Preconditioning Strategies for Models of Incompressible Flow

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Incompressible Navier-Stokes Equations

$$\alpha\, u_t - \nu\, \nabla^2 u + (u \cdot \text{grad})u + \text{grad}\, p = f$$
$$-\text{div}\, u = 0$$

α=0 !   steady state problem

α=1 !   evolutionary problem

Discretization and linearization → Matrix equation

$$\mathcal{A}x = b$$

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Goal: Robust general solution algorithms
Easy to implement
Derived from subsidiary building blocks
Adaptible to a variety of scenarios
(steady / evolutionary / Stokes / Boussinesq)

Outline

1. General approach
   preconditioning for saddle point problems
2. Relation to traditional approaches
   projection methods
   SIMPLE
3. Details for Navier-Stokes equations
4. Analytic / experimental results
5. Potential for more general problems

Preliminary: Steady Stokes Equations

$$-\nabla^2 u + \operatorname{grad} p = f$$
$$-\operatorname{div} u = 0$$

Rusten &Winther, 1992
Silvester & Wathen, 1993

Algebraic equation: $\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$ A = discrete vector Laplacian

Symmetric indefinite ! **MINRES** algorithm is applicable

Preconditioning operator: $\begin{pmatrix} A & 0 \\ 0 & Q_S \end{pmatrix}$

Generalized eigenvalue problem: $\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \lambda \begin{pmatrix} A & 0 \\ 0 & Q_S \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix}$

Case C=0: $Au + B^Tp = \lambda\, Au$, $\lambda \neq 1$) $u = 1/(\lambda-1)\, A^{-1}B^Tp$

$Bu = \lambda\, Q_S p$ $BA^{-1}B^Tp = \lambda(\lambda-1)\, Q_S p$

$BA^{-1}B^{T}p=\mu\ Q_{S}p, \quad \mu=\lambda(\lambda-1)$

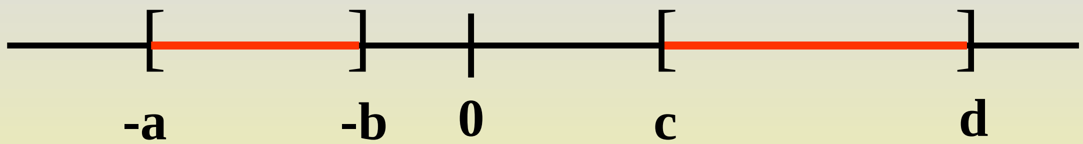
Verfürth, 1984: For Q_S=pressure mass matrix,

$\mu 2\ [a_s,b_s]$

independent of discretization parameter *h*

Under mapping $\mu \mapsto \lambda=1§\ (1+4\mu)^{1/2}$,

λ ½



Convergence bound for **MINRES**: $$\| r_k \| \le 2 \left(\frac{1-\sqrt{(bc)/(ad)}}{1+\sqrt{(bc)/(ad)}} \right)^{k/2} \| r_0 \|$$

Computational requirements, for $\begin{pmatrix} A & 0 \\ 0 & Q_S \end{pmatrix}^{-1}$ times a vector

Poisson solve: can be approximated, e.g. with multigrid

Mass matrix solve: cheap

Generalize to Navier-Stokes Equations

Linearization via Picard iteration (slightly inaccurate notation):

$$\frac{\alpha}{\Delta t} u^{(m+1)} - \nu \nabla^2 u^{(m+1)} + (u^{(m)} \cdot \mathrm{grad}) u^{(m+1)} + \mathrm{grad}\, p^{(m+1)} = f^{(m)}$$
$$- \mathrm{div}\, u^{(m+1)} = 0$$

Discretization $\longrightarrow \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$

Analogue of Stokes strategy: preconditioner $\begin{pmatrix} F & 0 \\ 0 & Q_S \end{pmatrix}$

Same analysis ! $BF^{-1}B^T p = \mu\, Q_S p, \quad \mu = \lambda(\lambda - 1)$

) seek approximation Q_S to Schur complement

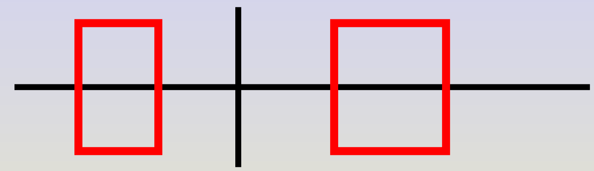
N.B. Same question arises for other strategies for linearization

Suppose Q_S ¼ $BF^{-1}B^T$ so that eigenvalues of

$$BF^{-1}B^T p = \mu\, Q_S p$$

are tightly clustered.

Under mapping $\mu \mapsto \lambda = 1$§ $(1+4\mu)^{1/2}$, eigenvalues λ are clustered in two regions, one on each side of imaginary axis



Can improve this:

Observation: symmetry is important for Stokes solver
MINRES: optimal with fixed cost per step
) need block diagonal preconditioner

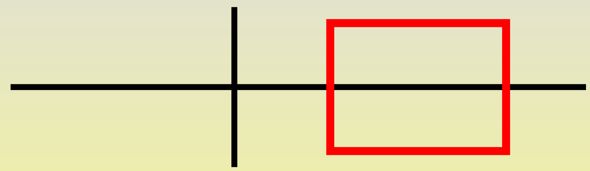
For Navier-Stokes: don't have symmetry
need Krylov subspace method for nonsymmetric matrices (e.g. **GMRES)**

