GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

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Abstract. In two recent papers ([GZ1] [GZ2]), we provided solutions to the well-known unsolved problem of constructing sufficiency classes of functions in $\mathbb{H}[\mathbb{R}^3]^3$ and $\mathbb{V}[\mathbb{R}^3]^3$, which would allow global, in time, strong solutions to the three-dimensional Navier-Stokes equations. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. In both previous papers, our solution was restricted to functions defined on a bounded open domain of class $\mathbb{C}^3$ contained in $\mathbb{R}^3$. In this paper, we study this problem for functions defined on all of $\mathbb{R}^3$. We prove that, under appropriate conditions, there exists a positive constant a and a number $\mathbf{u}_+$, depending only on the domain, the viscosity, the body forces and the eigenvalues of the "Hermite" Stokes operator (defined below) such that, for all functions in a dense set $\mathbb{D}$ contained in the closed ball $\mathbb{B}(\mathbb{R}^3)$ of radius $(1/2)\mathbf{u}_+$ in $\mathbb{H}[\mathbb{R}^3]^3$, the Navier-Stokes equations have unique strong solutions in $\mathbb{C}^1\left((0,\infty),\mathbb{H}[\mathbb{R}^3]^3\right)$.

Introduction

Let $\mathbb{L}^2[\mathbb{R}^3]^3$ be the real Hilbert space of square integrable functions on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, and let $\mathbb{H}_0[\mathbb{R}^3]^3$ be the completion of the set of functions in

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$\{\mathbf{u} \in \mathbb{C}_0^\infty[\mathbb{R}^3]^3 \mid \nabla \cdot \mathbf{u} = 0\}$ which vanish at infinity with respect to the inner product of $\mathbb{L}^2[\mathbb{R}^3]^3$, and let $\mathbb{V}_0[\mathbb{R}^3]^3$ be the completion of the above functions which vanish at infinity with respect to the inner product of $\mathbb{H}_0^1[\mathbb{R}^3]$, the functions in $\mathbb{H}_0[\mathbb{R}^3]^3$ with weak derivatives in $(\mathbb{L}^2[\mathbb{R}^3])^3$. The global in time classical Navier-Stokes initial-value problem (on $\mathbb{R}^3$ and all $T > 0$) is to find functions $\mathbf{u} : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$
\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0, T) \times \mathbb{R}^3, \\
&\nabla \cdot \mathbf{u} = 0 \text{ in } (0, T) \times \mathbb{R}^3 \text{ (in the weak sense)}, \\
&\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t, \mathbf{x}) = 0 \text{ on } (0, T) \times \mathbb{R}^3, \\
&\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3.
\end{aligned} \tag{1}
$$

The equations describe the time evolution of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ and the pressure p of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient ν in terms of a given initial velocity $\mathbf{u}_0(\mathbf{x})$ and given external body forces $\mathbf{f}(\mathbf{x}, t)$. (Note that our third condition, $\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t, \mathbf{x}) = 0$ on $(0, T) \times \mathbb{R}^3$, is natural in this case since it is well-known that $\mathbb{H}_0^k[\mathbb{R}^3]^3 = \mathbb{H}^k[\mathbb{R}^3]^3$ (see Stein [S] or [SY].)

Purpose

Let $\mathbb{P}$ be the (Leray) orthogonal projection of $(\mathbb{L}^2[\mathbb{R}^3])^3$ onto $\mathbb{H}_0[\mathbb{R}^3]^3$ and define the Stokes operator by: $\mathbf{A}\mathbf{u} =: -\mathbb{P}\Delta\mathbf{u}$, for $\mathbf{u} \in D(\mathbf{A}) \subset \mathbb{H}_0^2[\mathbb{R}^3]^3$, the domain of $\mathbf{A}$. Let $\mathbf{B}\mathbf{u} =: 1/2\mathbb{P}(-\Delta + |\mathbf{x}|^2)\mathbf{u}$ for $\mathbf{u} \in D(\mathbf{B})$. We call $\mathbf{B}$ the Hermite-Stokes operator. The purpose of this paper is to prove that there exists a number $\mathbf{u}_+$, depending only on $\mathbf{A}$, $\mathbf{B}$, f, ν and $\mathbb{R}^3$, such that, for all functions in $\mathbb{D} = D(\mathbf{A}) \cap \mathbb{B}(\mathbb{R}^3)$, where

$\mathbb{B}(\mathbb{R}^3)$ is the closed ball of radius $\mathbf{u}_+$ in $\mathbb{H}_0(\mathbb{R}^3)^3$, the Navier-Stokes equations have unique strong solutions in $\mathbf{u} \in L^{\infty}_{\text{loc}}[[0,\infty); \mathbb{V}_0(\mathbb{R}^3)^3] \cap \mathbb{C}^1[(0,\infty); \mathbb{H}_0(\mathbb{R}^3)^3$.

Preliminaries

In terms of notation and convention, we follow Sell and You [SY]. In order to simplify notation, we let $\mathbb{H}$ denote $\mathbb{H}_0[\mathbb{R}^3]^3$ and $\mathbb{V}$ denote $\mathbb{V}_0[\mathbb{R}^3]^3$. Our use of the Fourier transform follows the definition of Rudin [RU]: $\mathfrak{F}(h) = \frac{1}{[2\pi]^{3/2}} \int_{\mathbb{R}^3} e^{i\mathbf{x}\cdot\mathbf{y}} h(\mathbf{y}) d\mathbf{y}$, so that no factors of 2π appear in the transform pairs. In order to simplify our proofs, we always assume that all functions $\mathbf{u}$, $\mathbf{v}$ are in $D(\mathbf{A})$ and, as in [GZ2], we take $c = max\{c_i\}$, where c_i is one of the nine positive constants that appear on pages 363-367 in [SY]. It will also be convenient to use the fact that the norms of $\mathbb{V}$ and $\mathbb{V}^{-1}$ are equivalent in their respective graph norms relative to $\mathbb{H}$.

