An Introduction to Quantum Information

by Carl J. Williams

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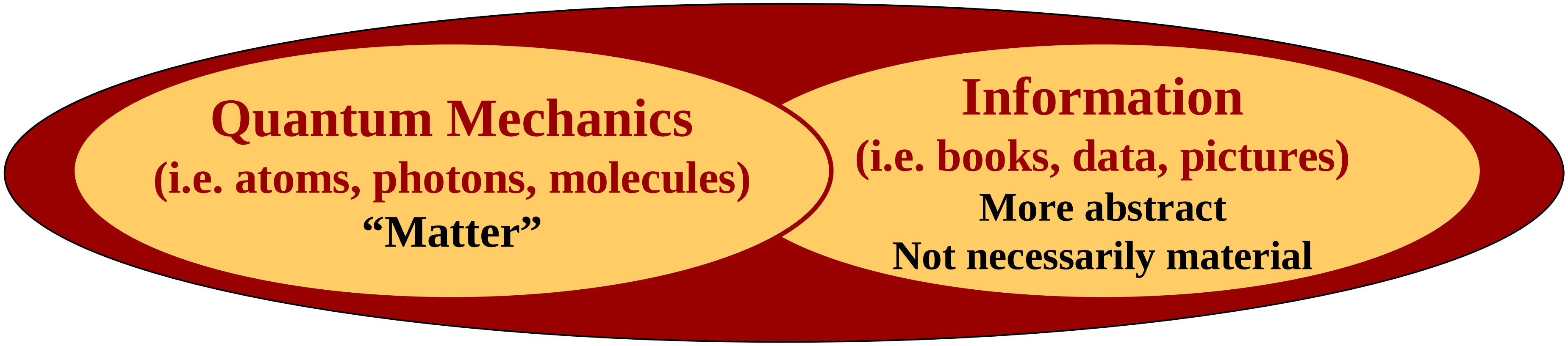
I. What is Quantum Information?

A radical departure in information technology, more fundamentally different from current IT than the digital computer is from the abacus.

A convergence of two of the 20th Century's great revolutions



Quantum Mechanics
(i.e. atoms, photons, molecules)
"Matter"

Information
(i.e. books, data, pictures)
More abstract
Not necessarily material

- **Ø A quantum computer if it existed could break all present-day public key encryption systems**
- **Ø Quantum encryption can defeat any computational attack**

Quantum Information may be Inevitable

The limits of miniturization:

At atomic scale sizes quantum mechanics rules

- **Since objects and electronic components continue to be miniaturized, inevitably we will reach feature sizes that are *atomic* in scale**
- **In general, attempts to make *atomic-size* circuits behave classically will fail due to their inability to dissipate heat and their quantum character**

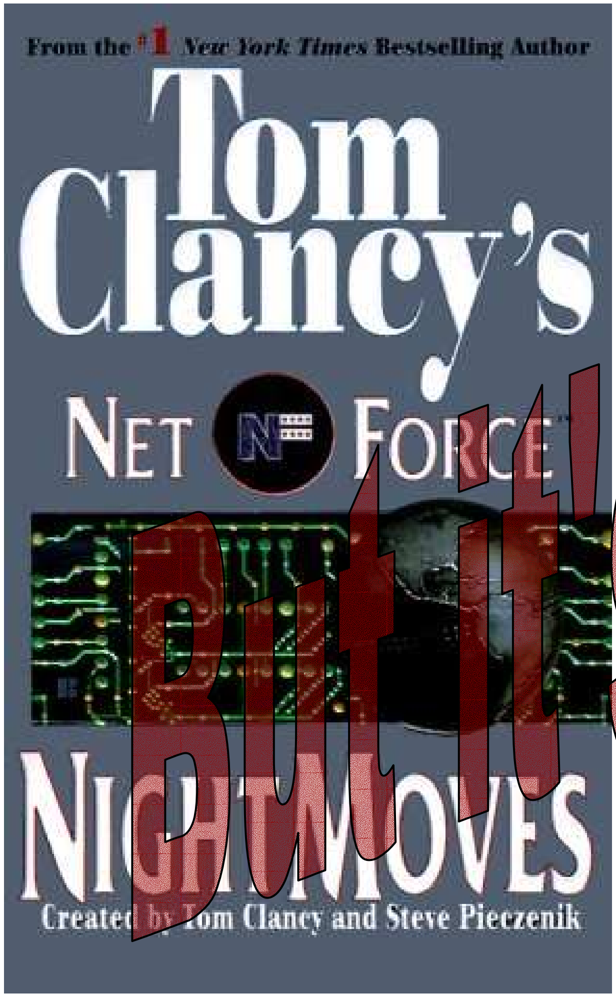
Thus quantum information may be inevitable!

Ø **Clearly, at the smallest scale, we need to take full advantage of quantum properties.**

Ø **This *emphasizes a different* view of why quantum information is useful and also show why it *may ultimately lead* to quantum engineering.**

Belief: Quantum Information and Quantum Engineering will have a *tremendous economic impact* in the 21st Century

II. Introduction


From the #1 New York Times Bestselling Author
Tom Clancy's
Net Force
NF
Night Moves
Created by Tom Clancy and Steve Pieczenik

"Using Shor's quantum factorization algorithm, one can see that factoring a large number can be done by a QC – quantum computer – in a very small fraction of the time the same number would take using ordinary hardware. A problem that a SuperCray might labor over for a few million years can be done in seconds by my QC. So for a practical matter like code breaking, the QC is vastly superior."

…

"**Wineland** and **Monroe** worked out the single quantum gate by trapping beryllium ions. …"

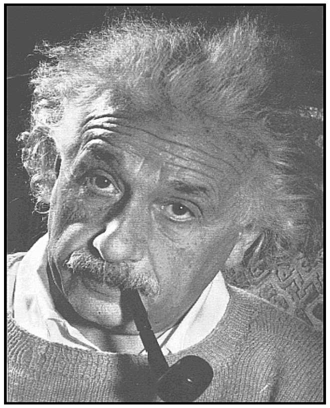
But it's not fiction

20th Century in Review

At the beginning of the 20th century a series of crises had taken place in physics – the *old* physics (now called classical physics) predicted numerous absurdities. At first *ad hoc* fixes were made to the *classical* theory – but the theory became untenable.

In the 1920's this crises gave way to a quantum mechanics – a new theory appropriate at the smallest scales (atomic, nuclear). Quantum mechanics reduces to classical physics under the appropriate conditions while removing the absurdities.

- Foundations of Quantum Mechanics
  - Planck: Planck's Constant
  - Einstein: Photoelectric Effect, Light Quanta, Special Relativity, $E=mc^2$, General Relativity
  - deBroglie: Wave-Particle Duality
  - Heisenberg: Uncertainty Principle, Matrix Mechanics
  - Schrödinger: Wave Equation



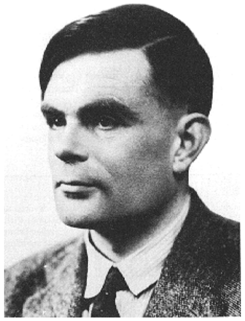
Note – that Einstein, one of the fathers of quantum mechanics, died believing that quantum mechanics was incomplete.

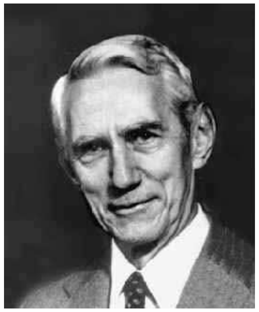
20th Century in Review (2)

Modern information theory originates in the 1930’s with the concept of a Turing machine capable of running a program or algorithm. The Church-Turing hypothesis then asserts that there exists an equivalent algorithm of similar complexity that can run on a Universal Turing Machine.

The discovery of the transistor in 1947, followed by integrated electronics, leads to the computer revolution and Moore’s law.

In the late 1940’s, Shannon defines the concept of a unit of information, which is given physical limitations by Landauer.

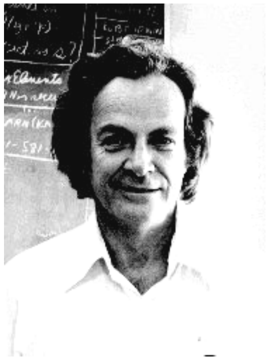




- Foundations of Information Theory
  - Church-Turing: Computability, Universality
  - von Neumann: Concept of a computer
  - Bardeen, Brattain, & Shockley: Transistor
  - Shannon: Information Measures
  - Landauer: Physical Limitations of Information; explanation for Maxwell’s Demon
  - Bennett: Reversible Turing Machine

History of Quantum Information

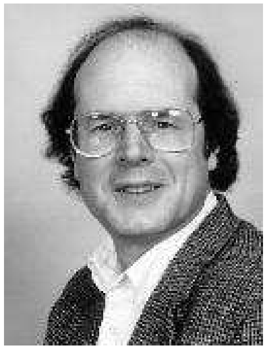
- Foundations
  - Benioff: Quantum Turing Machine
  - Feynman, Deutsch: Concept of Quantum computation
  - Landauer, Zurek: Physics of information
  - Bennett, DiVincenzo, Ekert , Lloyd: Concept of Quantum information science
  - Shor: Q. Factoring and discrete log algorithm
  - Preskill, Shor, Gottesman, Steane: Quantum error correction, Fault tolerant QC
  - Lloyd: Quantum simulators and Universal QC



Richard Feynman

Peter Shor

- From Theory to Experiment
  - Bennett, Gisin, Hughes: Demonstration of quantum cryptography
  - Wineland and Kimble: Demonstration of Qubits and quantum logic



Charles Bennett

How can we use Quantum Information?

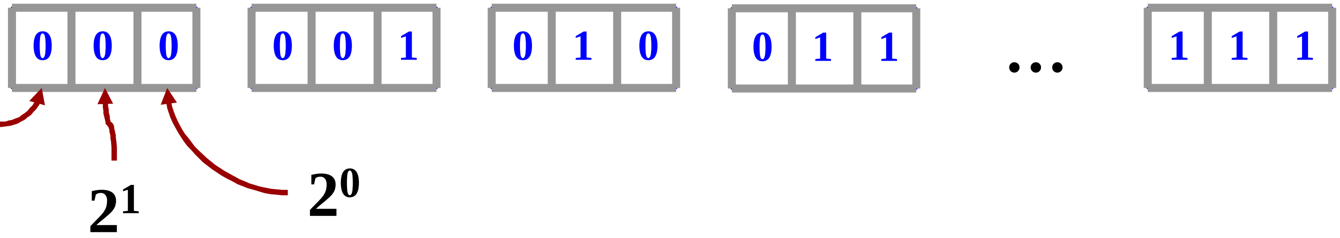
- **Quantum Communication - 100% physically secure**
  - **Quantum key distribution – generation of classical key material**
  - **Quantum Teleportation**
  - **Quantum Dense Coding**
- **Universal Quantum Logic: *all* quantum computations – *i.e. any arbitrary* unitary operations – may be efficiently constructed from 1- and 2-qubit gates**
- **Quantum Algorithms**
  - **Factorization of large primes (Shor's algorithm)**
  - **Searching large databases (Grover's algorithm)**
  - **Quantum Fourier Transforms**
  - **Potential attack of NP problems**
  - **Simulation of large-scale quantum systems**
- **Quantum Measurement – improved accuracy**
  - **Heisenberg limit $\propto$1/N vs Shot-Noise limit $\propto$1/Sqrt(N)**
  - **Better Atomic Clocks**
- **Quantum Engineering – specialized quantum devices**

Scaling of Quantum Information

- Classically, information stored in a bit register: a 3-bit register stores one number, from 0 – 7.

0	1	0

e.g.



0	0	0	0	0	1	0	1	0	0	1	1	…	1	1	1

- Quantum mechanically, a 3-qubit register can store all of these numbers in an arbitrary superposition:

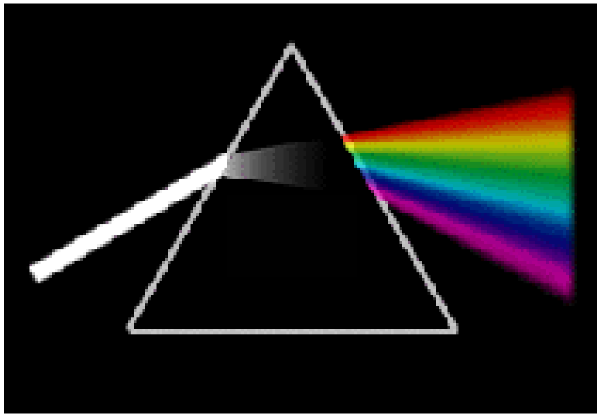
$$a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f\,|101\rangle + g|110\rangle + h|111\rangle$$

$|\cdots\rangle$ à Dirac Notation for the quantum state vector

- **Result:**
  - Classical: one N-bit number
  - Quantum: 2^N (all possible) N-bit numbers

III. The Quantum Primer

- Schrödinger’s Equation and Dirac Notation
- Light as Waves and Photons
- Quantum Nature of Matter: Atoms
- Superposition
- Quantum Measurement
- Quantum Interference
- Entanglement



Quantum Theory Summary

Quantum theory is the branch of physics that describes waves and particles at the smallest scale and lowest energies. This theory is based on the observation that changes in the energy of atoms and molecules occurs in discrete quantities known as quanta. This includes the electromagnetic field which consists of individual quanta of various frequencies known as photons.

The classical or Newtonian limit (which describes everyday phenomena) is typically recovered when a complex quantum system consisting of many parts becomes massive and/or its energy becomes large (many quanta).

Non-relativistic quantum mechanics gives rise to Schrödinger's wave equation. The key components of this equation, which in turn *fully describes the system*, are the Hamiltonian H that governs the interactions of the quantum system and the wavefunction$\Psi(r,t)$ that describes the state or wavefunction of the system. The latter is often denoted by the ket $\left|\Psi(t)\right\rangle$.

Schrödinger Equation

Schrödinger's wave equation is a first order differential equation that describes the time evolution of a quantum system under a Hamiltonian *H*. The Hamiltonian *H* is the operator equivalent of the total energy of the system which can be represented as the sum of the kinetic and potential energies of the system.

$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} = H(\vec{r},t)\,\Psi(\vec{r},t)$$

$\hbar$ is Planck's constant

Probability of being at position *r* at time *t*

$$\rho(\vec{r},t) = \Psi^*(\vec{r},t)\,\Psi(\vec{r},t)$$

Total integrated probability at time *t*

$$\langle \Psi(t) | \Psi(t) \rangle = \int \Psi^*(\vec{r},t)\,\Psi(\vec{r},t)\; d\vec{r} = \int \rho(\vec{r},t)\; d\vec{r}$$

Note: In general one does not put arguments inside of bras $\left| label \right\rangle$.