Alternative: block triangular preconditioner $\begin{pmatrix} F & B^T \\ 0 & -Q_S \end{pmatrix}$

! Generalized eigenvalue problem $\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \mu \begin{pmatrix} F & B^T \\ 0 & -Q_S \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix}$

$Fu + B^T p = \mu \, Fu$, $\mu \neq 1$) $u = -F^{-1}B^T p$

$Bu = -\mu \, Q_S p$ $BF^{-1}B^T p = \mu \, Q_S p$

μ ½ {1}[

Theorem (Fischer, Ramage, Silvester, Wathen):

For preconditioned GMRES iteration, let

p_0 be arbitrary and $u_0 = F^{-1}(f - B^T u_0)$ () $r_0 = (0, w_0)$)

$(u_k, p_k)_T$ be generated with block triangular preconditioner,

$(u_k, p_k)_D$ be generated with block diagonal preconditioner.

Then $(u_{2k}, p_{2k})_D = (u_{2k+1}, p_{2k+1})_D = (u_k, p_k)_T$.

Computational requirements, to implement block triangular preconditioner:

$$\text{Compute } \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} F & B^T \\ 0 & -Q_S \end{pmatrix}^{-1} \begin{pmatrix} v \\ q \end{pmatrix}$$

Solve $Q_s\, r = -q$, then solve $Fw = v - B^T r$

The only difference from block diagonal solve:
matrix-vector product $B^T\, r$ (negligible)

For second step: convection-diffusion solve:
can be approximated, e.g. with multigrid

For first step: something new needed: Q_S

One more interpretation:

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix} \begin{pmatrix} F & B^T \\ 0 & -(BF^{-1}B^T + C) \end{pmatrix}$$

$$) \qquad \begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} F & B^T \\ 0 & -(BF^{-1}B^T + C) \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}$$

$$¼ \begin{pmatrix} Q_F & B^T \\ 0 & -Q_S \end{pmatrix}^{-1}$$

Shows what is needed for stabilized discretization:

$$Q_S ¼ BF^{-1}B^T + C$$

Relation to Projection Methods

"Classical" O(Δt) projection method (Chorin 1967, Temam 1969):

Step 1: $$\frac{u^{(*)} - u^{(m)}}{\Delta t} - \nu \nabla^2 u^* + (u^{(m)} \cdot \mathrm{grad}) u^{(m)} = f$$

$$\left(\tfrac{1}{\Delta t} I - \nu \nabla^2\right) u^* = f - (u^{(m)} \cdot \mathrm{grad}) u^{(m)} + \tfrac{1}{\Delta t} u^{(m)}$$

In matrix form: $$\left(\tfrac{1}{\Delta t} M + \nu A\right) u^* = f - N u^{(m)} + \tfrac{1}{\Delta t} M u^{(m)}$$

Step 2: $$\begin{pmatrix} \tfrac{1}{\Delta t} I & \nabla \\ -\nabla & 0 \end{pmatrix} \begin{pmatrix} u^{(m+1)} \\ p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \tfrac{1}{\Delta t} u^* \\ 0 \end{pmatrix}$$

In matrix form: $$\begin{pmatrix} \tfrac{1}{\Delta t} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u^{(m+1)} \\ p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \tfrac{1}{\Delta t} M u^* \\ 0 \end{pmatrix}$$

Performed via pressure-Poisson solve

Substitute u^* from Step 1 into Step 2 :

$$\begin{pmatrix} \frac{1}{\Delta t}M+\nu A & \left(\frac{1}{\Delta t}M+\nu A\right)\left(\frac{1}{\Delta t}M\right)^{-1}B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} u^{(m+1)} \\ p^{(m+1)} \end{pmatrix}=\begin{pmatrix} f-Nu^{(m)}+\frac{1}{\Delta t}Mu^{(m)} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\Delta t}M+\nu A & 0 \\ B & -B\left(\frac{1}{\Delta t}M\right)^{-1}B^T \end{pmatrix}\begin{pmatrix} I & \left(\frac{1}{\Delta t}M\right)^{-1}B^T \\ 0 & I \end{pmatrix}$$

Perot, 1993

Contrast: update derived purely from linearization & discretization:

$$\begin{pmatrix} \frac{1}{\Delta t}M+\nu A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} u^{(m+1)} \\ p^{(m+1)} \end{pmatrix}=\begin{pmatrix} f-Nu^{(m)}+\frac{1}{\Delta t}Mu^{(m)} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\Delta t}M+\nu A & 0 \\ B & -B\left(\frac{1}{\Delta t}M+\nu A\right)^{-1}B^T \end{pmatrix}\begin{pmatrix} I & \left(\frac{1}{\Delta t}M+\nu A\right)^{-1}B^T \\ 0 & I \end{pmatrix}$$

$$\text{Error}: \quad B^T-\left(\tfrac{1}{\Delta t}M+\nu A\right)\left(\tfrac{1}{\Delta t}M\right)^{-1}B^T=-(\Delta t)\nu\, M^{-1}AB^T=O(\Delta t)$$

For higher order accuracy in time and related approaches:

Dukowicz & Dvinsky 1992

Perot 1993

Quarteroni, Saleri &Veneziani 2000

Henriksen & Holman 2002

Relation to SIMPLE

Patankar & Spaulding, 1972

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ B & -BF^{-1}B^T \end{pmatrix} \begin{pmatrix} I & F^{-1}B^T \\ 0 & I \end{pmatrix}$$