The Stokes Operator

It is known that $\mathbf{A}$ is a nonnegative linear operator which generates an analytic contraction semigroup. It follows that the fractional powers $\mathbf{A}^{1/2}$ and $\mathbf{A}^{-1/2}$ are well defined. Moreover, it is also known (cf., [SY], [T1]) that the norms $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}$ and $\left\|\mathbf{A}^{-1/2}\mathbf{u}\right\|_{\mathbb{H}}$ are equivalent to the corresponding norms induced by the Sobolev space $(H^1[\mathbb{R}^3])^3$, so that:

$$\|\mathbf{u}\|_{\mathbb{V}} \equiv \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}} \text{ and } \|\mathbf{u}\|_{\mathbb{V}^{-1}} \equiv \left\|\mathbf{A}^{-1/2}\mathbf{u}\right\|_{\mathbb{H}}. \tag{2}$$

In addition, $\mathbf{A}$ is an isomorphism from $D(\mathbf{A}) \xrightarrow{onto} D(\mathbf{A}^{-1})$. Furthermore, the embeddings $\mathbb{V} \to \mathbb{H} \to \mathbb{V}^{-1}$ are continuous, and it is easy to see that $\mathbf{A}^{-1}$ is the projection of an operator represented by the Riesz potential, mapping $D(\mathbf{A}^{-1})$

onto $D(\mathbf{A})$ (see Stein [S]). Applying the Leray projection to equation (1), with $\mathbf{C}(\mathbf{u},\mathbf{u}) = \mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$, we can recast equation (1) in the standard form:

$$
\begin{aligned}
&\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} - \mathbf{C}(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t) \text{ in } (0,T)\times\mathbb{R}^3,\\
&\nabla\cdot\mathbf{u} = 0 \text{ in } (0,T)\times\mathbb{R}^3,\\
&\lim_{\|\mathbf{x}\|\to\infty} \mathbf{u}(t,\mathbf{x}) = 0 \text{ on } (0,T)\times\mathbb{R}^3,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3,
\end{aligned} \tag{3}
$$

where we have used the fact that the orthogonal complement of $\mathbb{H}[\mathbb{R}^3]$ relative to $(\mathbb{L}^2)[\mathbb{R}^3])^3$ is $\{\mathbf{v} \;:\; \mathbf{v} = \nabla q,\ q \in (H^1[\mathbb{R}^3])^3\}$ to eliminate the pressure term (see Galdi [GA] or [SY, T1, T2]). Theorem 1 below will be used to get our basic estimate in Theorem 3. This result is a simple extension of the bounded domain case first proved by Constantin and Foiaş [CF].

Theorem 1. *Let* $\alpha_i, 1\le i\le 3,$ *satisfy* $0\le\alpha_1\le 3,\ 0\le\alpha_2\le 2,\ 0\le\alpha_3\le 3,$ *with* $\alpha_1+\alpha_2+\alpha_3\ge 3/2$ *and*

$$(\alpha_1,\alpha_2,\alpha_3)\notin\{(3/2,0,0),(0,3/2,0),(0,0,3/2)\}.$$

Then there is a positive constant $c = c(\alpha_i)$ *such that*

$$|\langle \mathbf{C}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H}}| \leqslant c\left\|\mathbf{A}^{\alpha_1/2}\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}^{(1+\alpha_2)/2}\mathbf{v}\right\|_{\mathbb{H}}\left\|\mathbf{A}^{\alpha_3/2}\mathbf{w}\right\|_{\mathbb{H}}.$$

We shall make use of the following interpolation inequality: (see Sell and You [SY], page 363)

$$\|\mathbf{A}^{\gamma}\mathbf{u}\|_{\mathbb{H}} \leqslant c\,\|\mathbf{A}^{\alpha}\mathbf{u}\|_{\mathbb{H}}^{\theta}\left\|\mathbf{A}^{\beta}\mathbf{u}\right\|_{\mathbb{H}}^{(1-\theta)}$$

for all $\mathbf{u}\in D(\mathbf{A}^{\alpha})$, where $\gamma = \theta\alpha + (1-\theta)\beta,\ \alpha,\beta,\gamma\in\mathbb{R},\ 0\le\theta\le 1$ and $\beta\leqslant\alpha$.

The Hermite-Stokes Operator

The operator $\hat{\mathbf{B}} = 1/2(-\Delta + |\mathbf{x}|^{\mathbf{2}})$ is the three-dimensional version of the standard harmonic oscillator operator, which generates the Hermite functions (products of the Hermite polynomials by $e^{-x^2/2}$) as eigenfunctions for the eigenvalue problem on $\mathbb{R}$, (see Hermite [HR], Appell and Kamé de Fériet [AK], and Magnus, Oberhettinger and Soni [MOS]). It is easy to show directly, by separation of variables, that the solution to the 3-dimensional problem is the product of the solutions to the 1-dimensional problem, while the eigenvalues for the 3-dimensional Hermite polynomials are the sums of those for the 1-dimensional polynomials. Furthermore, $\hat{\mathbf{B}}$, and hence $\mathbf{B} = \mathbb{P}\hat{\mathbf{B}}$, is positive with a compact inverse, while $\mathbf{A}$ has an unbounded inverse on $\mathbb{H}_0(\mathbb{R}^3)^3$. It turns out that $\hat{\mathbf{B}}$ is "natural" for $\mathbb{R}^3$ in the sense that it is the only positive self-adjoint (sectorial) operator of lowest degree that is invariant under both rotations and Fourier transformations. (This is actually true for $\mathbb{R}^n,\ n \geq 1$.)