Schrödinger Equation (2)

The Hamiltonian *H* for the system can typically be written as

$$H\left(\vec{r},t\right)=-\frac{\hbar^2}{2m}\nabla^2+V\left(\vec{r},t\right)$$

where *m* is the mass, $V\left(\vec{r},t\right)$ is the potential, and the ∇^2 in the kinetic energy term. Basically *H* describes the quantum systems interactions.

If the potential *V* is time independent with the result that *H* is time independent, one obtains the time independent Schrödinger equation. This is a second-order partial differential equation sometimes referred to as an eigenvalue equation:

$$H\left|\Psi_E\right\rangle=\mathrm{E}\left|\Psi_E\right\rangle$$

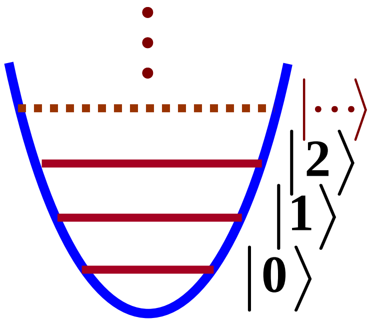
In general one does not need to know about transistors to understand classical computers. Similarly one does not need to know about *H* to understand quantum computers.

Example: Schrödinger's Equation

For a time independent problem, Schrödinger equation's can be written:

$$H\left|\Psi_n\right\rangle = \mathrm{E_n}\left|\Psi_n\right\rangle$$

For the special case of a 1-dimensional harmonic oscillator, the Hamiltonian is given by:



Harmonic Oscillator

$$H(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

$$E_n = (n + 1/2)\hbar\omega$$

$$\left\langle \xi \middle| \Psi_n \right\rangle = \exp\left(-\xi^2/2\right) H_n(\xi) \quad \text{where} \quad \xi = \sqrt{m\omega/\hbar}\, x$$

where $H_n(\xi)$ is a Hermite polynomial and Ψ_n satisfies:

$$\left\langle \Psi_n \middle| \Psi_k \right\rangle = \int_{-\infty}^{+\infty} \exp\left(-\xi^2\right) H_n(\xi) H_k(\xi)\, d\xi = 2^n n! \sqrt{\pi}\, \delta_{nk}$$

Normalized Wavefunctions

Convention in quantum mechanics is to use normalized wavefunctions since the total integrated density of a quantum system should be 1 – *i.e.*

$$\left\|\Psi(t)\right\| = \left\langle \Psi(t) \middle| \Psi(t) \right\rangle^{1/2} = 1$$

Thus in the example from the previous page, a normalized Ψ_n can be written as:

$$\left\langle \xi \middle| \Psi_n \right\rangle = \frac{2^{-n/2}}{\sqrt{n!\sqrt[4]{\pi}}} \exp\left(-\xi^2/2\right) H_n(\xi)$$

So that $\left\langle \Psi_n \middle| \Psi_k \right\rangle = \delta_{nk}$

Moreover for any quantum system, the state kets $|\alpha\rangle$ and $|\beta\rangle$ represent the same quantum state if they differ only by a non-zero multiplicative constant

$\lambda \in \square$ $\qquad$ $|\alpha\rangle = \lambda|\beta\rangle$

Dirac Notation

The elements, wavefunctions, eigenfunctions, or state vectors that are the solution of Schrödinger's equation form an *orthonormal* set. These state vectors are called ket vectors and are individually denoted as $\left|label_i\right\rangle$ or $\left|i\right\rangle$. The set of all such vectors $\left\{\left|i\right\rangle\right\}$ *span* an abstract vector space referred to mathematically as the Hilbert Space H.

A Hilbert Space H is very much like ordinary cartesian space (*x*,*y*,*z*). The square-of-the-length *l* of a vector from the origin *O* to an arbitrary point *i* given by the point (x_i,y_i,z_i) is:

$$l^2 = x_i^2 + y_i^2 + z_i^2 = \begin{pmatrix} x_i & y_i & z_i \end{pmatrix}\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

In Dirac notation and quantum mechanics one would label the state $\left|i\right\rangle$ and the length-squared or inner product would be denoted: $l^2 = \left\langle i \middle| i \right\rangle$ or $l = \left\langle i \middle| i \right\rangle^{1/2}$

Dirac Notation (2)

In normal cartesian space the unit vectors

$$\hat{x} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},\ \hat{y} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ \text{and}\ \hat{z} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

form an orthonormal set that spans the space.

Orthonormal because:

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

and

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

or $\langle \hat{x} | \hat{x} \rangle = \langle \hat{y} | \hat{y} \rangle = \langle \hat{z} | \hat{z} \rangle = 1$ and $\langle \hat{x} | \hat{y} \rangle = \langle \hat{y} | \hat{x} \rangle = \langle \hat{x} | \hat{z} \rangle = \langle \hat{y} | \hat{z} \rangle = 0$

Spans the space because an arbitrary vector $|u\rangle$ can be written:

$|u\rangle = a|\hat{x}\rangle + b|\hat{y}\rangle + c|\hat{z}\rangle$ and in normalized form $|\hat{u}\rangle$ as:

$$|\hat{u}\rangle = \frac{a|\hat{x}\rangle + b|\hat{y}\rangle + c|\hat{z}\rangle}{\sqrt{a^2 + b^2 + c^2}}$$

Quantum Mechanics for Mathematicans

The wavefunctions (previously denoted $|\Psi\rangle$) and quantum bits or qubits that arise from quantum mechanics live in a Hilbert space H (which may be finite and in the specific case of a single qubit: 2-dimensional). A Hilbert space H is a vector space over the complex numbers ÷ with a complex valued inner product. A complex valued inner product is a map: $(-,-):$ H × Hfi ÷ from H × H into the complex numbers ÷ such that:

Mathematics	Quantum Mechanics
$1)\ (u,u)=0 \text{ iff } u=0$	$1)\ \langle u\|u\rangle=0 \text{ iff } u=0$
$2)\ (u,v)=(v,u)^*$	$2)\ \langle u\|v\rangle=\langle v\|u\rangle^*$
$3)\ (u,v+w)=(u,v)+(u,w)$	$3)\ \langle u\|v+w\rangle=\langle u\|v\rangle+\langle u\|w\rangle$
$4)\ (u,\lambda v)=\lambda(u,v)$	$4)\ \langle u\|\lambda v\rangle=\lambda\langle u\|v\rangle$
$4')\ (\lambda u,v)=\lambda^*(u,v)$	$4')\ \langle\lambda u\|v\rangle=\lambda^*\langle u\|v\rangle$

* – denotes complex conjugation

Quantum Mechanics for Mathematicans

The wavefunctions (previously denoted $|\Psi\rangle$) and quantum bits or qubits that arise from quantum mechanics live in a Hilbert space H (which may be finite and in the specific case of a single qubit: 2-dimensional). A Hilbert space H is a vector space over the complex numbers ÷ with a complex valued inner product. A complex valued inner product is a map: $(-,-)$: H × Hfi ÷ from H × H into the complex numbers ÷ such that:

$$1)\ \ (\boldsymbol{u},\boldsymbol{u}) = \mathbf{0} \text{ iff } \boldsymbol{u} = \mathbf{0}$$

$$2)\ \ (\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{v},\boldsymbol{u})^{*}$$

$$3)\ \ (\boldsymbol{u},\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{u},\boldsymbol{w})$$

$$4)\ \ (\boldsymbol{u},\lambda\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v})$$

$$4')\ \ (\lambda\boldsymbol{u},\boldsymbol{v}) = \lambda^{*}(\boldsymbol{u},\boldsymbol{v})$$

*** denotes complex conjugate**

Math. Def^n for Dirac Notation

The elements or state vectors of the Hilbert Space H are called **ket** vectors and are denoted as $\left|label_1\right\rangle$. The elements of the dual space H* are called **bra** vectors and are denoted $\left\langle label_2\right|$. More formally, the linear functional $\left\langle label_2\right|$ is a linear operation which associates a complex number with every ket $\left|label_1\right\rangle$. This set of linear functionals defined on the kets $\left|label_1\right\rangle$ constitutes a vector space called the dual space of H and is denoted H*.

The complex inner product, denoted by a bra-c-ket is

$$\left\langle label_1 \middle| label_2\right\rangle = \left(\left|label_1\right\rangle, \left|label_2\right\rangle\right)$$

There is a isomorphic mapping on H (assuming it is finite dimensional) that maps it into H* defined by $\left|label\right\rangle \mapsto \left(\left|label\right\rangle, -\right)$ and denoted by the bra $\left\langle label\right|$.

All linear properties shown on the previous slide apply!

Qubits, Basis Sets, and Superposition

In most of the following we will concern ourselves with *quantum bits* or "*qubits*" that like classical bits have only two *elementary orthonormal basis* states. Thus even though quantum systems may have many states we will focus on the two lowest states. These states we we will denote hereafter as the abstract basis vectors $\left|\mathbf{0}\right\rangle$ and $\left|\mathbf{1}\right\rangle$, where

$$\left\langle \mathbf{0} \middle| \mathbf{0} \right\rangle = \left\langle \mathbf{1} \middle| \mathbf{1} \right\rangle = \mathbf{1} \quad \text{and} \quad \left\langle \mathbf{0} \middle| \mathbf{1} \right\rangle = \left\langle \mathbf{1} \middle| \mathbf{0} \right\rangle = \mathbf{0}$$

Although the original Hilbert Space *H* may have been d-dimensional, only the 2-dimensional *H spanned* by $\left\{\left|\mathbf{0}\right\rangle, \left|\mathbf{1}\right\rangle\right\}$ are relevant for quantum information. An arbitrary state $\left|\Psi\right\rangle$ can thus be represented as a *superposition* of $\left|\mathbf{0}\right\rangle$ and $\left|\mathbf{1}\right\rangle$

$$\left|\Psi\right\rangle = \alpha\left|\mathbf{0}\right\rangle + \beta\left|\mathbf{1}\right\rangle \qquad \{\alpha,\beta\}\in \square \quad \text{where} \quad \alpha^*\alpha + \beta^*\beta = \mathbf{1}$$

since $\left\langle \Psi \middle| \Psi \right\rangle = \mathbf{1} = \left(\alpha^*\left\langle \mathbf{0}\right| + \beta^*\left\langle \mathbf{1}\right|\right)\left(\alpha\left|\mathbf{0}\right\rangle + \beta\left|\mathbf{1}\right\rangle\right)$

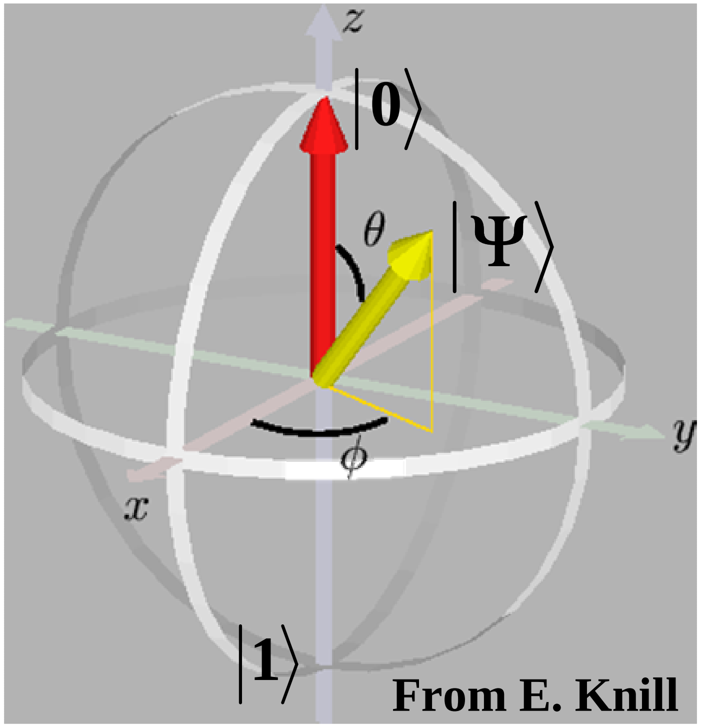
Consequently, the resulting single qubit *H* is equivalent to the vector space $\leq^2$.

Bloch Sphere: A Pictorial Qubit

The state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, which is an arbitrary superposition of the qubit basis sets $|0\rangle$ and $|1\rangle$, can be represented using the Bloch sphere. Assuming $|\Psi\rangle$ is normalized, then it is obvious that

$$\langle\Psi|\hat{O}|\Psi\rangle = \langle\Phi|\hat{O}|\Phi\rangle$$

for an arbitrary operator $\hat{O}$, if $|\Phi\rangle = e^{i\chi}|\Psi\rangle$ – *i.e.* $|\Psi\rangle$ and $|\Phi\rangle$ represent the same state since they differ at most by a constant.