¼ $$\begin{pmatrix} Q_F & 0 \\ B & -B\hat{F}^{-1}B^T \end{pmatrix} \begin{pmatrix} I & \hat{F}^{-1}B^T \\ 0 & I \end{pmatrix}$$

Q_F: approximate convection-diffusion solve

$\hat{F}$: diagonal part of F

Many variants

Perspective of New Approach

- Take on Schur complement directly
- Separate time discretization from algebraic algorithm
- Enable flexible treatment of time discretization, linearization
  - Allow choice of linearization
  - Allow large time steps / CFL numbers if circumstances warrant

Approximation for the Schur Complement (I)

Kay, Loghin, Wathen 2002

Suppose the gradient and convection-diffusion operators approximately commute ($w = u^{(m)}$):

$$\nabla(-\nu \nabla^2 + w \cdot \nabla)_p \approx (-\nu \nabla^2 + w \cdot \nabla)_u \nabla$$

↑ Require pressure convection-diffusion operator

Discrete analogue: $M_u^{-1} B^T M_p^{-1} F_p = M_u^{-1} F \, M_u^{-1} B^T$

$$\Rightarrow \quad BF^{-1}B^T = \underbrace{BM_u^{-1}B^T}_{\leftarrow A_p \rightarrow} F_p^{-1} M_p$$

In practice: don't have equality, instead $\boxed{Q_S \equiv A_p F_p^{-1} M_p}$

Requirements: Poisson solve A_p^{-1}

Mass matrix solve M_p^{-1} } for Q_S^{-1}

+ Convection-diffusion solve F^{-1}

Evolutionary Equations

$$u_t - \nu\, \nabla^2 u + (u \cdot \mathrm{grad}) u + \mathrm{grad}\, p = f$$
$$- \mathrm{div}\, u = 0$$

Backward Euler:

$$\frac{u^{(m+1)} - u^{(m)}}{\Delta t} - \nu\, \nabla^2 u^{(m+1)} + (u^{(m)} \cdot \mathrm{grad}) u^{(m+1)} + \mathrm{grad}\, p^{(m+1)} = f$$
$$- \mathrm{div}\, u^{(m+1)} = 0$$

Linearized 2nd order Crank-Nicolson (Simo & Armero):

$$\frac{u^{(m+1)} - u^{(m)}}{\Delta t} + \tfrac{1}{2}(-\nu\, \nabla^2 u^{(m+1)} + (w^{(m)} \cdot \mathrm{grad}) u^{(m+1)}) + \mathrm{grad}\, p^{(m+1)} =$$
$$f - \tfrac{1}{2}(-\nu\, \nabla^2 u^{(m)} + (w^{(m)} \cdot \mathrm{grad}) u^{(m)})$$
$$- \mathrm{div}\, u^{(m+1)} = 0$$

$$w^{(m)} = 1.5\, u^{(m)} - .5\, u^{(m-1)}$$

Matrix structure after discretization

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}, \quad F = \alpha M_v + \nu A + N \qquad \alpha \begin{cases} \text{larger for CN} \\ \text{than for BE} \end{cases}$$

Considerations are the same

Preconditioner $\begin{pmatrix} Q_F & B^T \\ 0 & Q_S \end{pmatrix},$

$$Q_F \approx F, \quad Q_S \approx S$$
$$F_p = \alpha M_p + \nu A_p + N_p$$
$$Q_S^{-1} = M_p^{-1} F_p A_p \approx S^{-1}$$

Convection-diffusion solve easier than for steady state

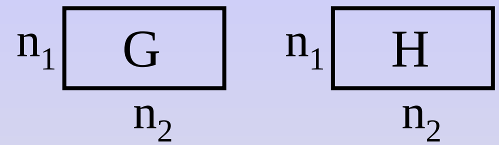
For large α (small Δt):

$$F = \alpha M_v + \nu A_p + N \sim \tfrac{h^d}{\Delta t} I \Rightarrow \qquad BF^{-1}B^T \approx \boxed{\tfrac{\Delta t}{h^d} BB^T}$$
$$Q_S = A_p F_p^{-1} M_p \sim (BM_u^{-1}B^T)\left(\tfrac{\Delta t}{h^d} I\right)(h^d I) \quad = \quad \boxed{\tfrac{\Delta t}{h^d} BB^T}$$

Approximation for the Schur Complement (II)

Elman 1999

Consider simple observation in linear algebra:
let G, H be rectangular matrices



Consider $H^T (GH^T)^{-1}G$, maps R^{n_1} to *range*(H^T)
fixes *range*(H^T)

$H^T (GH^T)^{-1}G^T = I$ on *range*(H^T)

Take $G=BF^{-1}$, $H=B$) $B^T (BF^{-1}B^T)^{-1}BF^{-1} = I$ on *range*(B^T)

$B^T (BF^{-1}B^T)^{-1}B = F$ on *range*$(F^{-1}B^T)$

Suppose *range*(B^T) ½ *range*$(F^{-1}B^T)$

) $B^T (BF^{-1}B^T)^{-1}B = F$ on *range*(B^T)

$(BB^T) (BF^{-1}B^T)^{-1} (BB^T)= BFB^T$

$(BF^{-1}B^T)^{-1}$¼ $(BB^T)^{-1}(BFB^T) (BB^T)^{-1}$ ´ Q_S^{-1}

Recapitulating: Two ideas under consideration

Preconditioners $\begin{pmatrix} Q_F & B^T \\ 0 & -Q_S \end{pmatrix}$ for $\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$