We will have need of the fact that every function $\mathbf{h}(t) \in \mathbb{H}$ has an expansion in terms of the eigenfunctions of $\mathbf{B}$ so that, for example, $\mathbf{B}^{-\beta}\,\mathbf{h}(t) = \sum_{k=1}^{\infty} \lambda_k^{-\beta} h_k(t)\mathbf{e}^k(\mathbf{x})$ and, from here, it is easy to see that $\left\|\mathbf{B}^{-\beta}\,\mathbf{h}(t)\right\|_{\mathbb{H}} \leq \lambda_1^{-\beta} \|\,\mathbf{h}(t)\|_{\mathbb{H}}$, where λ_1^{-1} is the largest eigenvalue of $\mathbf{B}^{-1}$. We also need the following result for our basic Theorem.

Lemma 2. $D(\mathbf{A}) = \mathbf{D}(\mathbf{B})$.

Proof. If we define a norm on $D(\mathbf{A})$ by $\|\mathbf{u}\|_{\mathbf{A}} = \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}$, then $(D(\mathbf{A})\,,\ \|\cdot\|_{\mathbf{A}})$ is a Hilbert space. Now note that the Fourier transform $\mathfrak{F}(\cdot)$ is an isometric isomorphism on $(D(\mathbf{A})\,,\ \|\cdot\|_{\mathbf{A}})$ to $\left(D(\mathbb{P}\,|\mathbf{x}|^2)\,,\ \|\cdot\|_{\mathbf{A}}\right)$, since $\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} = \|\mathfrak{F}(\mathbf{A}\mathbf{u})\|_{\mathbb{H}} =$

$\left\| \mathbb{P} \left| \mathbf{x} \right|^2 \hat{\mathbf{u}} \right\|_{\mathbb{H}}$. It is now easy to see that $D(\mathbf{A}) = D(\mathbb{P} \left| \mathbf{x} \right|^2)$. From this, it follows that $D(\mathbf{A}) = D(\mathbf{B})$. □

It follows from the above lemma that $(\mathbf{AB})^{-\delta}$ is bounded for $\delta > 0$. The following estimate is equation 61.24.1 on page 366 in Sell and You [SY]. If we set $\alpha_1 = 1, \alpha_2 = 1/2$, and $\alpha_3 = 0$ in Theorem 1, along with the interpolation inequality, we get that

$$\left| \langle \mathbf{C}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{\mathbb{H}} \right| \leqslant c \left\| \mathbf{A}^{1/2} \mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{A} \mathbf{v} \right\|_{\mathbb{H}} \left\| \mathbf{w} \right\|_{\mathbb{H}}. \tag{4}$$

Theorem 3. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{H}$, *and let* $\varepsilon > 0$ *be arbitrary. Then, for* $\delta = 1/4 + \varepsilon/2$, *we have that:*

$$\left| \left\langle (\mathbf{AB})^{-(1+\delta)} \mathbf{C}(\mathbf{u}, \mathbf{v}), \mathbf{w} \right\rangle_{\mathbb{H}} \right| \leqslant c \lambda_1^{-(1+\delta)} \left\| \mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{v} \right\|_{\mathbb{H}} \left\| \mathbf{w} \right\|_{\mathbb{H}}. \tag{5}$$

Proof. Using the self-adjoint property of $\mathbf{A}$, and integration by parts, we have

$$\left\langle \mathbf{A}^{-\beta} \mathbf{C}(\mathbf{u}, \mathbf{v}), \mathbf{h} \right\rangle_{\mathbb{H}} = \left\langle \mathbf{C}(\mathbf{u}, \mathbf{v}), \mathbf{A}^{-\beta} \mathbf{h} \right\rangle_{\mathbb{H}} = - \left\langle \mathbf{C}(\mathbf{u}, \mathbf{A}^{-\beta} \mathbf{h}), \mathbf{v} \right\rangle_{\mathbb{H}}.$$

It now follows from Theorem 1 that:

$$\left| \left\langle \mathbf{A}^{-\beta} \mathbf{C}(\mathbf{u}, \mathbf{v}), \mathbf{h} \right\rangle_{\mathbb{H}} \right| \leqslant c \left\| \mathbf{A}^{\alpha_1/2} \mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{-\beta + (1+\alpha_2)/2} \mathbf{h} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{\alpha_3/2} \mathbf{v} \right\|_{\mathbb{H}}.$$

If we set $\beta = 1 + \delta, \ \alpha_1 = \alpha_3 = 0$, we have

$$\left| \left\langle \mathbf{A}^{-(1+\delta)} \mathbf{C}(\mathbf{u}, \mathbf{v}), \mathbf{h} \right\rangle_{\mathbb{H}} \right| \leqslant c \left\| \mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{v} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{(\alpha_2 - 1 - 2\delta)/2} \mathbf{h} \right\|_{\mathbb{H}}.$$

With $\delta = 1/4 + \varepsilon/2$, we get that, for the last term to reduce to $\left\| \mathbf{h} \right\|_{\mathbb{H}}$, we can set $\alpha_2 = 3/2 + \varepsilon$. It follows that the conditions of Theorem 1 are satisfied if $3/2 + \varepsilon < 2$. Thus, it suffices to assume that $\varepsilon < 1/2$, which we will do in the rest of the paper without comment. Our proof is completed by taking $\mathbf{h} = \mathbf{B}^{-\beta}\mathbf{w}$, and the fact that $\left\|\mathbf{B}^{-\beta}\mathbf{w}\right\|_{\mathbb{H}} \leq \lambda_1^{-\beta} \left\|\mathbf{w}\right\|_{\mathbb{H}}$. □