From E. Knill

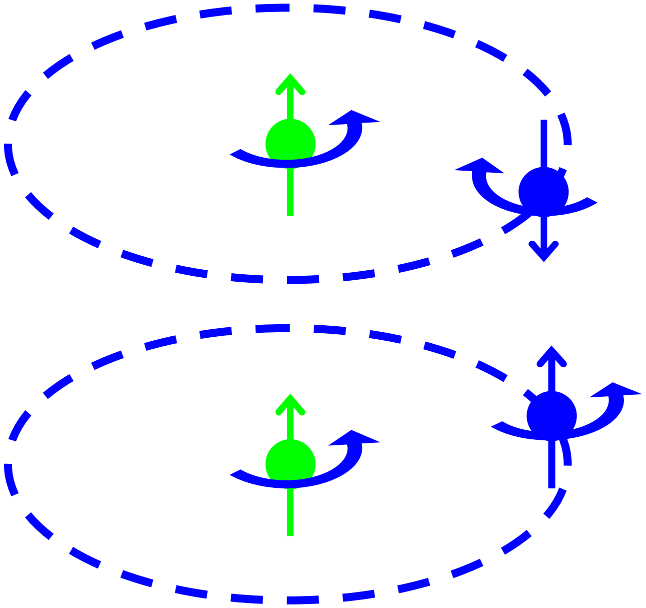
Thus $|\Psi\rangle = a|0\rangle + \beta'|1\rangle \quad a \in \mathbb{R}$ and $\beta' \in \mathbb{C}$ where $a^2 + \beta'^*\beta' = 1$

$|\Psi\rangle = a|0\rangle + e^{i\vartheta}b|1\rangle \qquad \{a,b\} \in \mathbb{R}$ where $a^2 + b^2 = 1$

which leads to: $|\Psi\rangle = \cos\dfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\dfrac{\theta}{2}|1\rangle$

Physical Representation of a Qubit

A one-electron atom:



lower energy state: $|0\rangle$

higher energy state: $|1\rangle$

An atom can be $|\mathbf{0}\rangle$ or it can be $|\mathbf{1}\rangle$ but it can also be $\dfrac{|\mathbf{0}\rangle+|\mathbf{1}\rangle}{\sqrt{\mathbf{2}}}$

ð *i.e.* -- quantum superpositions are possible

$$|\Psi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$$

Matrix Representations of Qubits

$$|\mathbf{0}\rangle = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} \text{ and } |\mathbf{1}\rangle = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

$$|\mathbf{\Psi}\rangle = \alpha|\mathbf{0}\rangle + \beta|1\rangle = \alpha\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \beta\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The "bra" $\langle \boldsymbol{label}|$ appropriate to the "ket" $|\boldsymbol{label}\rangle$ is given by the complex conjugate – transpose. Thus,

$$\langle \mathbf{0}| = (\mathbf{1} \quad \mathbf{0}) \text{ and } \langle \mathbf{1}| = (\mathbf{0} \quad \mathbf{1})$$

$$\langle \mathbf{\Psi}| = (\alpha^* \quad \beta^*)$$

As a result it is trivial to show:

$$\langle \mathbf{0}|\mathbf{0}\rangle = \langle \mathbf{1}|\mathbf{1}\rangle = \mathbf{1} \text{ and } \langle \mathbf{1}|\mathbf{0}\rangle = \mathbf{0} \ ;$$

$$\langle \mathbf{\Psi}|\mathbf{\Psi}\rangle = \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2$$

Projection Operators

A projection operator for the subspace spanned by the ket $|label\rangle$ is given by:

$$P_{label} = |label\rangle\langle label|$$

$$\hat{P}_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\mathrm{P}}_\Psi = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

Thus:

$$\hat{P}_0 |\Psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \alpha|0\rangle$$

$$\hat{P}_0 |\Psi\rangle = |0\rangle\langle 0|\Psi\rangle = |0\rangle\langle 0|\{\alpha|0\rangle + \beta|1\rangle\} = \alpha|0\rangle$$

$$\langle\Psi|\hat{P}_0|\Psi\rangle = \langle\Psi|0\rangle\langle 0|\Psi\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha^*\alpha = |\alpha|^2$$

Quantum Measurement

Quantum measurement is just a projection onto the measurement basis. Thus if we measure the state $|\Psi\rangle$ in the basis $\{|0\rangle, |1\rangle\}$, then the probability of getting $|0\rangle$ is:

$$\langle\Psi|\hat{P}_0|\Psi\rangle = \langle\Psi|0\rangle\langle 0|\Psi\rangle = |\alpha|^2$$

Assuming I obtained the measurement $|0\rangle$, then the new state of the system is:

$$\frac{\hat{P}_0|\Psi\rangle}{\sqrt{\langle\Psi|\hat{P}_0|\Psi\rangle}} = \frac{|0\rangle\langle 0|\Psi\rangle}{\sqrt{\langle\Psi|0\rangle\langle 0|\Psi\rangle}} = |0\rangle$$

Basically the term in the denominator, renormalizes the state. Repeating the measurement on this system will return the same result!

Quantum Observables for Experts

- Quantum observables are represented by linear Hermitian operators – *i.e.* $\hat{A} = \hat{A}^H = \hat{A}^\dagger = \hat{A}^{T*}$
- The eigenvalues a_j of an observable A are real

$$\hat{A}\left|\Psi_{a_j}\right\rangle = a_j \left|\Psi_{a_j}\right\rangle \qquad \text{or} \qquad \hat{A}\left|a_j\right\rangle = a_j\left|a_j\right\rangle$$

- For Hermitian operators one can write:

$$\hat{A} = \sum_{j=0}^{n-1} a_j \left|a_j\right\rangle\left\langle a_j\right| = \sum_{j=0}^{n-1} a_j P_{a_j}$$

- Moreover the projection operators are mutually orthogonal and complete

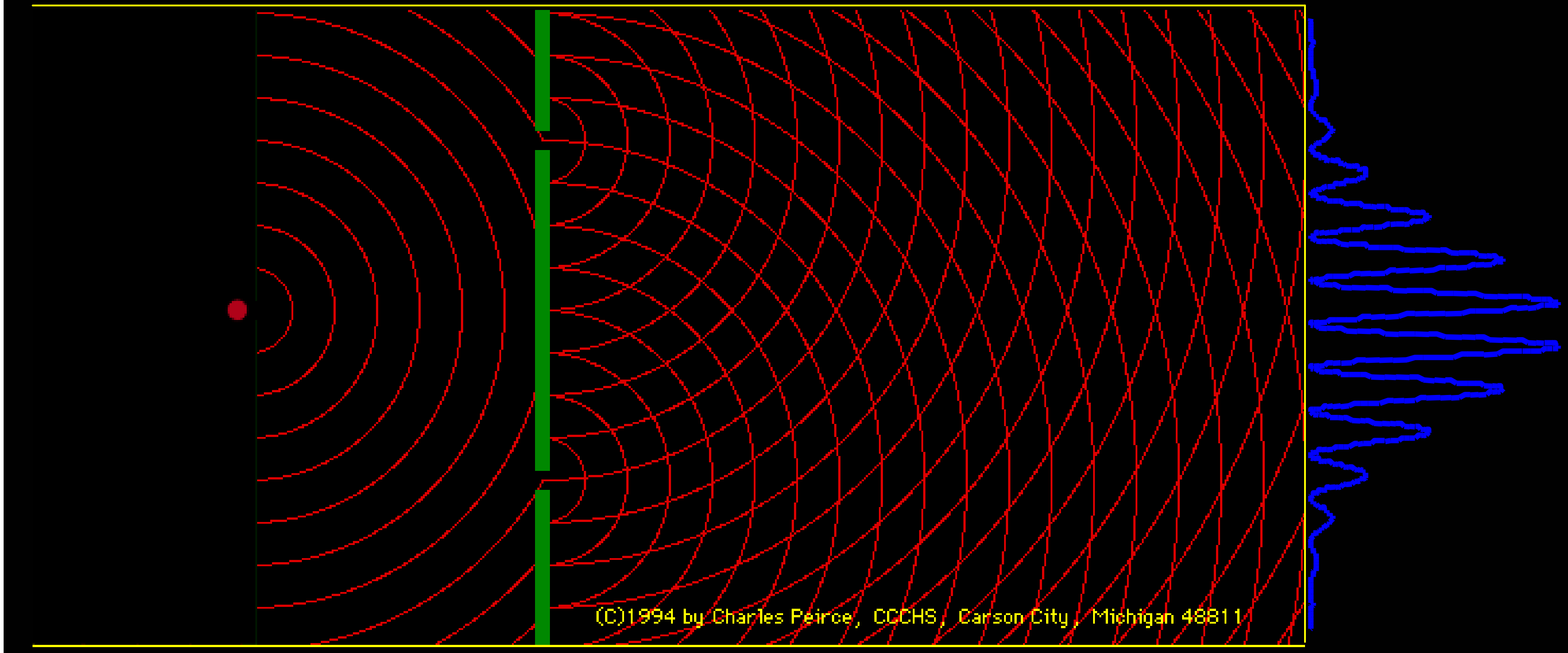
$$\sum_{j=0}^{n-1} P_{a_j} = \hat{I} = 1$$

- And finally an arbitrary state $\left|\Psi\right\rangle$ in H can be decomposed as

$$\left|\Psi\right\rangle = \sum_{j=0}^{n-1} P_{a_j}\left|\Psi\right\rangle = \sum_{j=0}^{n-1}\left|a_j\right\rangle\left\langle a_j\middle|\Psi\right\rangle$$

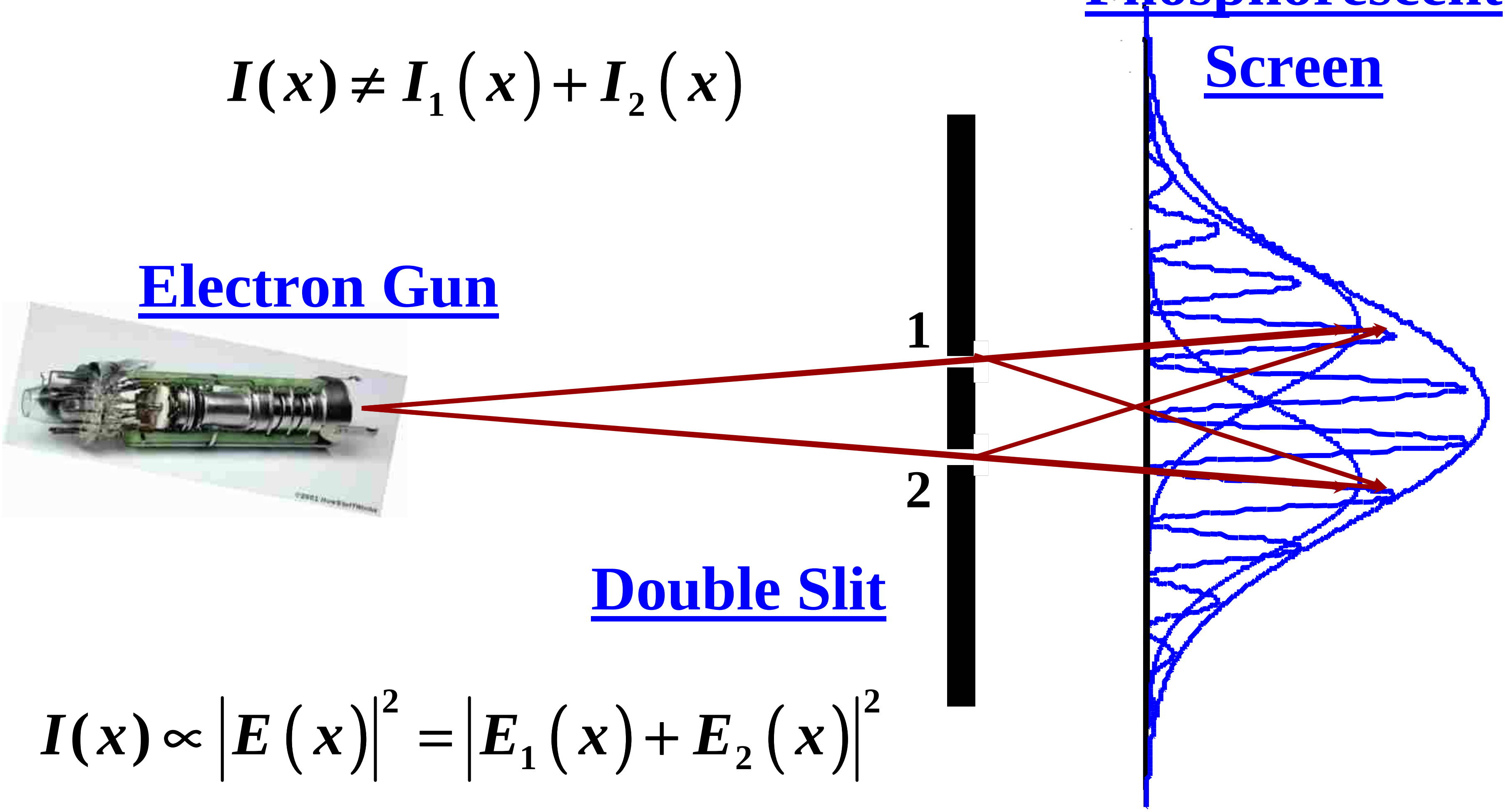
Quantum Interference

- Waves coming through two slits interfere


(C)1994 by Charles Peirce, CCCHS, Carson City, Michigan 48811

Quantum Particle Interference

$$I(x) \neq I_1(x) + I_2(x)$$

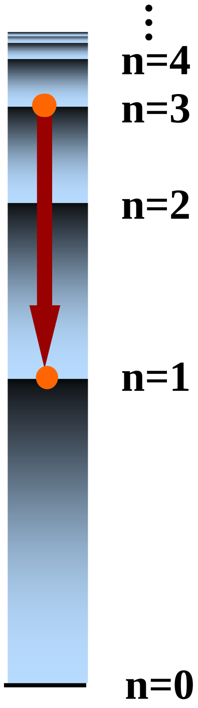


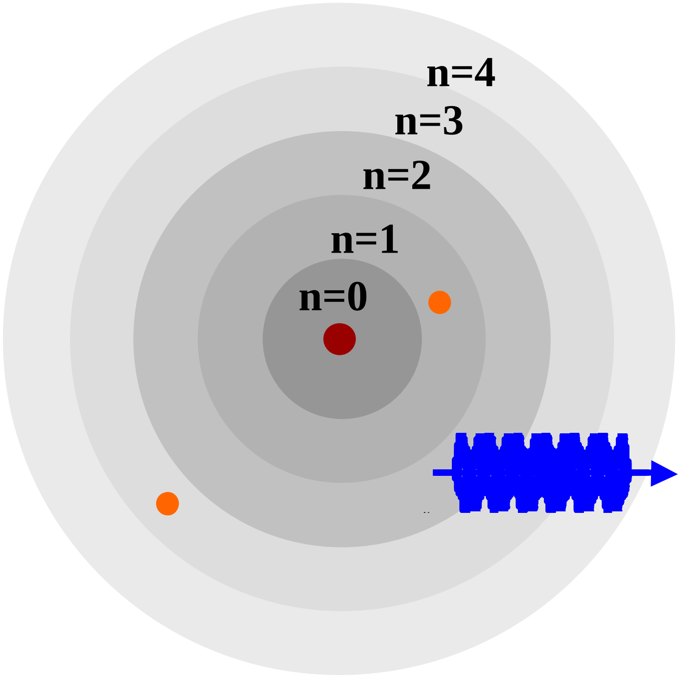
$$I(x) \propto \left|E(x)\right|^2 = \left|E_1(x) + E_2(x)\right|^2$$

where $E(x) = E_1(x) + E_2(x)$

Quantum and the Atom

- **The Atom**
  - Discrete Energy Levels
  - Alternative Representation
  - Transition
  - Spectrum





- Photons as particles
- Atoms as particles/waves
- Wave-Particle Duality (deBroglie waves)
- Wave both here and there

Superposition and Measurement

- **Quantum Superposition**

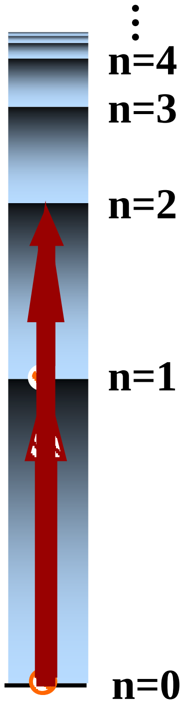
$$|\Psi\rangle = \left(\alpha|0\rangle + \beta|1\rangle\right)$$

