_1^\perp$ are **not** unit vectors in $L^2(S^2)$, as can be seen from (6.2). In particular,

$$\|u_m\|^2_{L^2(S^2)} = 2\pi \|\tilde{u}_m\|^2_{L^2([-1,1])} = 2\pi.$$

Therefore, the eigenvalues of (6.11) need to be divided by 2π in order to specify eigenvalues of the operator in $L^2(S^2)$.

Thus, since $\mathcal{N}_1^\perp \cap \mathcal{N}_2^\perp$ consists of the constant functions, which is the linear span of u_0, it follows that

$$\cos(\gamma(\mathcal{N}_1, \mathcal{N}_2)) = \sup_{m \in \mathbf{N}} |P_m(\cos\psi)|, \tag{6.12}$$

where ψ is the angle between w_1 and w_2, and P_m is the Legendre polynomials of degree m normalized so that $P_m(1) = 1$.

This result, coupled with Theorem 10 below, completes the proof of Theorem 9.

□

6.2 Supremum of Legendre polynomials

Theorem 10 *Let $P_m(x)$ be the Legendre polynomial of degree m with the usual normalization, i.e., $P_m(1) = 1$. Then for $m \geq 1$, $-1 \leq x \leq 1$,*

$$|P_m(x)| \leq \max\left(|P_1(x)|, |P_2(x)|, |P_3(x)|\right).$$

Moreover,

$$\max_{i=1,2,3}\left(|P_i(x)|\right) = \begin{cases} -P_2(x) & \text{if } 0 \leq x < \dfrac{\sqrt{6}-1}{5} \\ -P_3(x) & \text{if } \dfrac{\sqrt{6}-1}{5} \leq x < \dfrac{1}{\sqrt{5}} \\ P_1(x) & \text{if } \dfrac{1}{\sqrt{5}} \leq x \leq 1 \end{cases}$$

Proof: The proof is similar to the proof of Theorem 4 in Chapter IV, except here $\alpha = 1/2$ is fixed and is less than 1, so that theorem does not apply, and in fact the results are different. The reader is invited to examine Fig. 6.1 as motivation to the following discussion.

Since $P_m(-x) = (-1)^m P_m(x)$, it suffices to show the result for $0 \leq x \leq 1$. Next, let us write down the first several Legendre polynomials:

$$P_1(x) = x \tag{6.13}$$

$$P_2(x) = (3x^2 - 1)/2 \tag{6.14}$$

$$P_3(x) = (5x^3 - 3x)/2 \tag{6.15}$$

$$P_4(x) = (35x^4 - 30x^2 + 3)/8 \tag{6.16}$$

$$P_5(x) = (63x^5 - 70x^3 + 15x)/8 \tag{6.17}$$

The proof is broken down into 4 steps:

Step 1 Solve for the intersections of $|P_1|$, $|P_2|$ and $|P_3|$, and determine which function dominates over each domain.

Step 2 Show that $|P_4| \leq P_1 \vee (-P_2)$

Step 3 Show that $|P_5| \leq |P_1| \vee |P_2| \vee |P_3|$

Step 4 Produce an estimating function F_6 such that $|P_m| \leq F_6 < |P_1| \vee |P_2| \vee |P_3|$ for all $m \geq 6$

Step 1: Solving for intersections between the graphs of $|P_1|$, $|P_2|$ and $|P_3|$ on $[0, 1]$ yields

$$\begin{aligned} |P_1(x)| = |P_2(x)| \quad &\Longrightarrow \quad x \in \{\tfrac{1}{3}, 1\} \\ |P_1(x)| = |P_3(x)| \quad &\Longrightarrow \quad x \in \{0, \tfrac{1}{\sqrt{5}}, 1\} \\ |P_2(x)| = |P_3(x)| \quad &\Longrightarrow \quad x \in \{\tfrac{\sqrt{6}-1}{5}, \tfrac{\sqrt{6}+1}{5}\}. \end{aligned}$$

Therefore, the interval $[0, 1]$ should be broken into the 5 subintervals $[0, (\sqrt{6}-1)/5]$, $[(\sqrt{6}-1)/5, 1/3]$, $[1/3, 1/\sqrt{5}]$, $[1/\sqrt{5}, (\sqrt{6}+1)/5]$, and $[(\sqrt{6}+1)/5, 1]$.