Example 4. *If we use Theorem 1, with $\alpha_1 = 5/4$, $\alpha_2 = 1/4$, and $\alpha_3 = 0$, along with the interpolation inequality, and the fact that $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}} \leqslant \left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}$ we have that, for all $\mathbf{u}, \mathbf{v} \in D(\mathbf{A})$,*

$$
\begin{aligned}
\left\|\mathbf{C}(\mathbf{u},\mathbf{v})\right\|_{\mathbb{H}} &\leqslant c \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}^{3/4} \left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}^{1/4} \left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}}^{3/4} \left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}^{1/4} \\
&\leqslant c \left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}} \left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}.
\end{aligned} \tag{6}
$$

A better estimate is possible, but for our use, equation (6) will suffice.

Definition 5. *We say that the operator $\mathbf{J}(\cdot, t)$ is (for each t)*

(1) *0-Dissipative if $\langle \mathbf{J}(\mathbf{u}, t), \mathbf{u}\rangle_{\mathbb{H}} \leq 0$.*

(2) *Dissipative if $\langle \mathbf{J}(\mathbf{u}, t) - \mathbf{J}(\mathbf{v}, t), \mathbf{u} - \mathbf{v}\rangle_{\mathbb{H}} \leq 0$.*

(3) *Strongly dissipative if there exists an $\alpha > 0$ such that*

$$\langle \mathbf{J}(\mathbf{u}, t) - \mathbf{J}(\mathbf{v}, t), \mathbf{u} - \mathbf{v}\rangle_{\mathbb{H}} \leq -\alpha \left\|\mathbf{u} - \mathbf{v}\right\|_{\mathbb{H}}^2.$$

(4) *Uniformly dissipative if there exists a strictly monotone increasing function $a(t)$ with $a(0) = 0$, $\lim_{t\to\infty} a(t) = \infty$, and:*

$$\langle \mathbf{J}(\mathbf{u}, t) - \mathbf{J}(\mathbf{v}, t), \mathbf{u} - \mathbf{v}\rangle_{\mathbb{H}} \leq -a\left(\left\|\mathbf{u} - \mathbf{v}\right\|_{\mathbb{H}}\right) \left\|\mathbf{u} - \mathbf{v}\right\|_{\mathbb{H}}.$$

Note that, if $\mathbf{J}(\cdot, t)$ is a linear operator, definitions 1) and 2) coincide. Theorem 6 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35, page 887 in, Vol. IIB], while Theorem 7 is from Miyadera [M, p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]) .

Theorem 6. *Let $\mathbb{B}[\mathbb{R}^3]$ be a closed, bounded, convex subset of $\mathbb{H}[\mathbb{R}^3]$. If $\mathbf{J}(\cdot, t) : \mathbb{B}[\mathbb{R}^3] \to \mathbb{H}[\mathbb{R}^3]$ is closed and strongly dissipative for each fixed $t \geq 0$ then, for each $\mathbf{b} \in \mathbb{B}[\mathbb{R}^3]$, there is a $\mathbf{u} \in \mathbb{B}[\mathbb{R}^3]$ with $\mathbf{J}(\mathbf{u}, t) = \mathbf{b}$ (e.g., the range, Ran[$\mathbf{J}(\cdot, t)] \supset \mathbb{B}[\mathbb{R}^3]$).*

Theorem 7. *Let $\{\ \mathcal{A}(t), t \in I = [0, \infty)\}$ be a family of operators defined on $\mathbb{H}[\mathbb{R}^3]$ with domains $D(\mathcal{A}(t)) = D$, independent of t. We assume that $\mathbb{D} = D \cap \mathbb{B}[\mathbb{R}^3]$ is a closed convex set (in an appropriate topology):*

(1) *The operator $\mathcal{A}(t)$ is the generator of a contraction semigroup for each $t \in I$.*
(2) *The function $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables on $I \times \mathbb{D}$.*

Then, for every $\mathbf{u}_0 \in \mathbb{D}$, the problem $\partial_t \mathbf{u}(t, \mathbf{x}) = \mathcal{A}(t)\mathbf{u}(t, \mathbf{x})$, $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$, has a unique solution $\mathbf{u}(t, \mathbf{x}) \in \mathbb{C}^1(I; \mathbb{D})$.

M-Dissipative Conditions

Let us assume that $\mathbf{f}(t) \in L^{\infty}[[0, \infty); \mathbb{H}]$ and is Lipschitz continuous in t, with $\left\|\mathbf{f}(t) - \mathbf{f}(\tau)\right\|_{\mathbb{H}} \leq d\left|t - \tau\right|^{\theta}$, $d > 0$, $0 < \theta < 1$. With δ as in Theorem 3, we can rewrite equation (3) in the form:

$$\begin{gathered}\partial_t \mathbf{u} = \nu(\mathbf{AB})^{1+\delta}\mathbf{J}(\mathbf{u}, t) \text{ in } (0, T) \times \Omega, \\ \mathbf{J}(\mathbf{u}, t) = -\mathbf{B}^{-(1+\delta)}\mathbf{A}^{-\delta}\mathbf{u} - \nu^{-1}(\mathbf{AB})^{-(1+\delta)}\mathbf{C}(\mathbf{u}, \mathbf{u}) + \nu^{-1}(\mathbf{AB})^{-(1+\delta)}\mathbb{P}\mathbf{f}(t).\end{gathered} \tag{7}$$