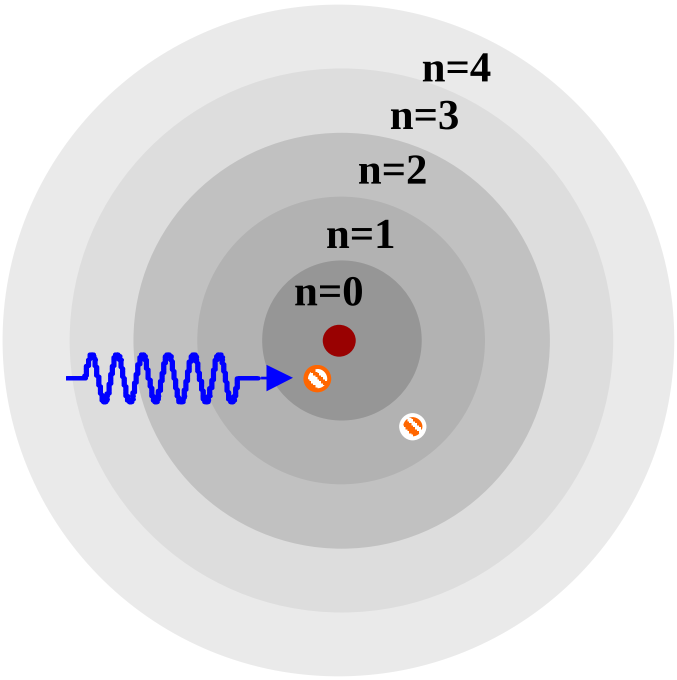
  - **Probability of being in "$|0\rangle$"**

$$\left|\langle 0|\Psi\rangle\right|^2 = \langle\Psi|0\rangle\langle 0|\Psi\rangle = \alpha\alpha^*$$

  - **Example a π/2 Pulse**

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$





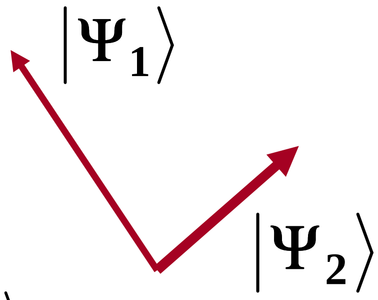
- **Quantum Measurement**

The act of observing or projecting a system into one of its natural states. Thus the system ends up in a new state $|\Psi_n\rangle$

Measurement in $|0\rangle$: $\left|\langle 0|\Psi\rangle\right|^2 = \langle\Psi|0\rangle\langle 0|\Psi\rangle$ **with probability:** $|\alpha|^2$

From 1-Qubit to 2-Qubits

Single Qubit: $|\Psi_1\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)$



2-Qubit State:

$$|\Psi_1\rangle \otimes |\Psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

basis set for particle 1 **basis set for particle 2**

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

denotes a 2-qubit basis state – *i.e.* $|00\rangle$

$$= \alpha_1\alpha_2|0\rangle_2 + \alpha_1\beta_2|1\rangle_2 + \beta_1\alpha_2|2\rangle_2 + \beta_1\beta_2|3\rangle_2$$

ð **product states span a 2-dimensional Hilbert space**

2-Qubit product states have the property that the product of the coefficients of the $|00\rangle$ and $|11\rangle$ **term equals the product of the** $|01\rangle$ and $|10\rangle$ **term!**

Are there a different class of 2-qubit states?

Quantum Entanglement

2-Qubit Entangled State (unfactorizable):

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_1\left|0\right\rangle_2 + \left|1\right\rangle_1\left|1\right\rangle_2\right) = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)$$

ð **not** a product state; can span a 4-dimensional Hilbert space
ð Entanglement creates a "shared fate" ** Schrodinger's Cat **

Another example of an unfactorizable 2-qubit state:

$$\left|\Psi\right\rangle_2 = \alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|10\right\rangle + \delta\left|11\right\rangle \text{ and } \alpha\delta \neq \beta\gamma$$

Note -- however if $\alpha\delta = -\beta\gamma$, then:

$$\left|\Phi\right\rangle_2 = \alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|10\right\rangle - \delta\left|11\right\rangle$$ **is factorizable!!**

Entanglement is a unique *quantum* resource:

" … fundamental resource of nature, of comparable importance to energy, information, entropy, or any other fundamental resource."

Nielsen & Chuang, Quantum Computation and Quantum Information

Tensor Products

Let ⁄$_1$ and ⁄$_2$ be two separate (possibly identical) quantum systems that have been independently prepared in states described by $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Assuming these two quantum systems ⁄$_1$ and ⁄$_2$ have not interacted since their preparation, then the combined wavefunction for the quantum system ⁄ can be represented as a tensor product – *i.e.* $|\Psi_{total}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$

$$|\Psi_1\rangle \otimes |\Psi_2\rangle \in H_1 \otimes H_2$$

More formally, given *n*-quantum systems, ⁄$_1$, ⁄$_2$, …, ⁄$_n$, characterized by the Hilbert spaces, H$_1$, H$_2$, …, H$_n$, respectively, then the multipartite quantum system ⁄ has a Hilbert space H given by: $H = \otimes_{j=1}^{n} H_j$

NOTE!! – However, the general state $|\Psi\rangle$ of ⁄ cannot be represented as tensor product of individual component wavefunctions $|\Psi_j\rangle$ – *i.e. generally* $|\Psi\rangle \neq \otimes_{j=1}^{n} |\Psi_j\rangle$

Matrix Representations of Tensors

2-Qubit Basis States:

$$|0\rangle_2 = |00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \quad |1\rangle_2 = |01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|2\rangle_2 = |10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; ; \quad |3\rangle_2 = |11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

A more general 2-Qubit Basis Product State:

$$|\Psi\rangle_2 = |\Psi_1\rangle \otimes |0\rangle = \{\alpha|0\rangle + \beta|1\rangle\} \otimes |0\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

Interesting *n*-particle Tensor States

The equal superposition of all possible (2^n) *n*-qubit states is a tensor product – Proof:

$$\left|\Psi\right\rangle_n = \left\{\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)\right\}^{\otimes n}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left\{\left|00\cdots00\right\rangle + \left|00\cdots01\right\rangle + \left|00\cdots10\right\rangle + \cdots + \left|11\cdots11\right\rangle\right\}$$

$$= \left(\frac{1}{2^{n/2}}\right)\left\{\left|0\right\rangle_n + \left|1\right\rangle_n + \left|2\right\rangle_n + \cdots + \left|2^n - 1\right\rangle_n\right\}$$

Note – in general an *n*-qubit state is defined by 2^n complex coefficients and therefore is defined by 4^n-2 real numbers since the overall phase is arbitrary and the total wavefunction should be normalized.

References Quantum Primer

A very good overall reference is *Quantum Computation and Quantum Information by M. A. Nielsen and I. L. Chuang*

For a general introduction to Quantum Mechanics see *Quantum Mechanics by C. Cohen-Tannoudji, B. Diu, and F. Laloë* (especially Chapters 2-4)

For a mathematical view of Quantum Mechanics see *Linear Operators for Quantum Mechanics by T. F. Jordan.*

For more on Dirac Notation see *The Principles of Quantum Mechanics by P. A. M. Dirac* (especially Chapter 1)

An overview written by a Mathematician – see *Quantum Computation: A Grand Mathematical Challenge …, Proceedings of Sympoisum in Applied Mathematics, v58, Chapter 1 by S. J. Lomonaco, Jr.*

An introduction to manipulating qubits that de-emphasizes physics: *arXiv:quant-ph/0207118 by N. D. Mermin*

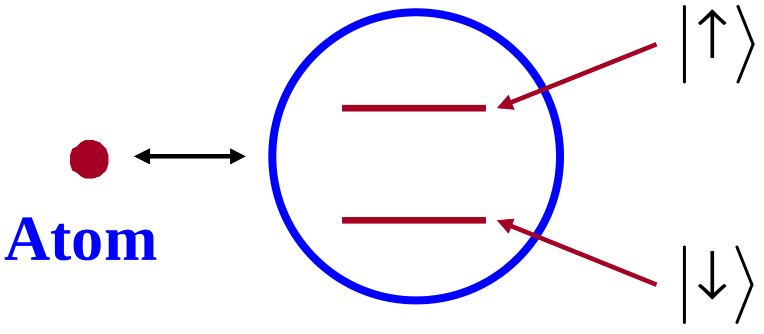
IV. Classical Bits vs. Quantum Bits

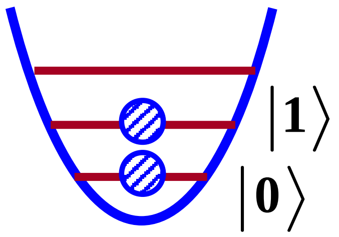
- **Classical Bits: two-state systems**

Classical bits: 0 (off) or 1 (on) (switch)

- **Quantum Bits *are also two-state (level) systems***

Note that almost all quantum systems have more than 2-states and thus a qubit is really using just 2-states of an *n*-state quantum system!





Internal State **Motional State**

ð But: Quantum Superpositions are possible

$$\Psi = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$
$$= \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$$

Scaling of Quantum Information

- Classically, information stored in a bit register: a 3-bit register stores one number, from 0 – 7. $\boxed{0}\boxed{1}\boxed{0}$

e.g. $\boxed{0}\boxed{0}\boxed{0}$ $\boxed{0}\boxed{0}\boxed{1}$ $\boxed{0}\boxed{1}\boxed{0}$ … $\boxed{1}\boxed{1}\boxed{1}$

- Quantum mechanically, a 3-qubit register can store all of these numbers in an arbitrary superposition:

$$\alpha|000\rangle+\beta|001\rangle+\chi|010\rangle+\delta|011\rangle+\varepsilon|100\rangle+\gamma|101\rangle+\eta|110\rangle+\kappa|111\rangle$$

- Result:
  - Classical: one N-bit number
  - Quantum: 2^N (all possible) N-bit numbers

Scaling of Quantum Information (2)

- **Consequence of Quantum Scaling**
  - **Calculate all values of f(x) at once and in parallel**
  - **Quantum Computer will provide Massive Parallelism**
- **But wait …**
  - **When I *"readout the result"* I obtain only one value of f(x)**
  - **For the previous 3-qubit example each value of f(x) appears with probability 1/8**
- **Thus must measure some global property of f(x)**
  - ***e.g.* periodicity**

Note!

300-qubit register has much more storage capacity than classically is in the whole universe

33-qubits has 1Gb of storage capacity

Analog vs. Quantum Computing

Is a quantum computer basically an analogue computer – (qubit coefficients are continuous)?

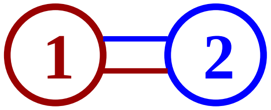
No!

- **Why Not? – Analog Computer**
  - Finite Resolution ⇒ must bin values
  - Scaling lost ⇒ equivalent to classical digital computer
    ⇒ classical Church-Turing hypothesis
- **Quantum Computer**
  - Add 1 qubit, double storage/memory capacity
  - Scaling is preserved ⇒ tensor product structure and entanglement

Einstein-Podolsky-Rosen Paradox

$|0_1\rangle|0_2\rangle+|1_1\rangle|1_2\rangle$

(1) Prepare 2-qubits in
an entangled state



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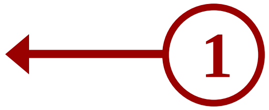
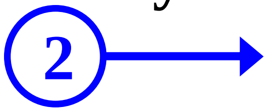
(2) Send qubit 1 with Alice to Paris
and qubit 2 with Bob to Tokyo





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before measurement, ① is both $|0_1\rangle$ and $|1_1\rangle$ (as is ② !)

But if you <u>measure</u> ① to be $|0_1\rangle$, then ② is <u>surely</u> $|0_2\rangle$

And you know it immediately, even if ② is light years away

No Cloning Theorem

Assume there exists a unitary operator $\hat{U}_{clone}$ that copies an arbitrary unknown quantum states into a standard or "null" state. Then for two arbitrary states $|\Psi\rangle$ and $|\Phi\rangle$ such that:

$$|\Psi\rangle \neq |\Phi\rangle \text{ and } \langle\Psi|\Phi\rangle \neq \mathbf{0}$$

one can then write:

$$\hat{U}_{clone}|\Psi\rangle|\mathbf{0}\rangle = |\Psi\rangle|\Psi\rangle$$

$$\hat{U}_{clone}|\Phi\rangle|\mathbf{0}\rangle = |\Phi\rangle|\Phi\rangle$$

Taking the Hermitian conjugate of the lower equation and equation and collecting the left and right sides one obtains:

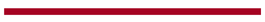
$$\langle\mathbf{0}|\langle\Phi|\hat{U}^{\dagger}_{clone}\hat{U}_{clone}|\Psi\rangle|\mathbf{0}\rangle = \langle\Phi|\langle\Phi||\Psi\rangle|\Psi\rangle$$

$$\langle\mathbf{0}|\langle\Phi|\Psi\rangle|\mathbf{0}\rangle = \langle\Phi|\Psi\rangle^{2}$$

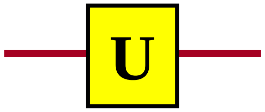
$$\mathbf{1} = \langle\Phi|\Psi\rangle$$

This is a clear contradictions and thus $\hat{U}_{clone}$ must not exist!

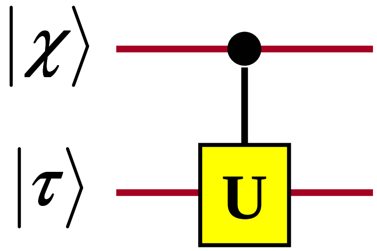
Quantum Circuits



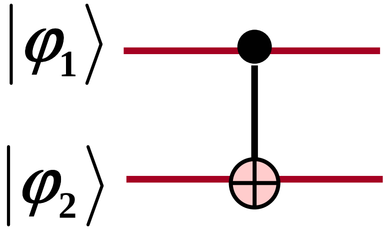
A timeline for a single qubit



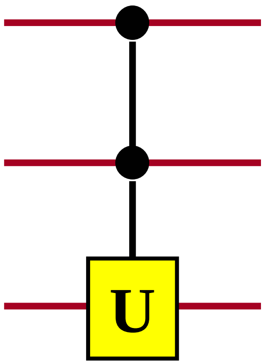
A gate on a single qubit



A controlled ● unitary gate where the state of the control $|\chi\rangle$ determines whether $\boxed{U}$ is applied



A controlled-not gate where the control flips the target



A controlled-controlled unitary gate where *iff* the two control qubits have a component "$|11\_\rangle$" is the unitary applied to the 3rd

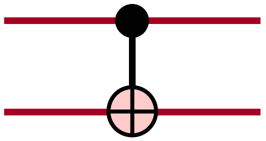
Note: Because of entanglement, one must be careful to interpret the circuit by linearly applying the appropriate set of gates on each of the individual components of the qubit bases functions $|0\rangle$ and $|1\rangle$ that span the H space – *i.e.* use the linear properties of the vector space.