Notice that $P_1(1) = P_2(1) = P_3(1) = 1$ and $P_1'(1) = 1 < P_2'(1) = 3 < P_3'(1) = 6$, so P_1 dominates in a neighborhood of 1. Therefore

$$P_1(x) = |P_1(x)| \geq |P_2(x)| \vee |P_3(x)| \qquad \text{for } x \in [1/\sqrt{5}, 1], \tag{6.18}$$

since $1/\sqrt{5}$ is the largest intersection point less that 1 which involves P_1.

Next note that $P_1(0) = P_3(0) = 0$, $P_2(0) = -1/2$, so $|P_2|$ dominates in some neighborhood of 0. In particular, $|P_2|$ must dominate $|P_1|$ and $|P_3|$ over the interval

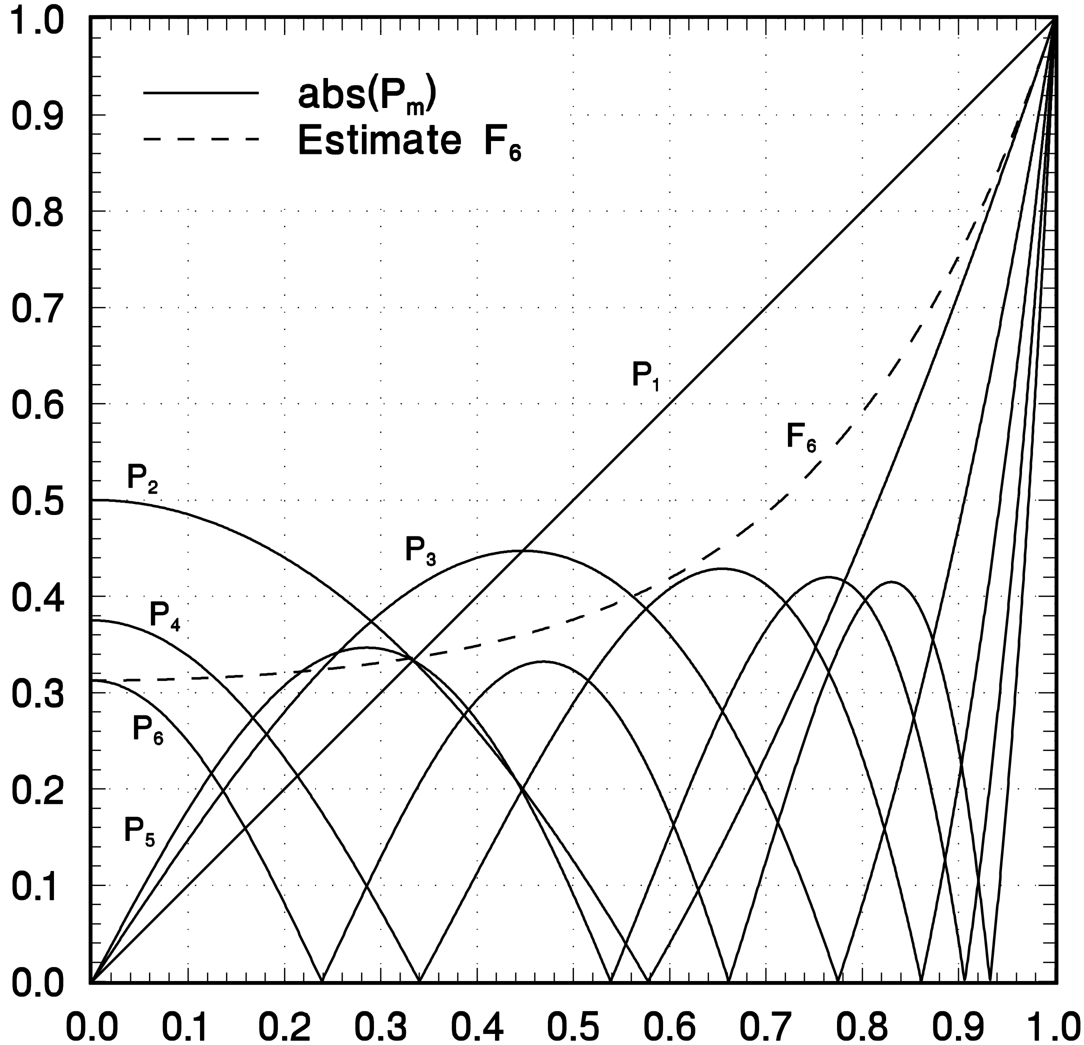

Figure 6.1: Comparison of Legendre polynomials $|P_m|$, m = 1–6 and estimate function F_6.