Approach

We begin with a study of the operator $\mathbf{J}(\cdot, t)$, for fixed t, and seek conditions depending on $\mathbf{A}, \mathbf{B}$, ν, and $\mathbf{f}(t)$ which guarantee that $\mathbf{J}(\cdot, t)$ is m-dissipative for each t. Clearly $\mathbf{J}(\cdot, t) : D[(\mathbf{AB})^{(\mathbf{1}+\delta)}] \xrightarrow{onto} D[(\mathbf{AB})^{(1+\delta)}]$ and, since $\nu(\mathbf{AB})^{(1+\delta)}$

is a closed positive (m-accretive) operator (so that $-(\mathbf{AB})^{(1+\delta)}$ generates a linear contraction semigroup), we expect that $\nu(\mathbf{AB})^{(1+\delta)}J(\cdot,t)$ will be m-dissipative for each t.

Theorem 8. *For $t \in I = [0,\infty)$ and, for each fixed $\mathbf{u} \in \mathbb{H}$, $\mathbf{J}(\mathbf{u},t)$ is Lipschitz continuous, with $\|\mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},\tau)\|_{\mathbb{H}} \leq d' \left|t-\tau\right|^{\theta}$, where $d' = d\nu^{-1}a^{-(1+\delta)}$, d is the Lipschitz constant for the function $\mathbf{f}(t)$ and $a^{-(1+\delta)} = \left\|(\mathbf{AB})^{-(1+\delta)}\right\|_{\mathbb{H}}$.*

Proof. For fixed $\mathbf{u} \in \mathbb{H}$,

$$\|\mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},\tau)\|_{\mathbb{H}} = \nu^{-1}\left\|(\mathbf{AB})^{-(1+\delta)}[\mathbb{P}\mathbf{f}(t) - \mathbb{P}\mathbf{f}(\tau)]\right\|_{\mathbb{H}}$$
$$\leq d\nu^{-1}a^{-(1+\delta)}\left|t-\tau\right|^{\theta} = d'\left|t-\tau\right|^{\theta}.$$

□

Main Results

Theorem 9. *Let $f = \sup_{t\in\mathbf{R}^+}\|\mathbb{P}\mathbf{f}(t)\|_{\mathbb{H}} < \infty$, then there exists a positive constant $\mathbf{u}_+$, depending only on f, $\mathbf{A}$, $\mathbf{B}$ and ν such that, for all $\mathbf{u}$ with $\|\mathbf{u}\|_{\mathbb{H}} \leq \mathbf{u}_+$, $\mathbf{J}(\cdot,t)$ is strongly dissipative.*

Proof. The proof of our first assertion has two parts. First, we require that the nonlinear operator $\mathbf{J}(\cdot,t)$ be 0-dissipative, which gives us an upper bound $\mathbf{u}_+$ in terms of the norm (e.g., $\|\mathbf{u}\|_{\mathbb{H}} \leqslant \mathbf{u}_+$). We then use this part, and the fact that $\|\mathbf{u}\|_{\mathbb{H}} \leqslant \|\mathbf{Au}\|_{\mathbb{H}}$, to show that $\mathbf{J}(\cdot,t)$ is strongly dissipative on the closed ball, $\mathbb{B}_+ = \{\mathbf{u} \in \mathbb{H} : \|\mathbf{Au}\|_{\mathbb{H}} \leqslant (1/2)\mathbf{u}_+\}$.

Part 1) From equation (5), we consider the expression

$$\begin{aligned}\left\langle \mathbf{J}(\mathbf{u},t), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} &= -\left\langle \mathbf{B}^{-1}(\mathbf{AB})^{-\delta}\mathbf{u}, (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} \\ &\quad + \nu^{-1}\left\langle -(\mathbf{AB})^{-(1+\delta)}\mathbf{C}(\mathbf{u},\mathbf{u}) + (\mathbf{AB})^{-(1+\delta)}\mathbb{P}\mathbf{f}(t), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} \\ &= -\left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}}^2 - \nu^{-1}\left\langle (\mathbf{AB})^{-(1+\delta)}\mathbf{C}(\mathbf{u},\mathbf{u}), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} + \nu^{-1}\left\langle (\mathbf{AB})^{-(1+\delta)}\mathbb{P}\mathbf{f}(t), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} \\ &= -\left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}}^2 - \nu^{-1}\left\langle \mathbf{C}((\mathbf{AB})^{-(1+\delta)}\mathbf{u},\mathbf{u}), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} + \nu^{-1}\left\langle (\mathbf{AB})^{-(1+\delta)}\mathbb{P}\mathbf{f}(t), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}}.\end{aligned}$$

It follows that

$$\begin{aligned}\left\langle \mathbf{J}(\mathbf{u},t), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}} &\leqslant -\left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}}^2 + \nu^{-1}\left|\left\langle \mathbf{C}((\mathbf{AB})^{-(1+\delta)}\mathbf{u},\mathbf{u}), (\mathbf{AB})^{-\delta}\mathbf{u}\right\rangle_{\mathbb{H}}\right| \\ &\quad + \nu^{-1}a^{-(1+\delta)}f\left\|(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}} \\ &\leqslant -\left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}}^2 + ca^{-\delta}(\nu\lambda_1^{(1+\delta)})^{-1}\left\|\mathbf{u}\right\|_{\mathbb{H}}^3 + \nu^{-1}a^{-(1+2\delta)}f\left\|\mathbf{u}\right\|_{\mathbb{H}}.\end{aligned}$$

In the last line, we used our estimate from Theorem 3. We now choose the first eigenvalue $\lambda_n,\ n \geq 1$, and number ω such that