Standard Single Qubit Gates

			Notes
• Hadamard	—$\boxed{\mathbf{H}}$—	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	• A very important & key 1-qubit gate
• Pauli-X	—$\boxed{\mathbf{X}}$—	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	• The basic 1-qubit bit-flip gate
• Pauli-Y	—$\boxed{\mathbf{Y}}$—	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	
• Pauli-Z	—$\boxed{\mathbf{Z}}$—	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	• A basic gate for a 1-qubit phase error
• Phase	—$\boxed{\mathbf{S}}$—	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	
• π/8	—$\boxed{\mathbf{T}}$—	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	

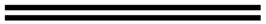
Common *n*-Qubit Gates

- Controlled-NOT

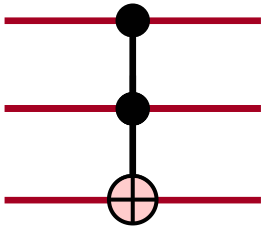


$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

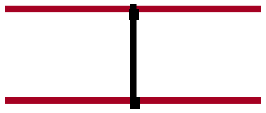
- Classical Bit



- Toffoli

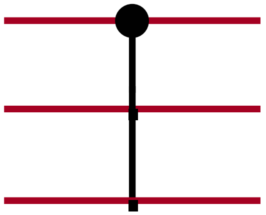


- Swap

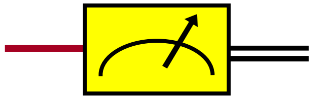


$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

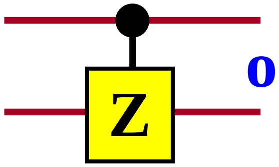
- Fredkin or controlled swap



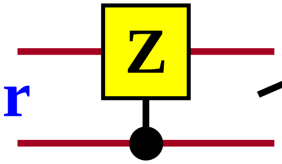
- Measurement



- Controlled-Z or controlled “phase”



or

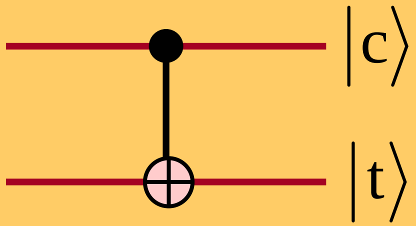


$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Example of CNOT Gate

Let : $|\Psi\rangle_{12} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{pmatrix}$$



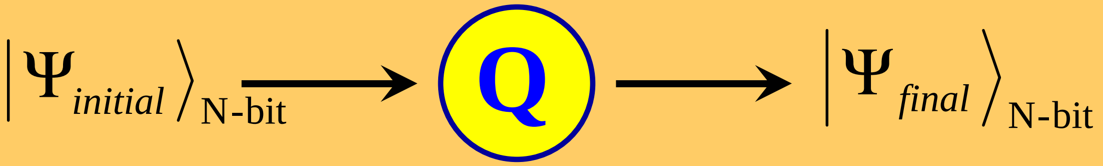
Circuit

$$\Rightarrow \quad |\Psi\rangle_{12} = \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

If $\alpha = \gamma = \frac{1}{\sqrt{2}}$ and $\beta = \delta = 0$; then $|\Psi\rangle_{12} = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle$

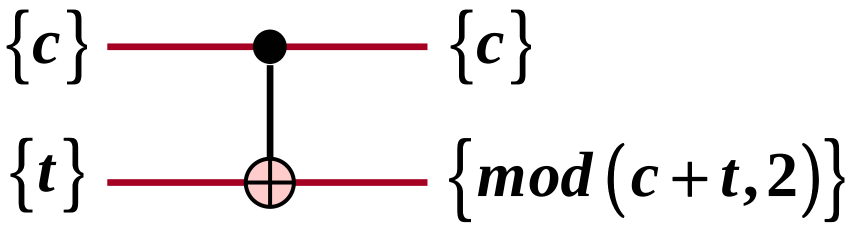
$$\Rightarrow \text{CNOT} \quad |\Psi\rangle_{12} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

1- & 2-Qubit Gates allow for all possible unitary operations


$|\Psi_{initial}\rangle_{\text{N-bit}}$
Q
$|\Psi_{final}\rangle_{\text{N-bit}}$

No Cloning Theorem – Revisited

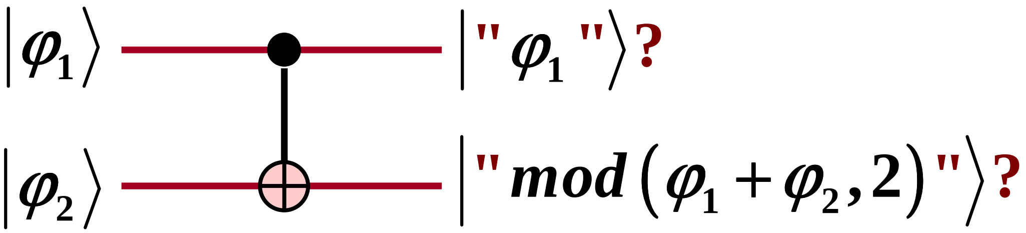
- Copying a Classical Bit



Truth Table

$\{c,t\}$		$\{c, mod(c+t,2)\}$
$\{00\}$		$\{00\}$
$\{01\}$		$\{01\}$
$\{10\}$	$\Rightarrow$	$\{11\}$
$\{11\}$		$\{10\}$

- Attempt to Copy a Quantum Bit:



Let: $|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|\varphi_2\rangle = |0\rangle$

Then:

$$|\varphi_1\varphi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

entangled state

Applications of Quantum Information

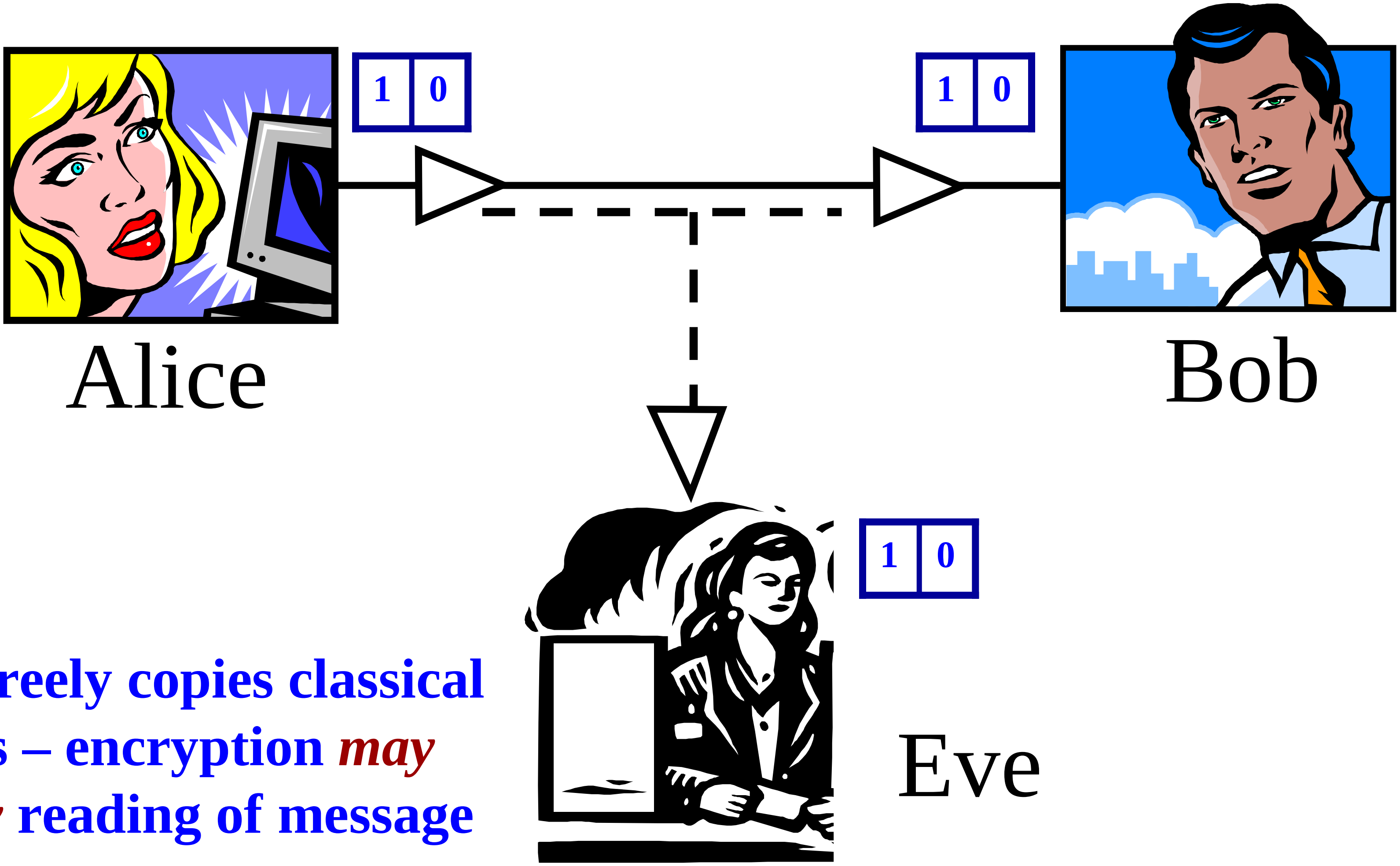
- **Quantum Communication - 100% physically secure**
  - Quantum cryptographic key exchange – generation of a one-time classical key for secure communication
  - Quantum Teleportation – requires "*entangled* photons"
- **Quantum Algorithms and Computing**
  - Factorization of large composite numbers
  - Searching large databases
  - Potential solution of computationally intractable (NP) problems
  - Simulation of large-scale quantum systems
- **Quantum Measurement – improved accuracy**
  - Beats classical limit on Signal to Noise $\propto 1/N$ vs $\propto 1/\mathrm{Sqrt}(N)$
  - Better Atomic Clocks Improved navigation
  - Metrology for Single Photon Sources and Detectors

Note: Quantum Computing requires larger register size and higher fidelity gates then either Quantum Communication or Quantum Measurement.

V. Quantum Communication

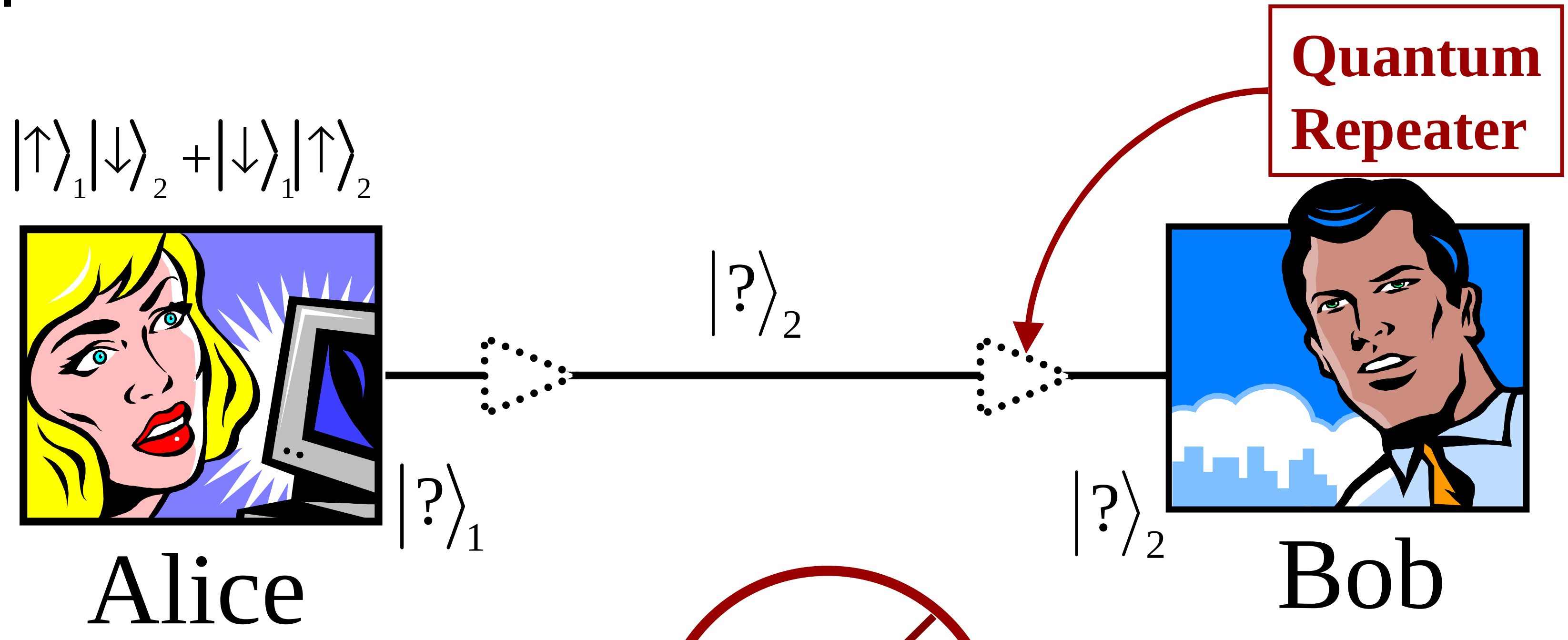
- Quantum Key Distribution – attenuated or single photon sources with known but arbitrary selected polarization and an authenticated classical channel
- Quantum Teleportation – *i.e.* “sending” of an unknown quantum state – requires shared Bell’s (entangled) states and an authenticated classical channel
- Dense Coding – requires shared Bell’s states
- Quantum Communication:
  - with attenuated sources is 100% physically secure and has been demonstrated over kilometer distances
  - in fibers over distances larger than ~100 km will require quantum repeaters
  - ~ 10 qubit quantum processors can serve as quantum repeaters

Classical Communication



Eve freely copies classical bits – encryption *may delay* reading of message

Quantum Communication

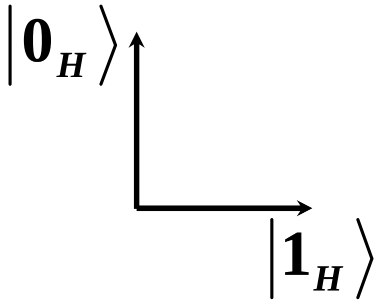

Quantum Repeater
$|\uparrow\rangle_1|\downarrow\rangle_2+|\downarrow\rangle_1|\uparrow\rangle_2$
$|?\rangle_2$
$|?\rangle_1$
$|?\rangle_2$
Alice
Bob

Eve can only obtain key bits by destroying them (no-cloning theorem). Eve presence is detected.