$[0, (\sqrt{6}-1)/5]$. Since $P_1(x) = x > 0$ if $x > 0$, it follows that $|P_2(x)| > 0$ for $0 \le x \le (\sqrt{6}-1)/5$, so $|P_2(x)| = -P_2(x)$ for $x \in [0, (\sqrt{6}-1)/5]$. This shows that

$$-P_2(x) = |P_2(x)| \ge |P_1(x)| \vee |P_3(x)| \qquad \text{for } x \in [0, (\sqrt{6}-1)/5]. \tag{6.19}$$

Also, $|P_1|$ and $|P_3|$ intersect only at $x = 0$, $1/\sqrt{5}$, and 1, so one dominates the other on the interval $[0, 1/\sqrt{5}]$. But $P_1(0) = P_3(0) = 0$ and $P_1'(0) = 1$, $P_3'(0) = -3/2$, so

$$-P_3(x) = |P_3(x) \ge |P_1(x)| \qquad \text{for } x \in [0, 1/\sqrt{5}].$$

Let us now determine $|P_2(x)| \vee |P_3(x)|$ on the interval $[(\sqrt{6}-1)/5, 1/\sqrt{5}]$. These two functions do not intersect on the interior of this interval, so one must dominate the other throughout. Moreover, as noted previously, $(\sqrt{6}-1)/5 < 1/3 < 1/\sqrt{5}$, and $|P_2(1/3)| = |P_1(1/3)| < -P_3(1/3)$. This shows that

$$-P_3(x) = |P_3(x)| \ge |P_1(x)| \vee |P_2(x)| \qquad \text{for } x \in [(\sqrt{6}-1)/5, 1/\sqrt{5}]. \tag{6.20}$$

Combining (6.18), (6.19), and (6.20) gives the result

$$\max_{i=1,2,3} (|P_i(x)|) = \begin{cases} -P_2(x) & \text{if } 0 \le x < \dfrac{\sqrt{6}-1}{5} \\ -P_3(x) & \text{if } \dfrac{\sqrt{6}-1}{5} \le x < \dfrac{1}{\sqrt{5}} \\ P_1(x) & \text{if } \dfrac{1}{\sqrt{5}} \le x \le 1 \end{cases} \tag{6.21}$$

Step 2: Next I show that $|P_4| \le P_1 \vee (-P_2)$. We have from (6.16) that $P_4(x) = (35x^4 - 30x^2 + 3)/8$, so define $g(z) = 35z^2 - 30z + 3$. Then for $z_0 > 0$, $g(z_0) = 0$ if and only if $P_4(\sqrt{z_0}) = 0$. The zeros for g are $z_0 = (15 \pm 2\sqrt{30}/35$, so set

$$\begin{aligned} r_1 &= \sqrt{(15 - 2\sqrt{30})/35} \\ r_2 &= \sqrt{(15 + 2\sqrt{30})/35}. \end{aligned}$$

Then $r_1 \approx 0.34$, $r_2 \approx 0.86$.

Now note that $P_4'(x) = (35x^3 - 15x)/2$, and $P_4''(x) = (105x^2 - 15)/2$. On the interval $[3/4, 1]$, $P_4''(x) \geq P_4''(3/4) = 705/32 > 0$, so $P_4'(x)$ is increasing on $[3/4, 1]$. This implies that $P_4'(x) \geq P_4'(3/4) = 225/128 > 1$. In particular, since $3/4 < r_2 < 1$, $P_1(1) = P_4(1) = 1$, and $P_1'(x) = 1 < P_4'(x)$ for $x \in [3/4, 1]$, it follows that

$$P_1(x) \geq P_4(x) = |P_4(x)| \qquad \text{for } x \in [r_2, 1]. \tag{6.22}$$

Next consider the interval $[r_1, r_2]$, on which $|P_4(x)| = -P_4(x)$. One can show that $-P_4(x)$ and $P_1(x)$ do not intersect on this interval. For example, consider $35x^4 - 30x^2 + 8x + 3 = 8(P_1(x) + P_4(x))$. One root of this polynomial is $x = -1$, and factoring out this root leaves

$$h(x) \equiv 35x^3 - 35x^2 + 5x + 3.$$

The minimum value of $h(x)$ on the interval $[r_1, r_2]$ occurs at $x_0 = (7 + 2\sqrt{7})/21$, and one can check that $h(x_0) > 0$. Thus $h(x)$ has no roots in $[r_1, r_2]$, which shows that $P_1(x) \neq -P_4(x)$ for all $x \in [r_1, r_2]$. Therefore,

$$P_1(x) \geq -P_4(x) = |P_4(x)| \qquad \text{for } x \in [r_1, r_2]. \tag{6.23}$$