1. F_p preconditioner: $Q_S = A_p F_p^{-1} M_p$

   Requirements: Poisson solve, mass matrix solve,
   F_p on pressure space
   Decisions on boundary conditions

2. BFBt preconditioner: $Q_S = (BB^T)(BFB^T)^{-1}(BB^T)$

   Requirements: Two Poisson solves
   C=0

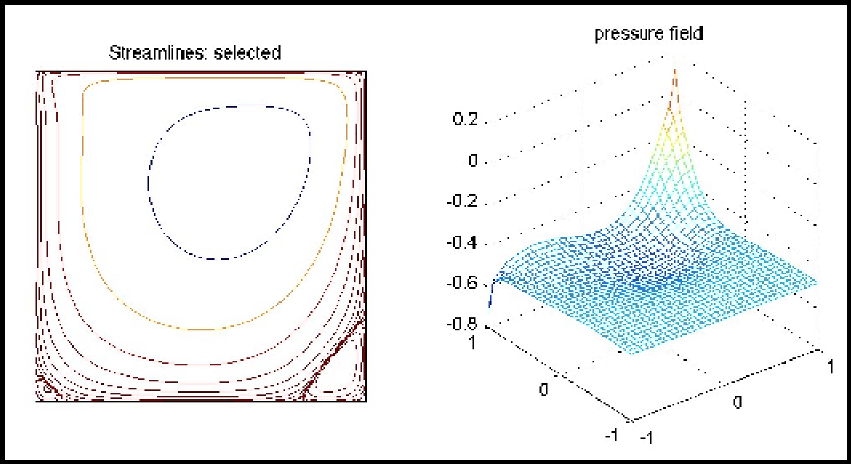
Requirement common to both: approximation of action of F^{-1}

Overarching philosophy: subsidiary operations

Poisson solve
convection-diffusion solve } are manageable

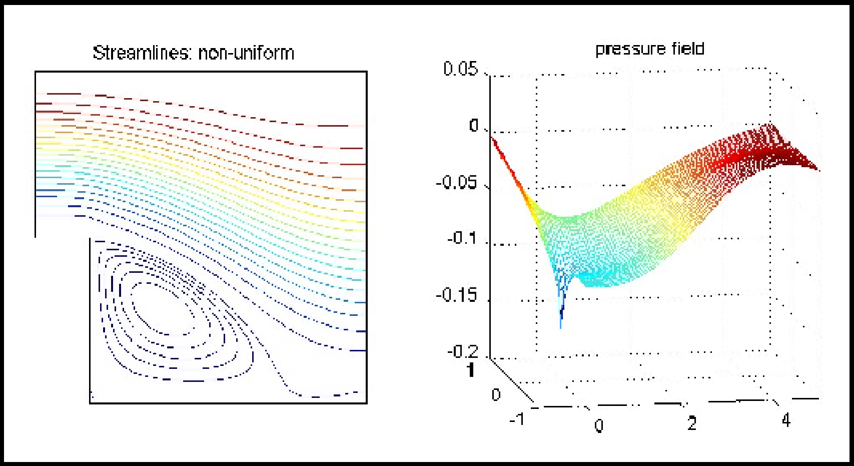
Benchmark problems

1. 2D Driven Cavity Problem



$u_1=u_2=0$, except $u_1=1$ at top

2. 2D Backward Facing Step



$u_1=u_2=0$, except

$u_1=1-y^2$ at inflow

$$\left.\begin{array}{l}\nu\,\partial u_1/\partial x=p \\ \partial u_2/\partial x=0\end{array}\right\}\text{ at outflow}$$

3. 3D Driven Cavity Problem

Various finite element / finite difference discretizations in space

Backward Euler or Crank-Nicolson in time

Experiment 1: 2D driven cavity problem on [0,1]£[0,1]

Discretization in space:

marker-and-cell finite differences (Harlow & Welch), h=1/128

Discretization in time:

backward Euler with various time steps

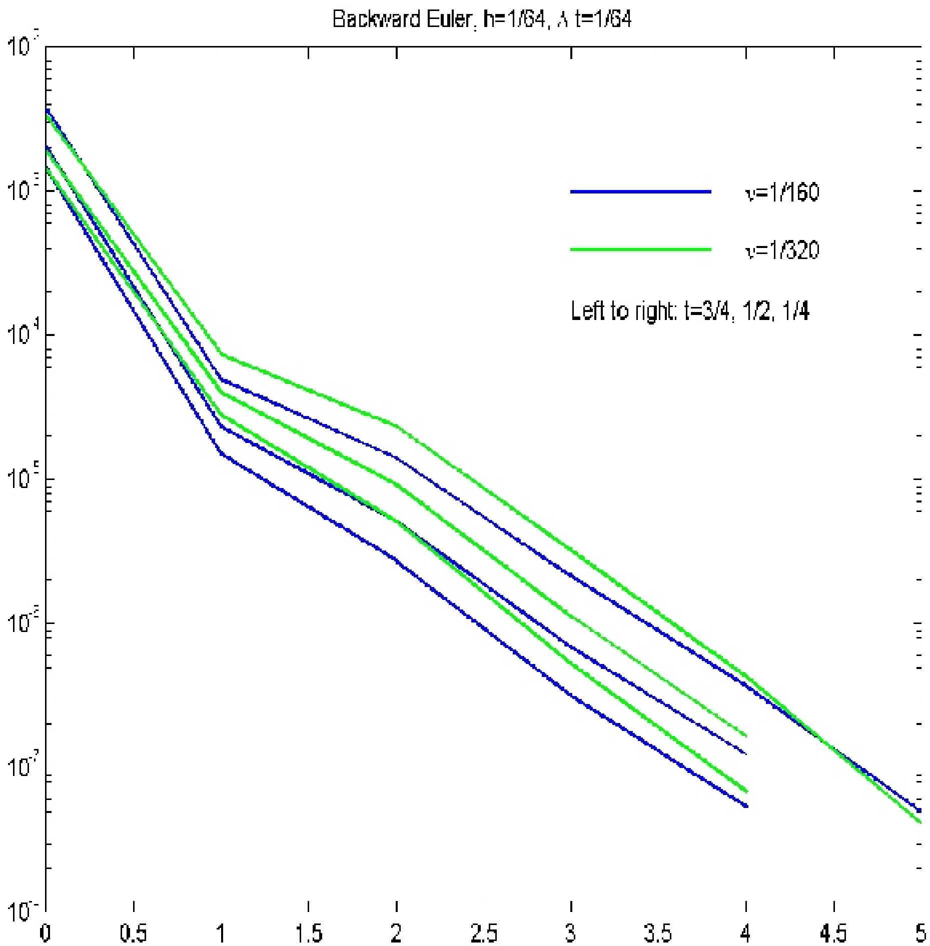
Integrate from t=0 to t=1

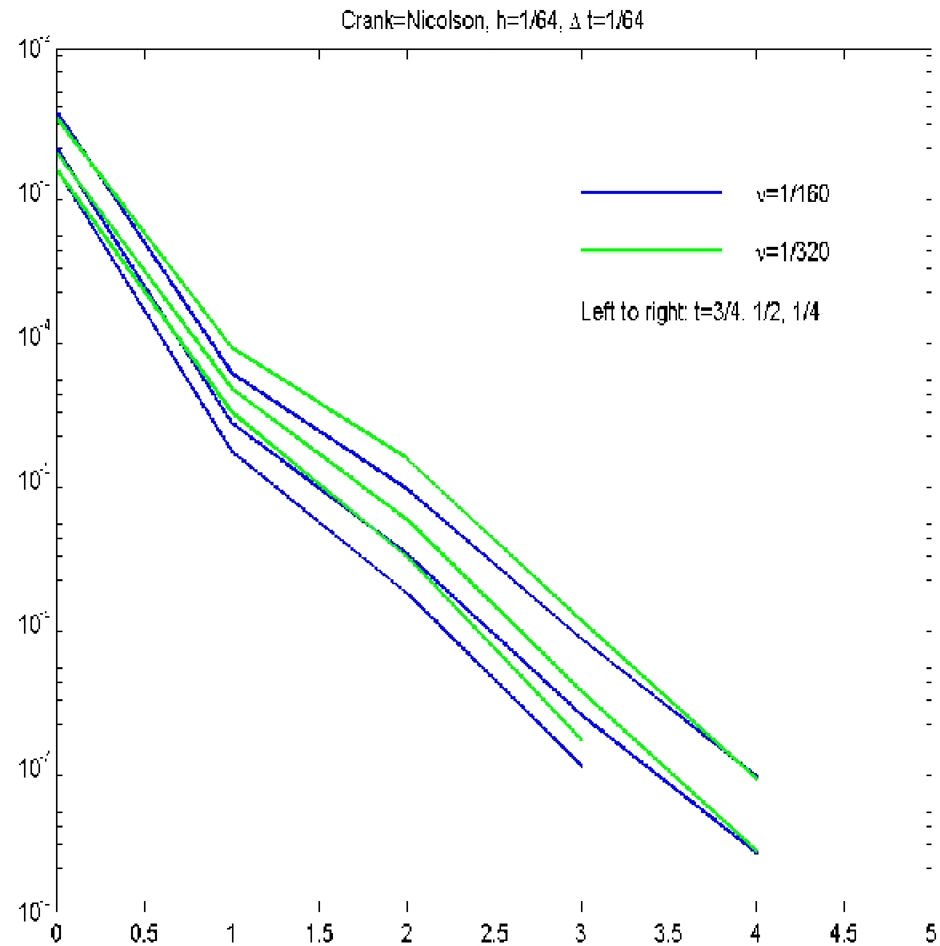
Solve implicit systems with F_p-preconditioned GMRES

Average iteration counts per linear solve

	ν				ν
Δt	**1/40**	**1/80**	**1/160**	**1/320**	**1/5000**
1/8	**6.9**	**8.4**	**9.3**	**9.9**	
1/16	**5.6**	**6.9**	**8.1**	**8.6**	
1/32	**4.0**	**5.1**	**6.2**	**6.9**	**4-5**
1/64	**2.9**	**3.6**	**4.3**	**5.0**	**3-4**