(1) $\lambda_n^{-1/2}a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}} \leqslant \left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}} \leqslant \lambda_1^{-1/2}a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}}$,

(2) $\lambda_1^{-\omega/2}a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}} \leqslant \left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}} \leqslant \lambda_1^{-1/2}a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}}$,

and let $\lambda_0^{-1} = \max\{\lambda_n^{-1}, \lambda_1^{-\omega}\}$. It then follows that $-\lambda_0^{-1}a^{-2\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}}^2 \geqslant -\left\|\mathbf{B}^{-1/2}(\mathbf{AB})^{-\delta}\mathbf{u}\right\|_{\mathbb{H}}^2$. Thus, $\mathbf{J}(\cdot,t)$ will be 0-dissipative if

$$-\lambda_0^{-1}a^{-2\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}}^2 + ca^{-\delta}(\nu\lambda_1^{(1+\delta)})^{-1}\left\|\mathbf{u}\right\|_{\mathbb{H}}^3 + (\nu a^{(1+2\delta)})^{-1}f\left\|\mathbf{u}\right\|_{\mathbb{H}} \leqslant 0,$$

so that

$$a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}}\left[c(\nu\lambda_1^{(1+\delta)})^{-1}\left\|\mathbf{u}\right\|_{\mathbb{H}}^2 - \lambda_0^{-1}a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}} + (\nu a^{(1+\delta)})^{-1}f\right] \leqslant 0. \tag{8}$$

Since $\left\|\mathbf{u}\right\|_{\mathbb{H}} > 0$, we have that $\mathbf{J}(\cdot,t)$ is 0-dissipative if

$$c(\nu\lambda_1^{(1+\delta)})^{-1}\left\|\mathbf{u}\right\|_{\mathbb{H}}^2 - \lambda_0^{-1}a^{-\delta}\left\|\mathbf{u}\right\|_{\mathbb{H}} + (\nu a^{(1+\delta)})^{-1}f \leqslant 0.$$

Solving, we get that

$$\mathbf{u}_{\pm} = \tfrac{\nu\lambda_1^{1+\delta}}{2c\lambda_0 a^{\delta}} \left\{ 1 \pm \sqrt{1 - (4c\lambda_0^2 f)\Big/(\nu^2 a^{(1-\delta)}\lambda_1^{(1+\delta)})} \right\} = \tfrac{\nu\lambda_1^{1+\delta}}{2c\lambda_0 a^{\delta}} \left\{ 1 \pm \sqrt{1-\gamma} \right\},$$

where $\gamma = (4c\lambda_0^2 f)\Big/(\nu^2 a^{(1-\delta)}\lambda_1^{(1+\delta)})$. Since we want real distinct solutions, we must require that

$$\gamma = (4c\lambda_0^2 f)\Big/(\nu^2 a^{(1-\delta)}\lambda_1^{(1+\delta)}) < 1 \Rightarrow \nu^2 a^{(1-\delta)}\lambda_1^{(1+\delta)} > 4c\lambda_0^2 f$$
$$\Rightarrow \ \nu > 2\lambda_0 a^{-(1-\delta)/2}\lambda_1^{-(1+\delta)/2}(cf)^{1/2}.$$

It follows that, if $\mathbb{P}\mathbf{f} \neq \mathbf{0}$, then $\mathbf{u}_- < \mathbf{u}_+$, and our requirement that $\mathbf{J}$ is 0-dissipative implies that, since our solution factors as $(\|\mathbf{u}\|_{\mathbb{H}} - \mathbf{u}_+)(\|\mathbf{u}\|_{\mathbb{H}} - \mathbf{u}_-) \leq 0$, we must have that:

$$\|\mathbf{u}\|_{\mathbb{H}} - \mathbf{u}_+ \leq 0, \ \ \|\mathbf{u}\|_{\mathbb{H}} - \mathbf{u}_- \geq 0.$$

First observe that terms of the form $(\mathbf{AB})^{-\delta}\mathbf{u}$ are dense. Then note that $\mathbf{J}(\mathbf{u}, t)$ is closed, and the dissipative nature of an operator is determined on a dense set. It follows that, for $\mathbf{u}_- \leq \|\mathbf{u}\|_{\mathbb{H}} \leq \mathbf{u}_+$, $\langle \mathbf{J}(\mathbf{u}, t), \mathbf{u}\rangle_{\mathbb{H}} \leq 0$. (It is clear that, when $\mathbb{P}\mathbf{f}(t) = \mathbf{0}, \mathbf{u}_- = \mathbf{0}$, and $\mathbf{u}_+ = \nu(c\lambda_0 a^{\delta})^{-1}\lambda_1^{(1+\delta)}$.)

Part 2): Now, for any $\mathbf{u}, \mathbf{v} \in \mathbb{H}$ with $\max(\ \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}, \|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}) \leq (1/2)\mathbf{u}_{+}$, we have that

$$\left\langle \mathbf{J}(\mathbf{u}, t) - \mathbf{J}(\mathbf{v}, t), (\mathbf{AB})^{-\delta}(\mathbf{u}-\mathbf{v}) \right\rangle_{\mathbb{H}} = -\left\| \mathbf{B}^{-\mathbf{1/2}}(\mathbf{AB})^{-\delta}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}}^{2}$$

$$- \nu^{-1} \left\langle (\mathbf{AB})^{-(1+\delta)}[\mathbf{C}(\mathbf{u}, \mathbf{u}-\mathbf{v}) + \mathbf{C}(\mathbf{v}, \mathbf{u}-\mathbf{v})], (\mathbf{AB})^{-\delta}(\mathbf{u}-\mathbf{v}) \right\rangle_{\mathbb{H}}$$