It remains to show that $P_4(x0 = |P_4(x)| \leq |P_2(x)| = -P_2(x)$ for $x \in [0, r_1]$ (the roots for $P_2(x)$ are $\pm 1/\sqrt{3} \approx \pm 0.58$). But for $x \geq 0$ we have

$$P_4(x) = -P_2(x) \Longrightarrow x = \sqrt{(9 + 2\sqrt{29})/35} > 3/4 > r_1.$$

Moreover, $-P_2(0) = 1/2 > 3/8 = P_4(0)$, so

$$-P_2(x) \geq P_4(x) = |P_4(x)| \qquad \text{for } x \in [0, r_1]. \tag{6.24}$$

Combining (6.22), (6.23), and (6.24) yields

$$|P_4(x)| \leq P_1(x) \vee (-P_2(x)) \text{ for } x \in [0, 1]. \tag{6.25}$$

Step 3: Here I show that $|P_5| < |P_1| \vee |P_2| \vee |P_3|$. It is easy to show

$$|P_5(x)| = |P_1(x)| \Longrightarrow x \in \{0, \pm\tfrac{1}{3}, \pm 1\}.$$

Since $P_5(1) = 15 > 1$, and $P_5(1) = P_1(1) = 1$, it follows that $P_1(x) > P_5(x)$ for some interval to the left of 1. Since the only intersections between $|P_1(x)|$ and $|P_5(x)|$ in $[0, 1]$ are at $x = 0$, $1/3$, and 1, it follows that

$$|P_5(x)| \leq P_1(x) \qquad \text{for } x \in [1/3, 1]. \tag{6.26}$$

On the interval $(0, 1/3]$, $|P_5(x)| \geq P_1(x) > 0$, so $|P_5(x)|$ has no roots in $(0, 1/3]$, which implies that $|P_5(x)| = P_5(x)$ for $x \in \{0, 1/3\}$. The maximum value of $P_5(x)$ in this interval occurs at $x_0 = \sqrt{(7 - 2\sqrt{7})/21} \approx 0.285$, and $P_5(x_0) < 1/6$. One can also easily check that $|P_2(x)|$ is decreasing on the interval $[0, (\sqrt{6} - 1)/5]$ and

$|P_3(x)|$ is increasing on the interval $[(\sqrt{6}-1)/5, 1/\sqrt{5}]$, so

$$-P_2\left(\frac{\sqrt{6}-1}{5}\right) = P_3\left(\frac{\sqrt{6}-1}{5}\right) \leq |P_2(x)| \vee |P_3(x)| \qquad \text{for } x \in [0, 1/\sqrt{5}]. \quad (6.27)$$

In particular, since $1/3 < 1/\sqrt{5}$, and $-P_2\left((\sqrt{6}-1)/5)\right) > 4/11 > 1/6$, we have

$$|P_5(x)| \leq |P_2(x)| \vee |P_3(x)| \qquad \text{for } x \in [0, 1/3], \tag{6.28}$$

which combines with (6.26) to prove

$$|P_5(x)| \leq |P_1(x)| \vee |P_2(x)| \vee |P_3(x)| \qquad \text{for } x \in [0, 1]. \tag{6.29}$$

Step 4: From Appendix A, formula (A.11), we have (using $\alpha = 1/2$)

$$P_m(\cos\theta) = \frac{1}{\pi}\int_0^\pi (\cos\theta + i\sin\theta\cos\phi)^m \, d\phi,$$

so define

$$F_m(\cos\theta) \equiv \frac{1}{\pi}\int_0^\pi \left((\cos\theta)^2 + (\sin\theta\cos\phi)^2\right)^{m/2} \leq |P_m(\cos\theta)|\,. \tag{6.30}$$