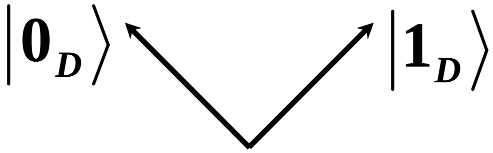
Basis for BB84

Two non-orthogonal Alphabets

Horizontal/Vertical



Diagonal

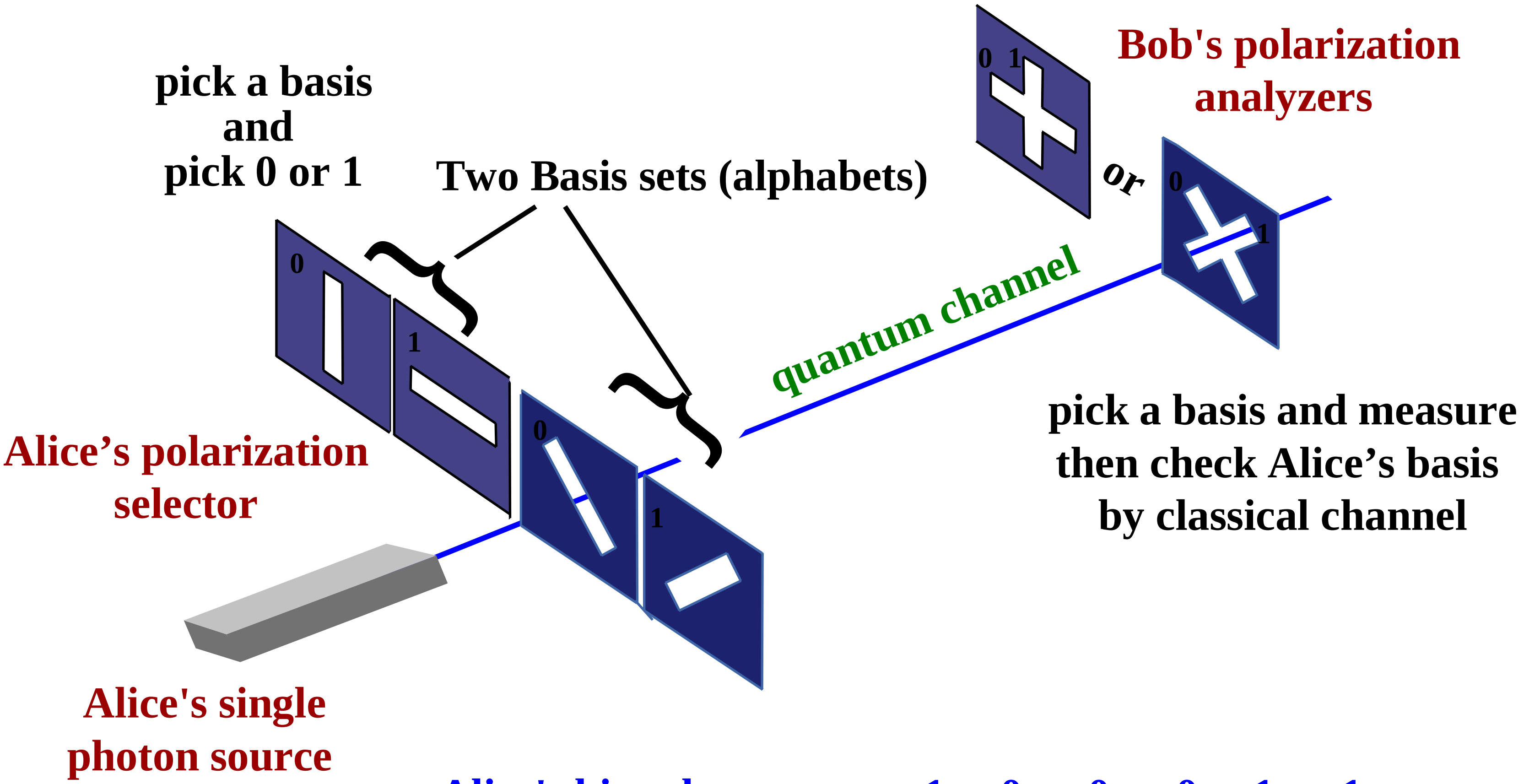


Relation between Basis Sets:

$$\langle \Psi_D | \Psi_H \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

If you measure either $|0_H\rangle$ or $|1_H\rangle$ in the diagonal basis you have a 50% probability of obtaining $|0_D\rangle$ or $|1_D\rangle$. Similarly if you measure $|0_D\rangle$ or $|1_D\rangle$ in the horizontal basis. Easily obtained using simple trigonometry.

BB84 Protocol Schematic



pick a basis
and
pick 0 or 1

Two Basis sets (alphabets)

Bob's polarization
analyzers

quantum channel

pick a basis and measure
then check Alice’s basis
by classical channel

Alice’s polarization
selector

Alice's single
photon source

Alice's bit value	1	0	0	0	1	1
Alice's polarization	/	|	\	\	/	—
Bob's polarization basis	×	×	+	×	+	+
Bob's result	1	1	0	0	1	1
Same basis?	Y	N	N	Y	N	Y
Transmitted key	1			0		1

BB84 Protocol

	0	1
+ basis (HV)	↕	↔
× basis (D)	⤡	⤢

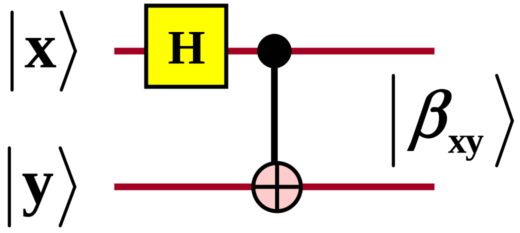
- **STEP 1 : Transmission - *quantum channel***
  - **Alice selects random key and transmits each bit using random basis**
  - **Bob measures each bit in random basis**
  - **Bob now has key, but only some are right**
- **STEP 2: Reconciliation - *classical channel***
  - **Bob tells Alice which bases he used (but not the data)**
  - **Alice tells Bob which bases match (the bits measured in the same bases should match – assuming no errors)**

BB84 Protocol (2)

- Only bits *transmitted* and *received* using same basis are used as key
- STEP 3: Detecting Eve - *classical channel*
  - Alice & Bob compare initial bits of key
  - If key does not match, then it has been compromised
  - If error rate > 25%, must assume Eve is present
  - In practice other sources of error must be accounted for. Error correction and privacy amplification can be applied for error rates < 25%.

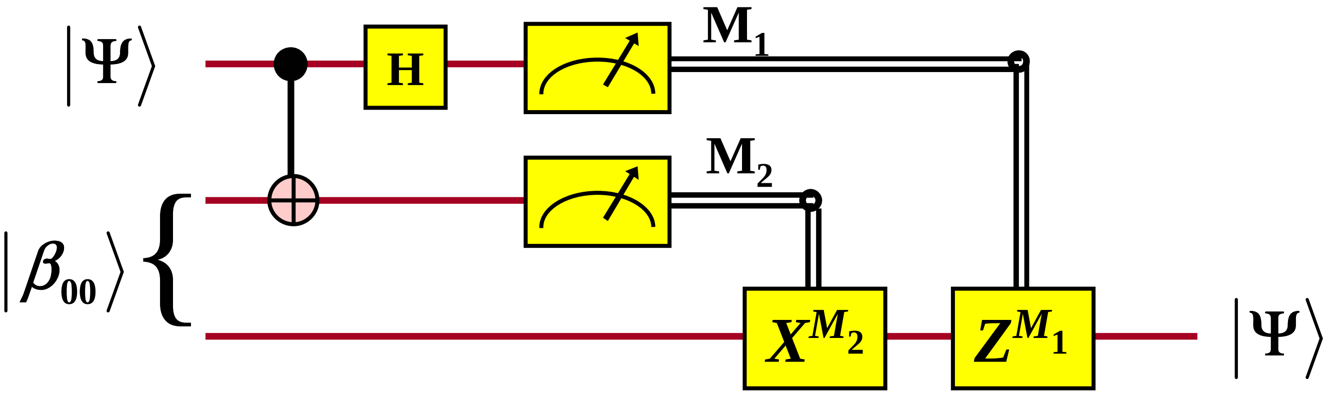
Bell States and Teleportation

- Making Bell States

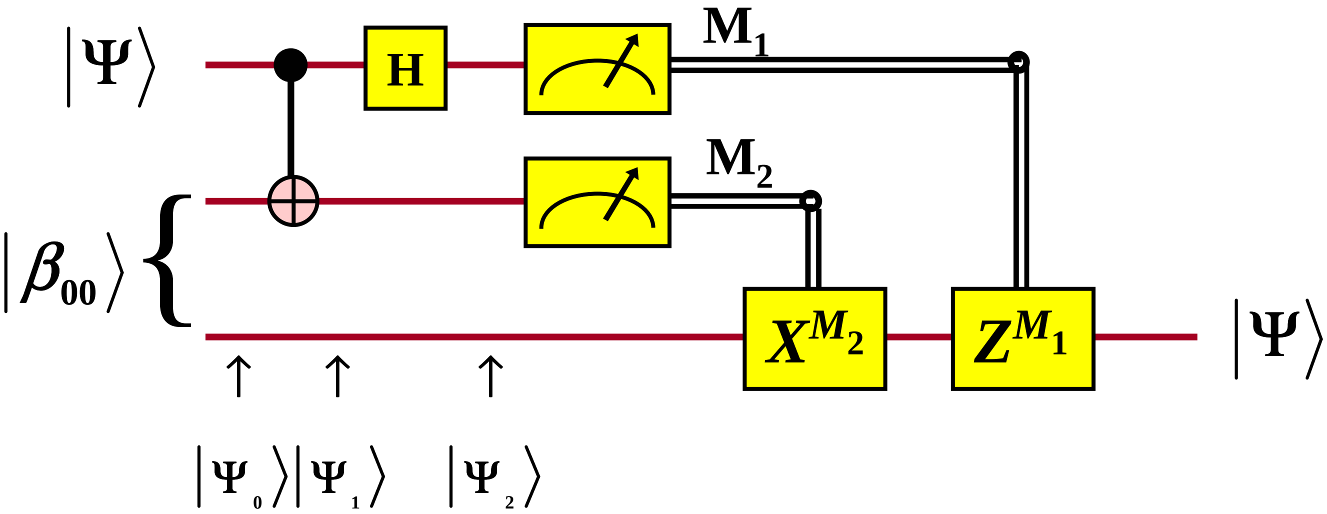


$$\begin{pmatrix}|00\rangle\\|01\rangle\\|10\rangle\\|11\rangle\end{pmatrix} \Rightarrow \begin{pmatrix}\left(|00\rangle+|11\rangle\right)/\sqrt{2}\\\left(|01\rangle+|10\rangle\right)/\sqrt{2}\\\left(|00\rangle-|11\rangle\right)/\sqrt{2}\\\left(|01\rangle-|10\rangle\right)/\sqrt{2}\end{pmatrix} \equiv \begin{pmatrix}|\beta_{00}\rangle\\|\beta_{01}\rangle\\|\beta_{10}\rangle\\|\beta_{11}\rangle\end{pmatrix}$$

- Teleportation



Analysis of Teleportation Circuit



$$|\Psi_0\rangle = |\Psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle \otimes \big(|00\rangle + |11\rangle\big) + \beta|1\rangle \otimes \big(|00\rangle + |11\rangle\big)\Big]$$

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle \otimes \big(|00\rangle + |11\rangle\big) + \beta|1\rangle \otimes \big(|10\rangle + |01\rangle\big)\Big]$$

$$|\Psi_2\rangle = \frac{1}{2}\Big[|00\rangle \otimes \big(\alpha|0\rangle + \beta|1\rangle\big) + |01\rangle \otimes \big(\alpha|1\rangle + \beta|0\rangle\big) +$$

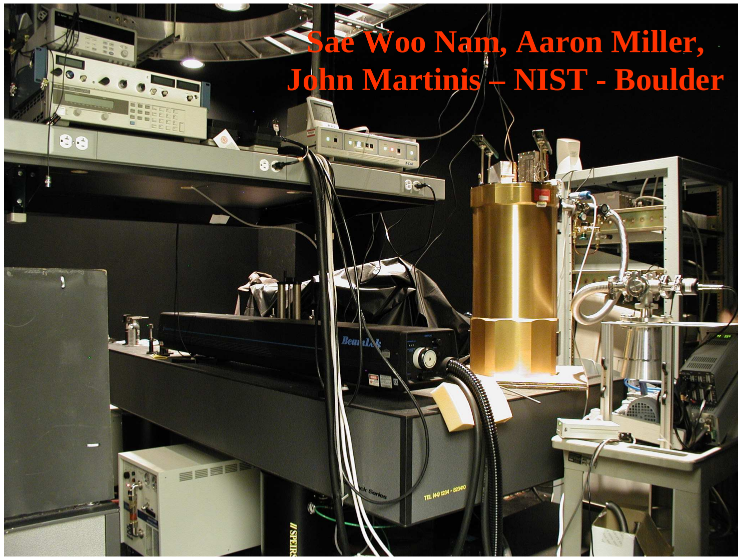
$$|10\rangle \otimes \big(\alpha|0\rangle - \beta|1\rangle\big) + |11\rangle \otimes \big(\alpha|1\rangle - \beta|0\rangle\big)\Big]$$