For same problem: GMRES behavior at t=1/4, 1/2, 3/4
h=1/64, ν = 1/160, 1/320

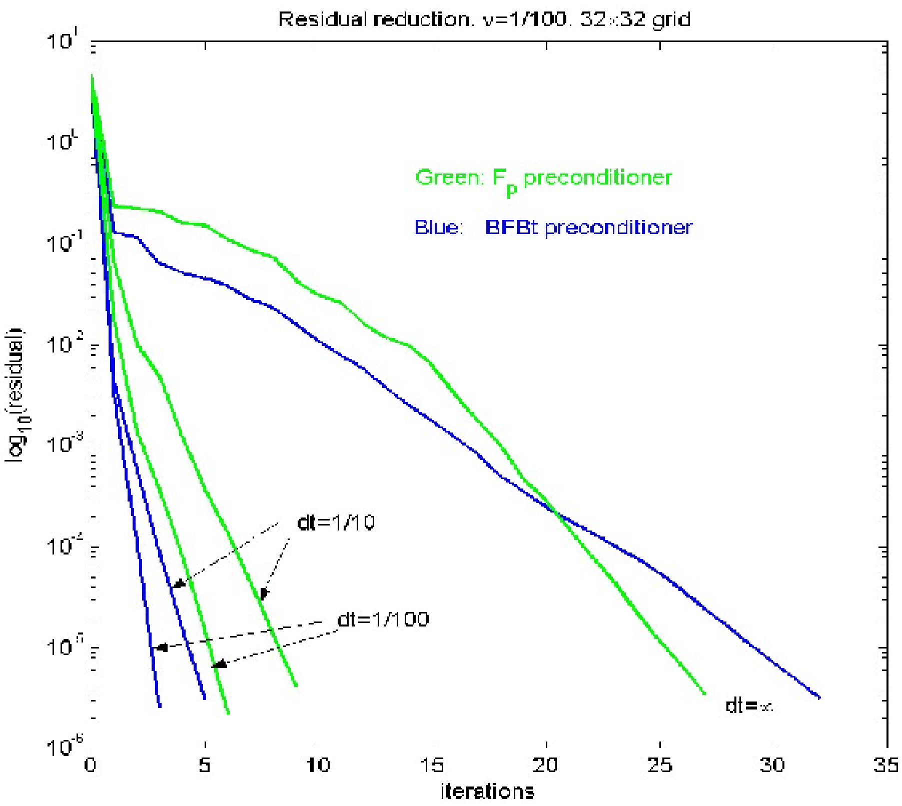


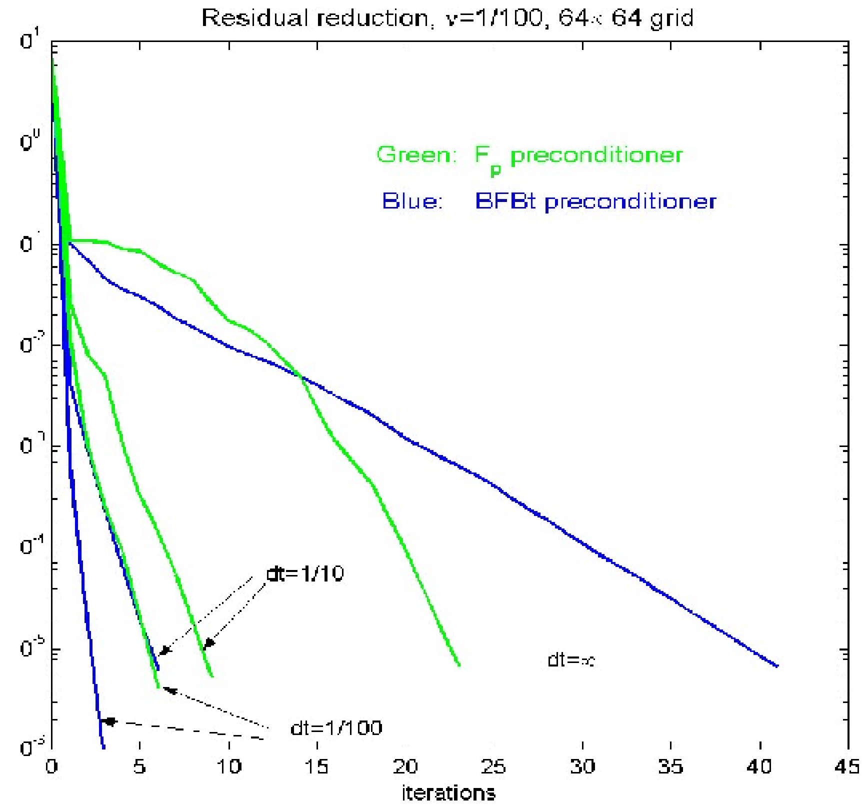


Experiment 2: 2D driven cavity flow on [-1,1]£[-1,1], Re=200

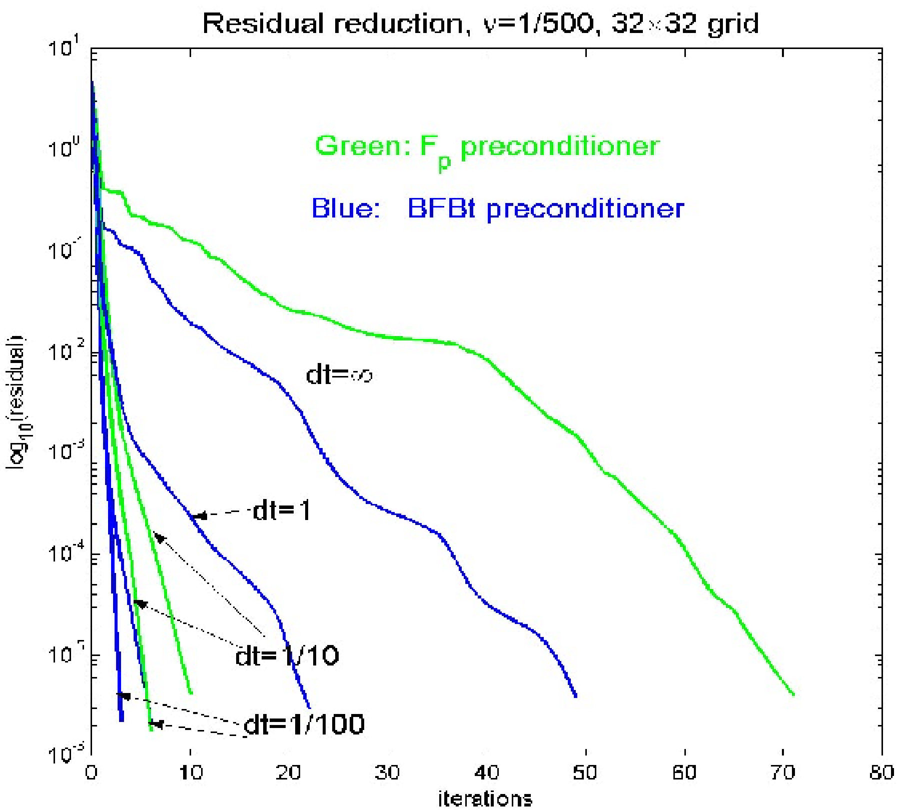
Discretization in space: Q_2-Q_1 finite elements
Discretization in time: backward Euler with various time steps