Note that F_m is decreasing as a function of m, so in fact

$$|P_m(x)| \leq F_6(x) \qquad \text{for } m \geq 6,\, x \in [0, 1]. \tag{6.31}$$

We want to show that $F_6 \leq |P_1| \vee |P_2| \vee |P_3|$. To do this, one can determine $F_6(x)$ explicitly by evaluating the integral in (6.30), which gives

$$F_6(\cos\theta) = \frac{1}{\pi}\sum_{k=0}^{3}\binom{3}{k} B(\tfrac{1}{2}, k+\tfrac{1}{2})(\cos\theta)^{6-2k}(\sin\theta)^{2k}$$

where $B(\cdot\,,\,\cdot)$ is the Beta function. Replacing $\cos\theta$ with x and simplifying leads to

$$F_6(x) = (5x^6 + 3x^4 + 3x^2 + 5)/16. \tag{6.32}$$

Since $F_6''(x) = 3(25x^4 + 6x^2 + 1)/8 > 0$, $-F_6(x)$ is convex, so $F_6(x)$ intersects $P_1(x) = x$ at no more than 2 points. Since $F_6'(1) = 3 > 1 = P_1'(1)$, it follows that $P_1(x) \geq F_6(x)$ for some interval to the left of 1. This, coupled with the fact that $F_6(1/\sqrt{5}) = 9/25 < 1/\sqrt{5} = P_1(1/\sqrt{5})$ show that

$$F_6(x) \leq P_1(x) \qquad \text{for } x \in [1/\sqrt{5}, 1]. \tag{6.33}$$

Furthermore, $F_6(x)$ is increasing as a function of x for $x > 0$, so

$$F_6(x) \leq F_6(1/\sqrt{5}) = 9/25 \qquad \text{for } x \in [0, 1/\sqrt{5}]. \tag{6.34}$$

Recall now (6.27), which describes the minimum value of $|P_2(x)| \vee |P_3(x)|$ on the interval $x \in [0, 1/\sqrt{5}]$. In particular,

$$|P_2(x)| \vee |P_3(x)| > \frac{4}{11} > \frac{9}{25} \qquad \text{for } x \in [0, 1/\sqrt{5}],$$

which combined with (6.34), (6.33), and (6.31) gives

$$|P_m(x)| \leq F_6(x) \leq |P_1(x)| \vee |P_2(x)| \vee |P_3(x)| \qquad \text{for } x \in [0, 1],\ m \geq 6. \tag{6.35}$$

And so finally, combining (6.21), (6.25), (6.29), and (6.35) completes the proof of Theorem 10.

□

Appendix A

Useful formulae

In this appendix I collect several formulae that are used in the preceding work. These formulae can be found in standard reference works (e.g., [6] or [12]).

Let $\Gamma(z)$ be the Gamma function, $B(w, z)$ the Beta function. The domain of $\Gamma(z)$ is $\mathbf{C} \setminus \{0, -1, -2, \ldots\}$. $B(w, z)$ is given (for w, z, $w + z$ in the domain of Γ) by

$$B(w, z) = B(z, w) = \Gamma(w)\Gamma(z)/\Gamma(w + z) \tag{A.1}$$

and also by

$$B(\frac{w+1}{2}, \frac{z+1}{2}) = 2\int_0^{\pi/2} (\sin t)^w (\cos t)^z \, dt \qquad (\Re w > -1,\ \Re z > -1), \tag{A.2}$$

or similarly by

$$B(\frac{w+1}{2}, \frac{z+2}{2}) = 2\int_0^1 y^w (1 - |y|^2)^{z/2} \, dy \qquad (\Re w > -1,\ \Re z > -2). \tag{A.3}$$

From this one can calculate the volume of the unit ball in $\mathbf{R}^N$, namely

$$V_N = \frac{\pi^{N/2}}{\Gamma(\frac{N}{2} + 1)} \tag{A.4}$$

Also useful is the duplication formula for the Gamma function:

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}. \tag{A.5}$$

Let us turn now to the Gegenbauer (Ultraspherical) polynomials, $C_m^{(\alpha)}(x)$. These are polynomials of degree m in x which satisfy the orthogonality condition

$$\int_{-1}^{1} C_k^{(\alpha)}(x)C_m^{(\alpha)}(x)(1-x^2)^{\alpha-\frac{1}{2}}\,dx = 0 \qquad (k \neq m,\ \alpha > -\tfrac{1}{2}). \tag{A.6}$$