Status of Quantum Communications

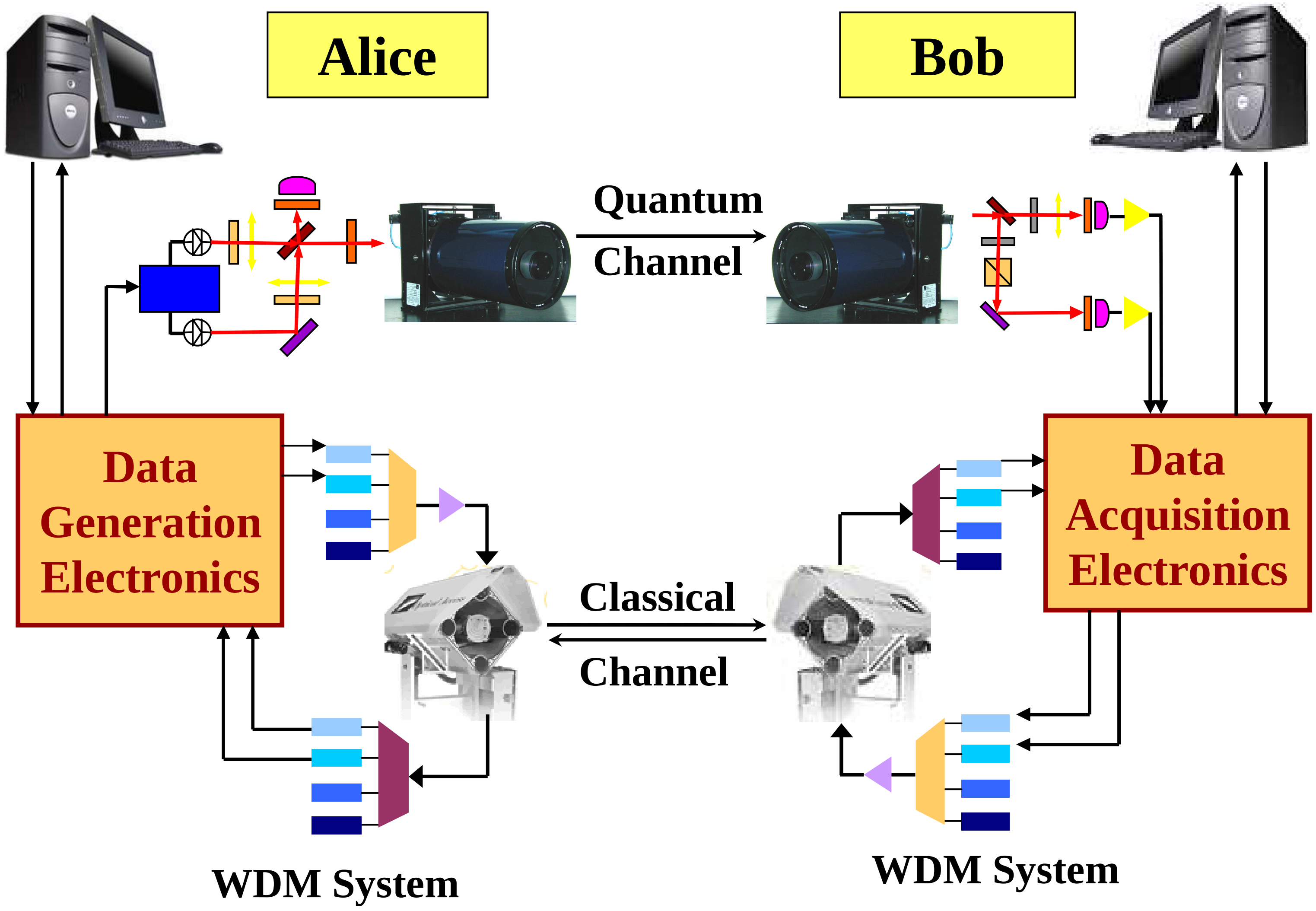
OH ALICE.. YOU'RE THE ONE FOR ME
BUT BOB.. IN A QUANTUM WORLD HOW CAN WE BE SURE?
ψ^+ or ψ^- ?

- State of the Art
  - **Free Space**
    - 10 km both day and night: LANL
    - 30 km night: Kurtseifer, Rarity
  - **Fiber over 65km**
    - LANL, Telcordia
    - U. Geneva: Gisin
    - MagiQ

- Wish List
  - **Single Photon Sources: Numerous Demonstrations**
  - **High Efficiency Single Photon Detectors**
  - **Quantum Repeaters**


Sae Woo Nam, Aaron Miller, John Martinis – NIST - Boulder

NIST Testbed Structure



1.25 GHz High-speed QKD

Quantum Communication Test-Bed

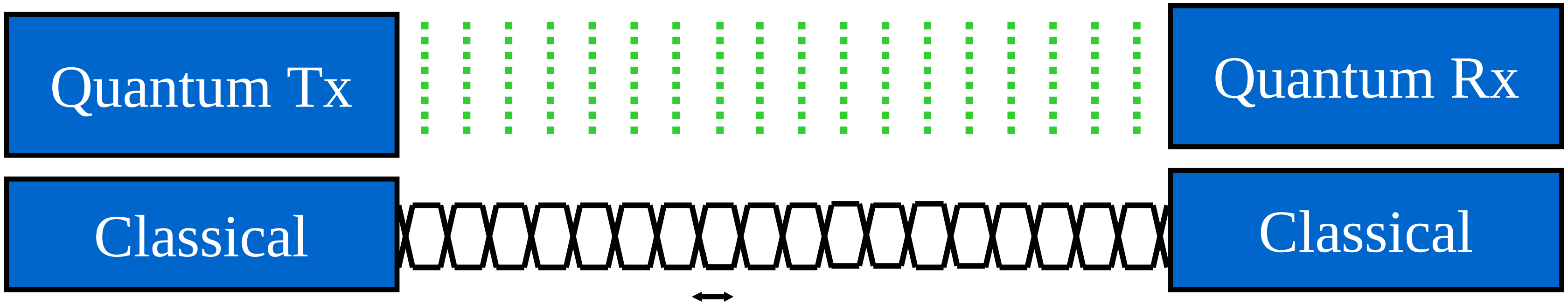
What is special about the NIST system?

- Dual Classical & Quantum Channels running at 1.25 GHz
- Network – Internet interfaced (Also BBN)
  - Security Protocols – SSL, Authentication
- Quantum Link
  - Attenuated VCSEL transmitters (initially)
  - 850 nm free space optics
  - Si avalanche detectors
- Two classical links near 1550 nm
  - 8B/10B encoded path for timing/framing
  - Dedicated gigabit ethernet channel
    - Sifting, Error correction, and Reconciliation
    - Privacy amplification

Joshua Bienfang, Bob Carpenter, Alex Gross, Ed Hagley, Barry Hershman, Richang Lu, Alan Mink, Tassos Nakassis, Xiao Tang, Jesse Wen, David Su, Charles Clark, Carl Williams

High-Speed Free-Space QKD

- Spectral, Spatial filter to ~ 10^6 solar photons/sec into Rx
  - (0.1 nm, 300 cm^2, 220 μrad)
- Gating:



1 nsec

8B/10B encoding/clock recovery

- No heralding pulse: all time bins are filled
- A 1 ns gate is equivalent to 1 GHz pulse rate
  - Gate shortens with increased pulse rate
  - Limited by detector jitter and recovery time

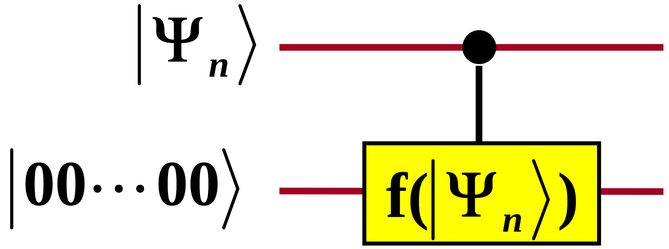
VI. Quantum Computing

- A Uniform Superpositions of all input states is easy:

$$|\Psi\rangle_n = \left\{\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\right\}^{\otimes n}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left\{|00\cdots00\rangle+|00\cdots01\rangle+|00\cdots10\rangle+\cdots+|11\cdots11\rangle\right\}$$

- Using *n*-additional qubits calculate the function *f* on $|\Psi\rangle_n$



The result is entanglement between $|\Psi\rangle_n$ and its function

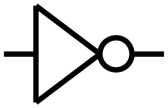
$$|\Psi f(\Psi)\rangle_n = \left(\frac{1}{2^{n/2}}\right)\left\{|0\rangle_n\left|f\left(|0\rangle_n\right)\right\rangle+|1\rangle_n\left|f\left(|1\rangle_n\right)\right\rangle+\cdots+|2^n-1\rangle_n\left|f\left(|2^n-1\rangle_n\right)\right\rangle\right\}$$

Classical *vs* Quantum Computation

Classical Computation

- Initialize state: “0”
- Logic:

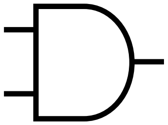
not



$0 \rightarrow 1$

$1 \rightarrow 0$

and



$00 \rightarrow 0$

$01 \rightarrow 0$

$10 \rightarrow 0$

$11 \rightarrow 1$

- Output result
- Logic errors:
  Error correction possible

Quantum Computation

- Initialize state: $\Psi_{in} = \left|000\cdots0\right\rangle$
- Logic:

1-qubit

$\left|0\right\rangle \rightarrow \left|1\right\rangle$

$\left|1\right\rangle \rightarrow \left|0\right\rangle$

$\left|0\right\rangle \rightarrow \left(\left|0\right\rangle + \left|1\right\rangle\right) / 2^{1/2}$

2-qubit controlled-not

$$\left.\begin{array}{l} \left|00\right\rangle \rightarrow \left|00\right\rangle \\ \left|01\right\rangle \rightarrow \left|01\right\rangle \\ \left|10\right\rangle \rightarrow \left|11\right\rangle \\ \left|11\right\rangle \rightarrow \left|10\right\rangle \end{array}\right\} \textit{linear + superposition}$$

control target

- Final state measurement

Measure qubits Þ $\Psi_f = \left|ijk\cdots l\right\rangle$

- Coherence: $\tau_{coherence} / \tau_{logic} \cong 10^4$

Q. Computation allows *non-classical* computation

Universal Quantum Logic

All quantum computations and all unitary operators may be efficiently constructed from 1- and 2- qubit logic gates

Single Qubit Operations/Gates

Arbitrary 1-qubit rotations: $|0\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle$

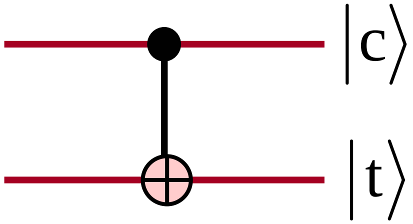
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \qquad \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Note: Although the standard paradigm for quantum computations relies on the ability to do arbitrary 1- qubit gates and almost any 2- qubit gates, alternatives exist

Universal Quantum Logic -- II

Most common 2-Qubit Gate: CNOT Gate

$$\begin{array}{c} |c,t\rangle \\ \begin{pmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{pmatrix} \end{array} \Rightarrow \begin{pmatrix} |00\rangle \\ |01\rangle \\ |11\rangle \\ |10\rangle \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

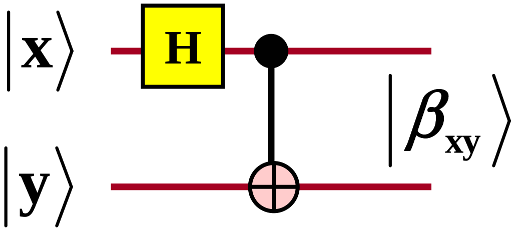


Operation Transformation Circuit

This gate is similar to addition modular 2 of classical gates but one should recall that this gate works on arbitrary superpositions

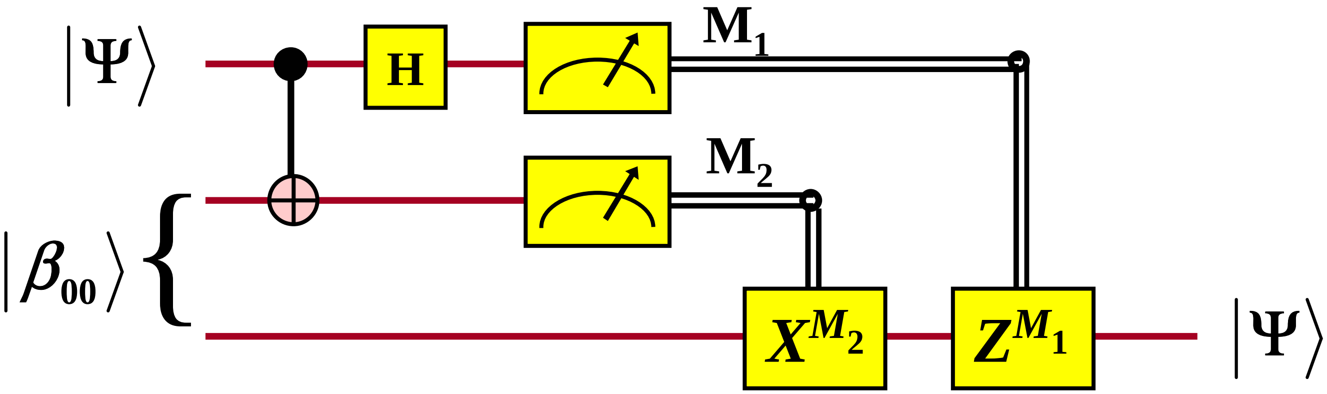
Bell States and Teleportation

- Making Bell States

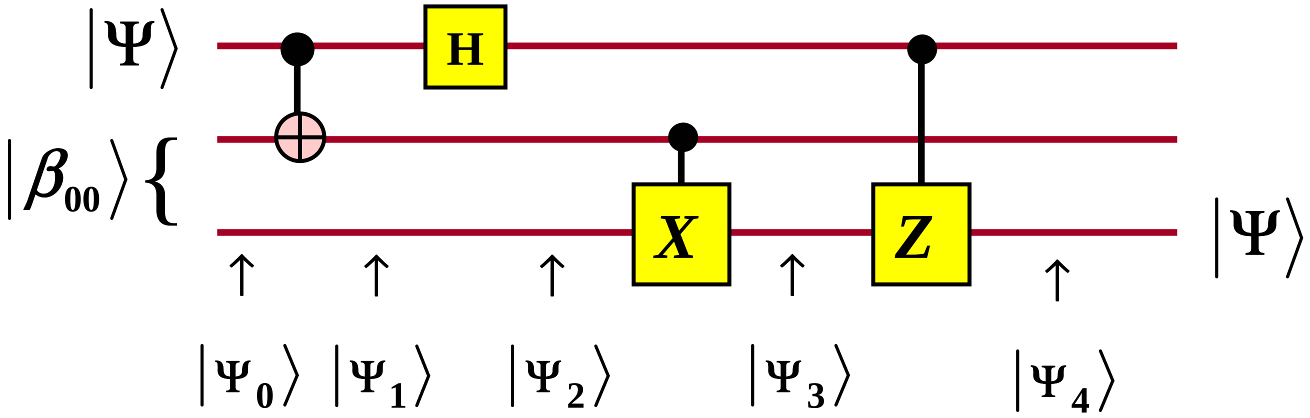


$$\begin{pmatrix}|00\rangle\\|01\rangle\\|10\rangle\\|11\rangle\end{pmatrix} \Rightarrow \begin{pmatrix}(|00\rangle+|11\rangle)/\sqrt{2}\\(|01\rangle+|10\rangle)/\sqrt{2}\\(|00\rangle-|11\rangle)/\sqrt{2}\\(|01\rangle-|10\rangle)/\sqrt{2}\end{pmatrix} \equiv \begin{pmatrix}|\beta_{00}\rangle\\|\beta_{01}\rangle\\|\beta_{10}\rangle\\|\beta_{11}\rangle\end{pmatrix}$$