$$\leqslant -\lambda_0^{-1} a^{-2\delta} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 + ca^{-\delta}\nu^{-1}\lambda_1^{-(1+\delta)} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 (\|\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{v}\|_{\mathbb{H}})$$

$$\leq -\lambda_0^{-1} a^{-2\delta} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 + ca^{-\delta}\nu^{-1}\lambda_1^{-(1+\delta)} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 \mathbf{u}_{+}$$

$$= -\lambda_0^{-1} a^{-2\delta} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 + ca^{-\delta}\nu^{-1}\lambda_1^{-(1+\delta)} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 \left(\tfrac{1}{2}\nu\lambda_1^{(1+\delta)}(c^{-1}a^{-\delta}\lambda_0^{-1}) \left\{ 1 + \sqrt{1-\gamma} \right\} \right)$$

$$= -\tfrac{1}{2}\lambda_0^{-1} a^{-2\delta} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2 \left\{ 1 - \sqrt{1-\gamma} \right\}$$

$$= -\alpha \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}^2, \ \alpha = \tfrac{1}{2}\lambda_0^{-1} a^{-2\delta} \left\{ 1 - \sqrt{1-\gamma} \right\}.$$

□

Theorem 10. *The operator $\mathcal{A}(t) = \nu\mathbf{A}^{(1+\delta)}\mathbf{J}(\cdot, t)$ is closed, uniformly dissipative and jointly continuous in $\mathbf{u}$ and t. Furthermore, for each $t \in \mathbf{R}^{+}$ and $\beta > 0$, $Ran[I - \beta\mathcal{A}(t)] \supset \mathbb{B}[\Omega]$, so that $\mathcal{A}(t)$ is m-dissipative on $\mathbb{D}$.*

Proof. Since $\mathbf{J}(\cdot, t)$ is strongly dissipative and closed on $\mathbb{B}$, it follows from Theorem 6 that $Ran[\mathbf{J}(\cdot, t)] \supset \mathbb{B}$.

To show that $\mathcal{A}(t) = \nu(\mathbf{AB})^{(1+\delta)}\mathbf{J}(\cdot, t)$ is uniformly dissipative for $\mathbf{u}, \mathbf{v} \in \mathbb{B}_{+}$, we have

$$\langle \mathcal{A}(t)\mathbf{u} - \mathcal{A}(t)\mathbf{v}, (\mathbf{u}-\mathbf{v}) \rangle_{\mathbb{H}} = -\nu \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H}}^{2}$$

$$- \langle (1/2)[\mathbf{C}(\mathbf{u}-\mathbf{v}, \mathbf{u}) + \mathbf{C}(\mathbf{u}-\mathbf{v}, \mathbf{v})], (\mathbf{u}-\mathbf{v}) \rangle_{\mathbb{H}}.$$

Now, from equation (4),

$$\begin{aligned}&\left|\langle[\mathbf{C}(\mathbf{u}-\mathbf{v},\mathbf{u})+\mathbf{C}(\mathbf{u}-\mathbf{v},\mathbf{v})],(\mathbf{u}-\mathbf{v})\rangle_{\mathbb{H}}\right| \\ &\qquad\qquad \leqslant c\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}}\left\{\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}+\|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}\right\}.\end{aligned}$$

We now use $-\lambda_0^{-1}a^{-\delta}\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}} \geqslant -\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}$, and the fact that the first eigenvalue of $\mathbf{B}$ is $1/2$, so that $\lambda_1^{1+\delta}<1$, to get:

$$\begin{aligned}&\langle\mathcal{A}(t)\mathbf{u}-\mathcal{A}(t)\mathbf{v},\mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}} \leqslant -\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}^2+\tfrac{1}{2}c\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}}\left\{\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}+\|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}\right\} \\ &\quad=\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}\left\{-\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}+\tfrac{1}{2}c\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}\left[\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}+\|\mathbf{A}\mathbf{v}\|_{\mathbb{H}}\right]\right\} \\ &\quad\leqslant\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}\left\{-\nu\lambda_0^{-1}a^{-\delta}+c\mathbf{u}_+\right\} \\ &\quad\leqslant\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}\left\{-\nu\lambda_0^{-1}a^{-\delta}+\tfrac{1}{2}\nu\lambda_1^{(1+\delta)}\lambda_0^{-1}a^{-\delta}\left[1+\sqrt{1-\gamma}\right]\right\} \\ &\quad<\tfrac{1}{2}\nu\lambda_0^{-1}a^{-\delta}\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{H}}\left\{-1+\sqrt{1-\gamma}\right\}<0.\end{aligned}$$

If we set $a\left(\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}}\right)=-\frac{1}{2}\nu\lambda_0^{-1}a^{-\delta}\left[-1+\sqrt{1-\gamma}\right]\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}$, we have that:

$$\langle\mathcal{A}(t)\mathbf{u}-\mathcal{A}(t)\mathbf{v},\mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}}\leqslant -a\left(\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}}\right)\|(\mathbf{u}-\mathbf{v})\|_{\mathbb{H}}.$$