These polynomials are standardized so that

$$\int_{-1}^{1} \left(C_m^{(\alpha)}(x)\right)^2 (1-x^2)^{\alpha-\frac{1}{2}}\,dx = \frac{\pi\Gamma(m+2\alpha)}{2^{2\alpha-1}m!(m+\alpha)\left(\Gamma(\alpha)\right)^2} \qquad (\alpha \neq 0). \tag{A.7}$$

Special cases of the Gegenbauer polynomial are the Chebyshev polynomials and the Legendre polynomials:

$$\text{Chebyshev polynomial of the first kind} \quad T_m(x) = mC_m^{(0)}(x)/2 \tag{A.8}$$

$$\text{Chebyshev polynomial of the second kind} \quad U_m(x) = C_m^{(1)}(x) \tag{A.9}$$

$$\text{Legendre polynomial} \quad P_m(x) = C_m^{(1/2)}(x). \tag{A.10}$$

Gegenbauer polynomials also have the integral representation

$$C_m^{(\alpha)}(\cos\theta) = \frac{\Gamma(m+2\alpha)}{2^{2\alpha-1}m!\left(\Gamma(\alpha)\right)^2}\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^m(\sin\phi)^{2\alpha-1}\,d\phi \qquad (\alpha > 0). \tag{A.11}$$

Expanding the term $(\cos\theta + i\sin\theta\cos\phi)^m$ and evaluating the resulting integrals (via (A.2)) gives the representation

$$C_m^{(\alpha)}(\cos\theta) = \frac{\Gamma(m+2\alpha)}{2^{2\alpha-1}m!\left(\Gamma(\alpha)\right)^2}\sum_{k=0}^{[m/2]}(-1)^k\binom{m}{2k}B(k+1/2,\alpha)(\cos\theta)^{m-2k}(\sin\theta)^{2k} \tag{A.12}$$

for $\alpha > 0$, where $[\,\cdot\,]$ denotes the greatest integer and $B(\cdot,\cdot)$ is the Beta function.

In particular, (A.12) shows that

$$\begin{aligned}
C_m^{(\alpha)}(1) &= \frac{\Gamma(m+2\alpha)B(\alpha,\frac{1}{2})}{2^{2\alpha-1}m!\,(\Gamma(\alpha))^2} \\
&= \frac{\Gamma(m+2\alpha)\sqrt{\pi}}{2^{2\alpha-1}\Gamma(m+1)\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})} \\
&= \frac{\Gamma(m+2\alpha)}{\Gamma(m+1)\Gamma(2\alpha)} \qquad \text{(by (A.5))} \\
&\stackrel{\text{def}}{=} \binom{m+2\alpha-1}{m} \qquad (\alpha > 0).
\end{aligned} \tag{A.13}$$

Also,

$$C_m^{(\alpha)}(0) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ (-1)^{m/2}\dfrac{\Gamma(\alpha+m/2)}{\Gamma(\alpha)\Gamma(1+m/2)} & \text{if } m \text{ is even,} \end{cases} \tag{A.14}$$

where the second line is achieved with two applications of the Gamma function duplication formula (A.5). Furthermore, notice that the second line can also be written

$$C_m^{(\alpha)}(0) = (-1)^{m/2}\binom{\alpha-1+m/2}{m/2} \qquad (m \text{ even}, \alpha > 0),$$

which should be compared to (A.13).

The equations A.12–A.14 do not hold for $\alpha = 0$, but do extend to other values of α (for example, to $-\frac{1}{2} < \alpha < 0$).

One can also use (A.12) to calculate $C_m^{(\alpha)}(\cos\theta)$ for small m:

$$C_0^{(\alpha)}(\cos\theta) = 1 \tag{A.15}$$

$$C_1^{(\alpha)}(\cos\theta) = 2\alpha\cos\theta \tag{A.16}$$

$$C_2^{(\alpha)}(\cos\theta) = \alpha[2(\alpha+1)(\cos\theta)^2 - 1]. \tag{A.17}$$

These polynomials can also be calculated via the better known relation

$$C_m^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[m/2]} (-1)^k \frac{\Gamma(\alpha + m - k}{k!(m-2k)!} (2x)^{m-2k} \qquad (\alpha > -\tfrac{1}{2},\ \alpha \neq 0), \quad \text{(A.18)}$$

which can be found in [6]. Let us use this relation to calculate $C_3^{(\alpha)}$:

$$\begin{aligned} C_3^{(\alpha)}(x) &= \frac{1}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha+3)}{6} 8x^3 - 2\Gamma(\alpha+2)x \right) \\ &= \frac{4(\alpha+2)(\alpha+1)\alpha}{3} x^3 - 2(\alpha+1)\alpha x. \end{aligned}$$

Thus

$$C_3^{(\alpha)}(\cos\theta) = \frac{2}{3}(\alpha+1)\alpha[2(\alpha+2)(\cos\theta)^3 - 3\cos\theta]. \qquad \text{(A.19)}$$

Appendix B

Supplemental results

In this appendix I present a proof of a known, "obvious" result which is not accessible in the literature. This result is needed in several places in the preceding work.