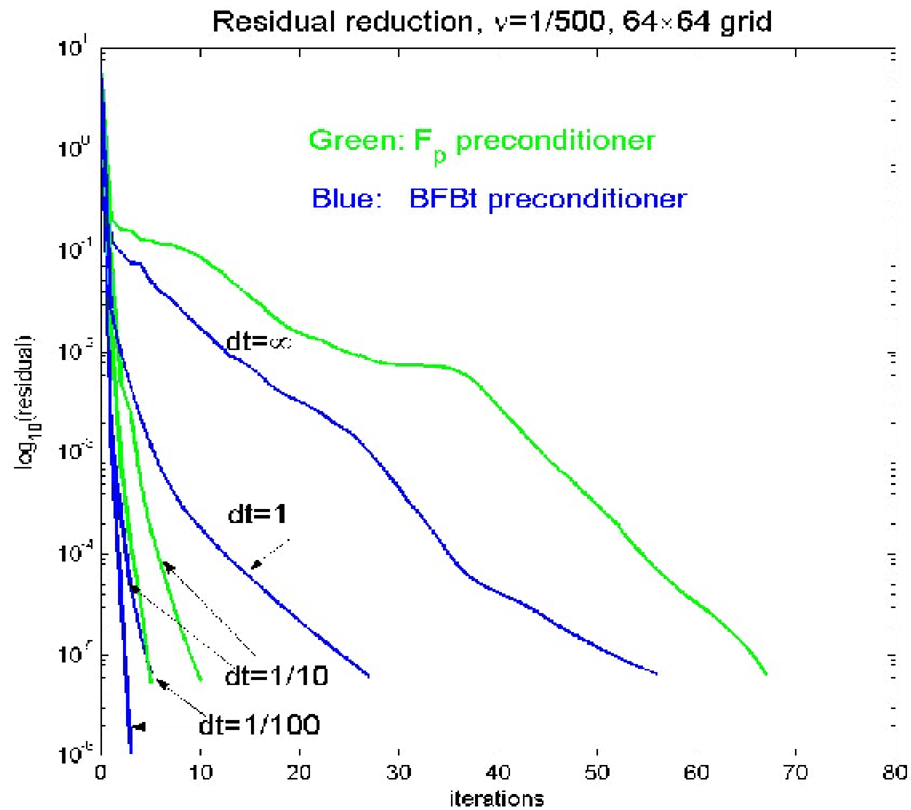
Iterations of GMRES at sample time / Picard step