- Teleportation



Teleportation without Measurement



$$|\Psi_0\rangle = |\Psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle \otimes \left(|00\rangle + |11\rangle\right) + \beta|1\rangle \otimes \left(|00\rangle + |11\rangle\right)\right]$$

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle \otimes \left(|00\rangle + |11\rangle\right) + \beta|1\rangle \otimes \left(|10\rangle + |01\rangle\right)\right]$$

$$\begin{aligned}|\Psi_2\rangle = \frac{1}{2}\Big[&|00\rangle \otimes \left(\alpha|0\rangle + \beta|1\rangle\right) + |01\rangle \otimes \left(\alpha|1\rangle + \beta|0\rangle\right) + \\ &|10\rangle \otimes \left(\alpha|0\rangle - \beta|1\rangle\right) + |11\rangle \otimes \left(\alpha|1\rangle - \beta|0\rangle\right)\Big]\end{aligned}$$

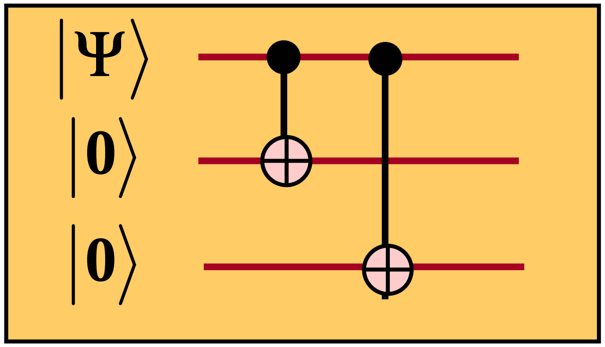
$$|\Psi_3\rangle = \frac{1}{2}\left[\left(|00\rangle + |01\rangle\right) \otimes \left(\alpha|0\rangle + \beta|1\rangle\right) + \left(|10\rangle + |11\rangle\right) \otimes \left(\alpha|0\rangle - \beta|1\rangle\right)\right]$$

$$|\Psi_4\rangle = \frac{1}{2}\left[\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \otimes \left(\alpha|0\rangle + \beta|1\rangle\right)\right]$$

Quantum Error Correction

e.g. -- Redundant Encoding $|0\rangle_L = |000\rangle$ and $|1\rangle_L = |111\rangle$

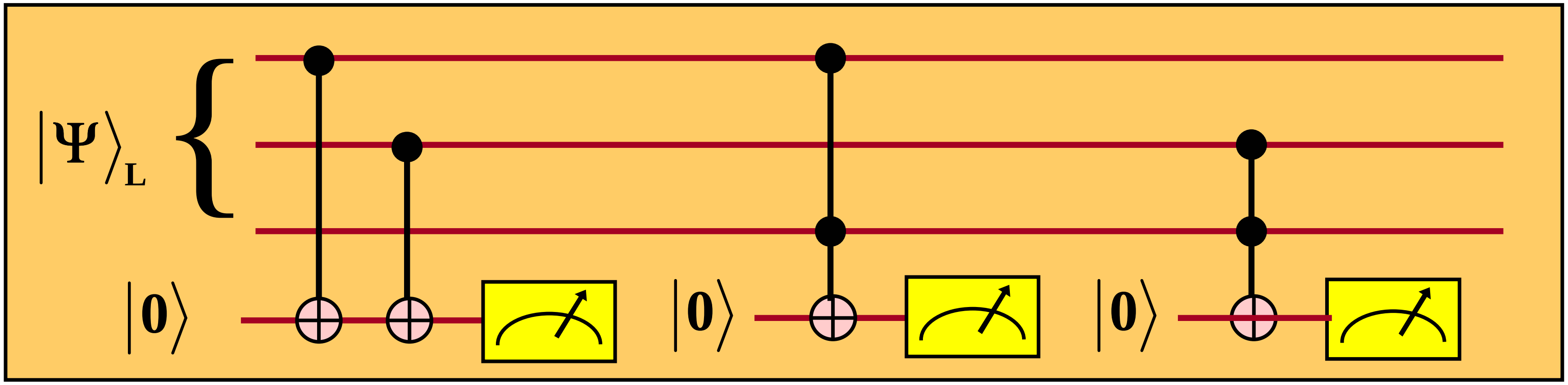
$$|\Psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$



Measure Error Syndrome

extract error information (measure *parity*)

preserve original quantum information



Quantum Computing appears impossible without Quantum Error Correction (Shor, Steane,...)

10^{-2} to 10^{-4} decoherence

opening bid:

depends on errors, could improve

Basis of Shor’s Algorithm

- N – number to be factored
- select a number x such that gcd(x,N)=1 (coprime)
- find r such that x^r=1 mod (N)
- Example: N=15, x=13

x^1 mod (15) = 13 x^2 mod (15) = 4 x^3 mod (15) = 7
x^4 mod (15) = 1 x^5 mod (15) = 13 x^6 mod (15) = 4

Þ r=4 and ∴ $x^r - 1 = 0$ or for r even
$(x^{r/2} - 1)(x^{r/2} + 1) = 0$ mod (N) = kN

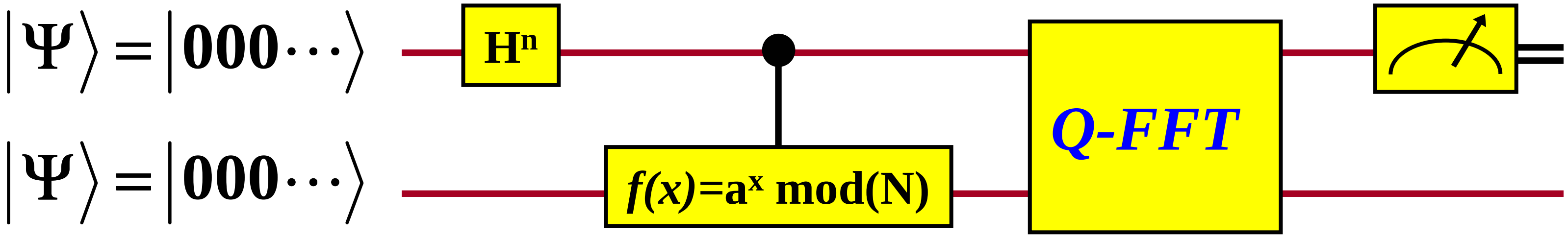
Þ factors are $(x^{r/2} \pm 1)$ mod (N)

e.g. x=4 x^1 mod (15) = 4 x^2 mod (15) = 1

e.g. x=7 x^1 mod (15) = 7 x^2 mod (15) = 4
x^3 mod (15) = 13 x^4 mod (15) = 1

Shor's Algorithm

- Select N such that N = p • q
- Find x such that gcd(x,N) = 1 (coprime)
- Run Shor's Algorithm


$|\Psi\rangle = |000\cdots\rangle$
$|\Psi\rangle = |000\cdots\rangle$
H^n
$f(x)$=a^x mod(N)
Q-FFT

- Measure first register and obtain an approximation to r
- factors are ($x^{r/2}$ ± 1) mod (N)

Quantum Information's Impact

- Revolutionary
  - Builds the physical foundation for information theory
  - Teaches us to examine the information content in real systems
  - Help us to develop a language to move quantum mechanics from a scientific to an engineering field
- Quantum Limited Measurement will become available
- 20th Century we used the particle/wave aspects of Quantum Mechanics: Televisions, CRT's, NMR …
- 21st Century we will use the coherence, entanglement, and tensor structure of quantum systems to build new, as yet unimagined, types of devices

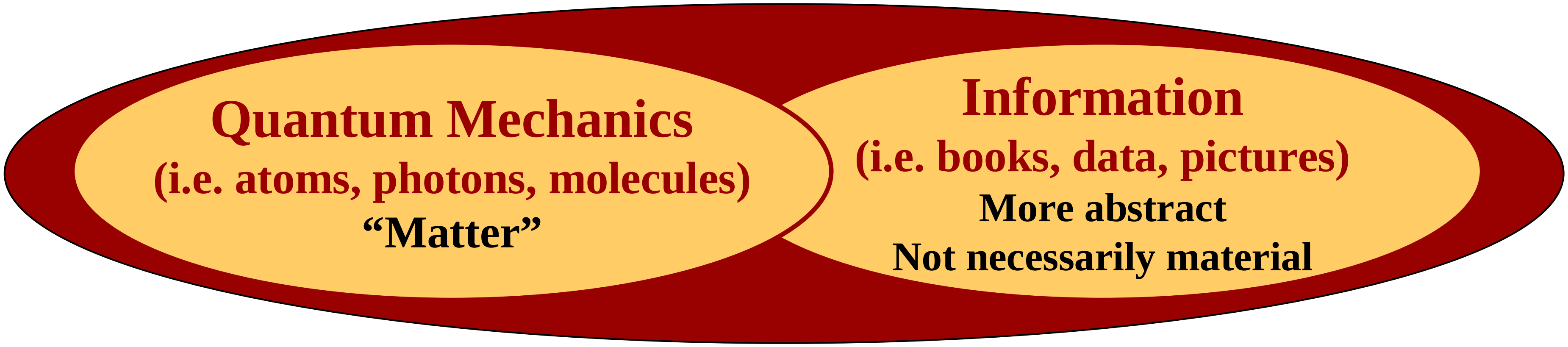
<u>Let me speculate</u>: Quantum engineering will come and will allow us to extend the Moore's Law paradigm based not on making things smaller but making them more powerful by using the laws of quantum mechanics.

VIII. Conclusions

What is Quantum Information?

A radical departure in information technology, more fundamentally different from current IT than the digital computer is from the abacus.

A convergence of two of the 20th Century's great revolutions



Quantum Mechanics

(i.e. atoms, photons, molecules)

"Matter"

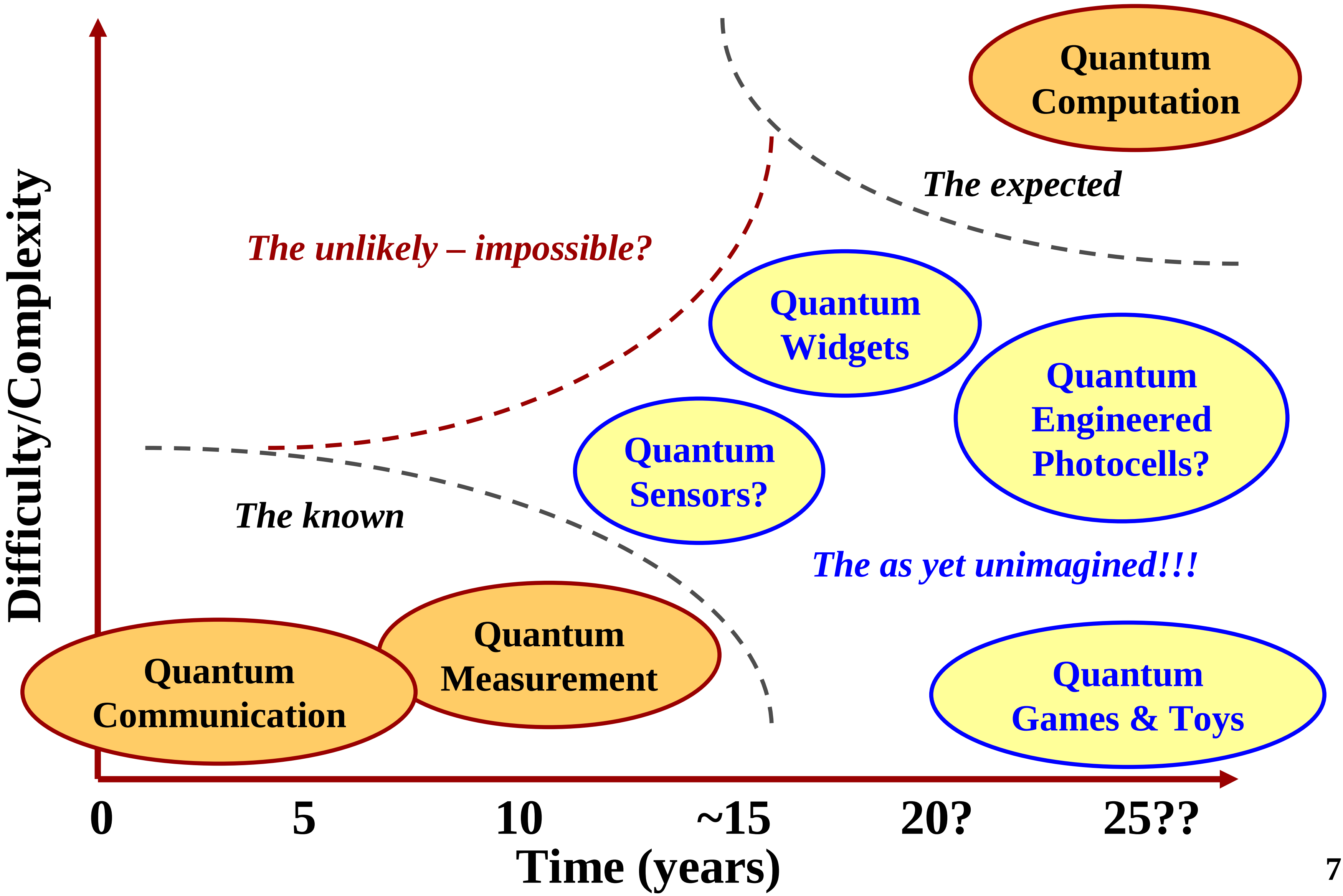
Information

(i.e. books, data, pictures)

More abstract

Not necessarily material

Quantum Information Timeline



Quantum Mechanics Summary

Quantum Mechanics at its simplest level reduces to solving a differential equation that determines the time evolution of quantum system. This equation includes the Hamiltonian *H* which describes a systems kinetic and potential energies. The solution of this equation is a wavefunction$\Psi(r,t)$ which can be more briefly written as the "ket" $\left|\Psi(t)\right\rangle$. The wavefunction along with *H*, *fully describes the system*.

This mathematical view of quantum mechanics has been confirmed experimentally – an untold number of times.

Note a "ket" is nothing but a vector. The same is true of a "bra" $\left\langle\Psi(t)\right|$.

The next few pages provides a "physics" and "mathematics" view of quantum mechanics. I will not do justice to either group. *The key point is that bra's and ket's are vectors*.