Theorem 11 *Suppose $f \in L^2([-1,1]^N)$ and suppose that there exists f_1 and f_2 with $f_1(z) = f_2(z) = f(z)$ a.e. such that*

$$\begin{aligned} f_1(z_1, z_2, \ldots, z_N) &= f_1(z_1, z_2, \ldots, z_k, 0, 0, \ldots, 0) \quad \forall z \in [-1,1]^N \\ f_2(z_1, z_2, \ldots, z_N) &= f_2(0, \ldots, 0, z_{m_1}, \ldots, z_{m_2}, 0, \ldots, 0) \quad \forall z \in [-1,1]^N. \end{aligned}$$

Then there exists f_3 with $f_3(z) = f(z)$ a.e. such that

$$f_3(z_1, z_2, \ldots, z_N) = f_3(0, \ldots, 0, z_{m_1}, \ldots, z_{\min(k,m_2)}, 0, \ldots, 0) \quad \forall z \in [-1,1]^N.$$

Proof: Let α be a multi-index of length N, i.e., $\alpha \in \mathbf{Z}_+^N$. The set $\{e^{-\pi i \alpha \cdot z}\}$ ($\alpha \cdot z \stackrel{\text{def}}{=} \alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_N z_N$) is a basis for $L^2([-1,1]^N)$. Consider

$$\left\langle f, e^{-\pi i \alpha \cdot z} \right\rangle = \int_{-1}^{1} \cdots \int_{-1}^{1} f(z) e^{-\pi i \alpha \cdot z} \, dz_1 \ldots dz_N$$

$$
\begin{aligned}
&= \int_{-1}^{1}\cdots\int_{-1}^{1} f_1(z_1,\dots,z_k,0,\dots,0)e^{-\pi i(\alpha_1 z_1+\cdots+\alpha_k z_k)}\,dz_1\dots dz_k \\
&\quad \times \int_{-1}^{1}\cdots\int_{-1}^{1} e^{-\pi i(\alpha_{k+1}z_{k+1}+\cdots+\alpha_N z_N)}\,dz_{k+1}\dots dz_N \\
&= 0 \quad \text{unless } \alpha_{k+1}=\cdots=\alpha_N=0.
\end{aligned}
$$

This shows that

$$f \in \text{lin span}\{e^{-\pi i\alpha\cdot z} \mid \alpha_{k+1}=\cdots=\alpha_N=0\}^{\text{cl}}.$$

If the representation f_2 is used instead of f_1 in the above argument then one obtains

$$f \in \text{lin span}\{e^{-\pi i\alpha\cdot z} \mid \alpha_1=\cdots\alpha_{m_1-1}=\alpha_{m_2+1}=\cdots=\alpha_N=0\}^{\text{cl}}.$$

It follows that f lies in the intersection of these two spaces, i.e.,

$$f \in \text{lin span}\{e^{-\pi i\alpha\cdot z} \mid \alpha_1=\cdots\alpha_{m_1-1}=\alpha_{1+\min(k,m_2)}=\cdots=\alpha_N=0\}^{\text{cl}}.$$

Therefore, there exists f_3 with $f_3(z)=f(z)$ a.e. such that

$$f_3(z_1,z_2,\dots,z_N)=f_3(0,\dots,0,z_{m_1},\dots,z_{\min(k,m_2)},0,\dots,0) \quad \forall z\in[-1,1]^N,$$

as desired. □

Corollary 2 *Let P_i, $i=1,2,3$ be the projections on $[-1,1]^N$ defined by*

$$
\begin{aligned}
P_1(z_1,z_2,\dots,z_N) &= (z_1,z_2,\dots,z_k,0,\dots,0) \\
P_2(z_1,z_2,\dots,z_N) &= (0,\dots,0,z_{m_1},\dots,z_{m_2},0,\dots,0) \\
P_3(z_1,z_2,\dots,z_N) &= (0,\dots,0,z_{m_1},\dots,z_{\min(k,m_2)},0,\dots,0).
\end{aligned}
$$