It follows that $\mathcal{A}(t)$ is uniformly dissipative. Since $-\mathbf{A}^{(1+\delta)}$ is m-dissipative, for $\beta>0$, $Ran(I+\beta(\mathbf{AB})^{(1+\delta)})=\mathbb{H}$. As $\mathbf{J}$ is strongly dissipative (in the ball of radius $\frac{1}{2}\mathbf{u}_+$) and closed, with $Ran[\mathbf{J}]\supset\mathbb{B}$, and $\mathbf{J}(\cdot,t):\mathbb{D}\xrightarrow{onto}\mathbb{D}$, $\mathcal{A}(t)$ is maximal dissipative (in the ball of radius $\frac{1}{2}\mathbf{u}_+$), and also closed, so that $Ran[I-\beta\mathcal{A}(t)]\supset\mathbb{B}$. It follows that $\mathcal{A}(t)$ is m-dissipative on $\mathbb{B}$ for each $t\in\mathbf{R}^+$ (since $\mathbb{H}$ is a Hilbert space). To see that $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables, let $\mathbf{u}_n,\mathbf{u}\in\mathbb{B}_+$, $\|\mathbf{A}(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H}}\to 0$, with $t_n, t \in I$ and $t_n \to t$. Then (see equation (6))

$$\|\mathcal{A}(t_n)\mathbf{u}_n - \mathcal{A}(t)\mathbf{u}\|_{\mathbb{H}} \leqslant \|\mathcal{A}(t_n)\mathbf{u} - \mathcal{A}(t)\mathbf{u}\|_{\mathbb{H}} + \|\mathcal{A}(t_n)\mathbf{u}_n - \mathcal{A}(t_n)\mathbf{u}\|_{\mathbb{H}}$$

$$= \|[\mathbb{P}\mathbf{f}(t_n) - \mathbb{P}\mathbf{f}(t)]\|_{\mathbb{H}} + \|\nu\mathbf{A}(\mathbf{u}_n - \mathbf{u}) + [\mathbf{C}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n) + \mathbf{C}(\mathbf{u}, \mathbf{u}_n - \mathbf{u})]\|_{\mathbb{H}}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} + \|\mathbf{C}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n) + \mathbf{C}(\mathbf{u}, \mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} + c\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}\,\{\|\mathbf{A}\mathbf{u}_n\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{u}\|_{\mathbb{H}}\}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}} + +2c\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}\,\mathbf{u}_+.$$

It follows that $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables. $\square$

Since $\mathbb{B}_+$ is the closure of $\mathbb{D} = D(\mathbf{A}) \cap \mathbb{B}$ equipped with the restriction of the graph norm of $\mathbf{A}$ induced on $D(\mathbf{A})$, it follows that $\mathbb{B}_+$ is a closed, bounded, convex set. We now have:

Theorem 11. *For each $T \in \mathbf{R}^+$, $t \in (0, T)$ and $\mathbf{u}_0 \in \mathbb{D} \subset \mathbb{B}$, the global in time Navier-Stokes initial-value problem in $\mathbb{R}^3$:*

$$\begin{aligned} &\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T)\times\mathbb{R}^3, \\ &\nabla\cdot\mathbf{u} = 0 \text{ in } (0,T)\times\mathbb{R}^3, \\ &\lim_{\|\mathbf{x}\|\to\infty} \mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\mathbb{R}^3, \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3, \end{aligned} \tag{9}$$

has a unique strong solution $\mathbf{u}(t, \mathbf{x})$, which is in $L^2_{loc}[[0,\infty);\mathbb{H}^2]$ and in $L^\infty_{loc}[[0,\infty);\mathbb{V}] \cap \mathbb{C}^1[(0,\infty);\mathbb{H}]$.

Proof. Theorem 7 allows us to conclude that, when $\mathbf{u}_0 \in \mathbb{D}$, the initial value problem is solved and the solution $\mathbf{u}(t, \mathbf{x})$ is in $\mathbb{C}^1[(0,\infty);\mathbb{D}]$. Since $\mathbb{D} \subset \mathbb{H}^2$, it follows that

$\mathbf{u}(t, \mathbf{x})$ is also in $\mathbb{V}$, for each $t > 0$. It is now clear that, for any $T > 0$,

$$\int_0^T \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{H}}^2 \, dt < \infty, \text{ and } \sup_{0<t<T} \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{V}}^2 < \infty.$$

This gives our conclusion. $\square$

Discussion

It is known that, if $\mathbf{u}_0 \in \mathbb{V}$, and $\mathbf{f}(t)$ is $L^\infty[(0,\infty), \mathbb{H}]$ then there is a time $T > 0$ such that a weak solution with this data is uniquely determined on any subinterval of $[0, T)$ (see Sell and You, page 396, [SY]). Thus, we also have that:

Corollary 12. *For each $t \in \mathbf{R}^+$ and $\mathbf{u}_0 \in \mathbb{D}$ the Navier-Stokes initial-value problem on $\mathbb{R}^3$:*

$$\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \ \textit{in} \ (0,T) \times \mathbb{R}^3, \\
&\nabla \cdot \mathbf{u} = 0 \ \textit{in} \ (0,T) \times \mathbb{R}^3, \\
&\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t,\mathbf{x}) = \mathbf{0} \ \textit{on} \ (0,T) \times \mathbb{R}^3, \\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \ \textit{in} \ \mathbb{R}^3.
\end{aligned} \tag{10}$$

has a unique weak solution $\mathbf{u}(t, \mathbf{x})$, which is in $L^2_{loc}[[0,\infty); \mathbb{H}^2]$ and in $L^\infty_{loc}[[0,\infty); \mathbb{V}] \cap \mathbb{C}^1[(0,\infty); \mathbb{H}]$.

Since we require that our initial data be in $\mathbb{H}^2$, the conditions for the Leray-Hopf weak solutions are not satisfied. However, it was an open question as to whether these solutions developed singularities, even if $\mathbf{u}_0 \in \mathbb{C}_0^\infty$ (see Giga [G] and references therein). The above Corollary shows that it suffices that $\mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^2$ to insure that the solutions develop no singularities.

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