(Note that $P_3 = P_1 \circ P_2$, and $P_3 = 0$ if $m_1 > k$.) Let X be a topological space, and let μ be a Borel measure (or the completion of one) on X. Let T be an invertible mapping from X to $[-1,1]^N$ (with the usual Lebesgue measure m) such that both T and T^{-1} are measurable and map sets of measure zero to sets of measure zero. Suppose that $g \in L^2(X,\mu)$, $g \circ T^{-1} \in L^2([-1,1]^N)$, and there exist $g_1(x) = g_2(x) = g(x)$ a.e. $[\mu]$ such that

$$g_i \circ T^{-1}(z) = g_i \circ T^{-1} \circ P_i(z) \quad \text{for all } z \in [-1,1]^N,\ i = 1,2. \tag{B.1}$$

Then there exists $g_3 \in L^2(X,\mu)$ with $g_3(x) = g(x)$ a.e. $[\mu]$ such that

$$g_3 \circ T^{-1} \circ P_3 \circ T(x) = g_3(x) \quad \text{for all } x \in X. \tag{B.2}$$

Proof: We can assume that g is a Borel function, since otherwise we can change g on a set of measure zero to make it one. Then $g \circ T^{-1}$ and $g_i \circ T^{-1}$, $i = 1,2$, are measurable functions on $[-1,1]^N$. Moreover, since T maps sets of measure zero to sets of measure zero, it follows that $g \circ T^{-1}(z) = g_i \circ T^{-1}(z)$ a.e. $[m]$, $i = 1,2$.

Let $f = g \circ T^{-1}$ and $f_i = g_i \circ T^{-1}$ for $i = 1,2$. Apply Theorem 11 to show the existence of f_3 with $f_3(z) = f(z)$ a.e. $[m]$ such that $f_3 \circ P_3(z) = f_3(z)$ for all $z \in [-1,1]^N$. Let $g_3 = f_3 \circ T$. Since T^{-1} also maps sets of measure zero to sets of measure zero, it follows that $g_3(x) = g(x)$ a.e. $[\mu]$. Moreover, for all $x \in X$,

$$\begin{aligned} g_3 \circ T^{-1} \circ P_3 \circ T(x) &= f_3 \circ P_3 \circ T(x) \\ &= f_3 \circ T(x) \\ &= g_3(x). \end{aligned}$$

$\square$

Bibliography

[1] P. Funk, “Über eine geometrische anwendung der abelschen integralgleichung,” *Math. Ann.*, **77**, 129–135 (1916).

[2] J. Radon, “Über die bestingmmung von funktionen durch ihre integralwerte längs gewisser mannigfaltigkeiten,” *Ber. Verh. Sächs. Akad. Wiss. Leipsiz, Math-Nat. kl.*, **69**, 262–277 (1917).

[3] F. Natterer, *The Mathematics of Computerized Tomography.* Wiley-Teubner, 1986.

[4] C. Hamaker and D. C. Solmon, “The angle between the null spaces of X rays,” *Journal of Mathematical Analysis and Applications*, **62**, 1–23 (1978).

[5] M. E. Davison and F. A. Grunbaum, “Tomographic reconstruction with arbitrary directions,” *Communications on Pure and Applied Mathematics*, **34**, 77–120 (1981).

[6] M. Abramowitz, I. Stegun, *et al.*, *Handbook of Mathematical Functions.* Dover, 1970.

[7] C. Davis and W. M. Kahan, “Some new bounds on perturbation of subspaces,” *Bulletin of the American Mathematical Society*, **75**, 863–868 (1969).

[8] W. Feller, *An Introduction to Probability Theory and Its Applications.* Vol. 1, John Wiley and Sons, Inc., third ed., 1968.

[9] J. B. Conway, *A Course in Functional Analysis.* Springer-Verlag, 1985.

[10] J. Dixmier, “Étude sur les variétés et les opérateurs de julia, avec quelques applications,” *Bulletin de la Société Mathématique de France*, **77**, 11–101 (1949).

[11] S. Helgason, *The Radon Transform.* Birkhäuser, 1980.

[12] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics.* Chelsea, 1949.

[13] M. I. Yadrenko, *Spectral Theory of Random Fields.* Optimization Software, 1983.