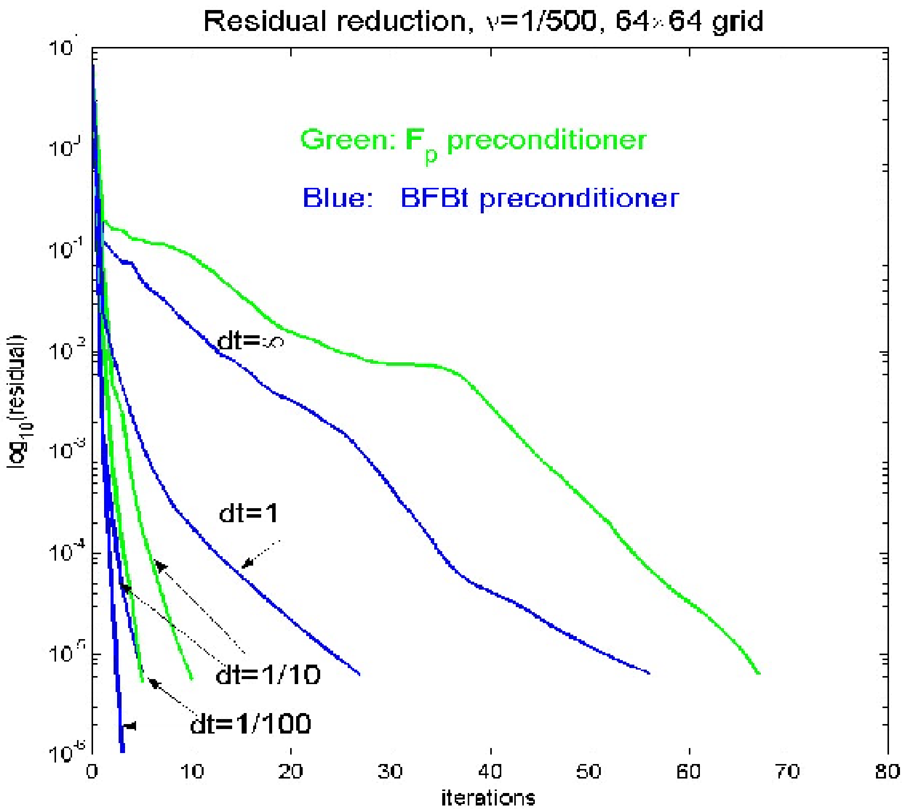


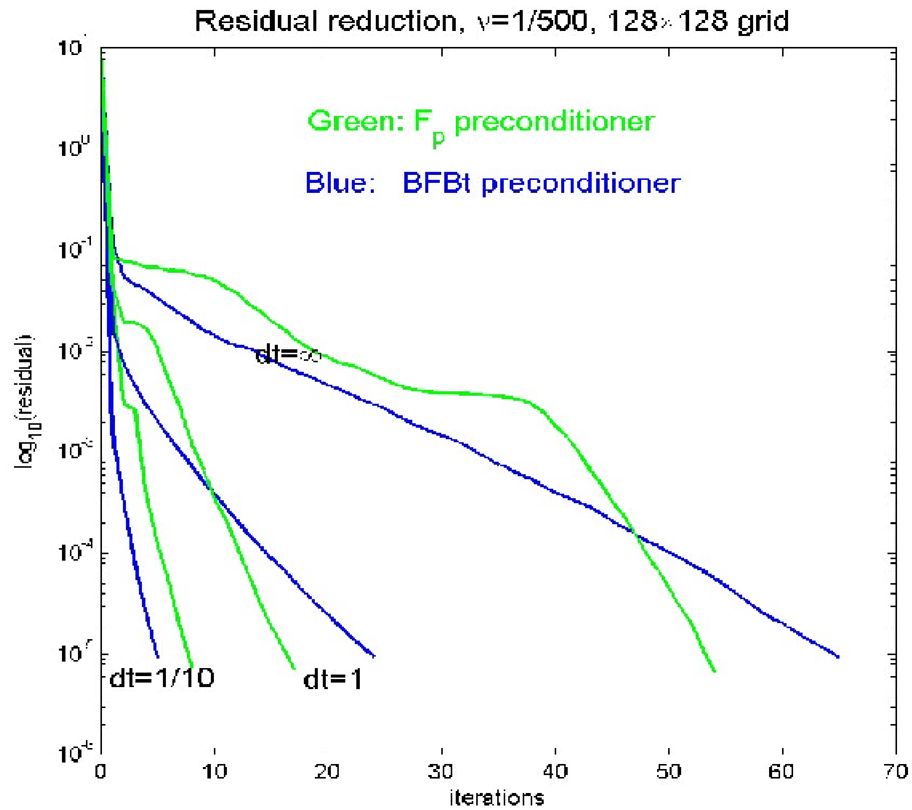
Experiment 2, continued: Re=1000





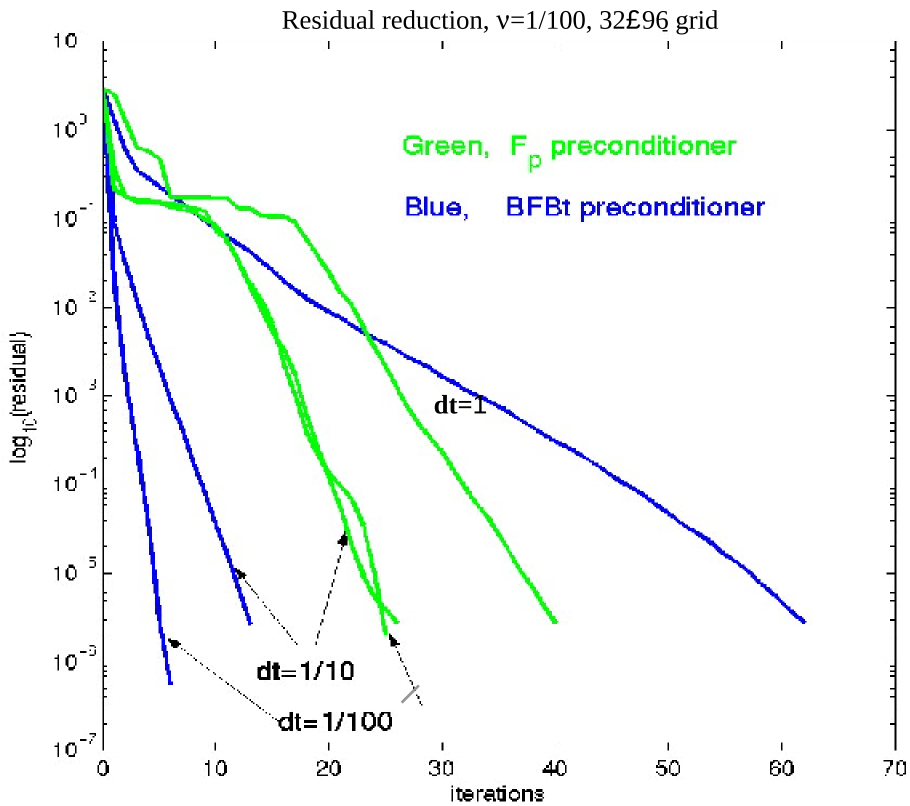
Experiment 2, continued: Re=1000

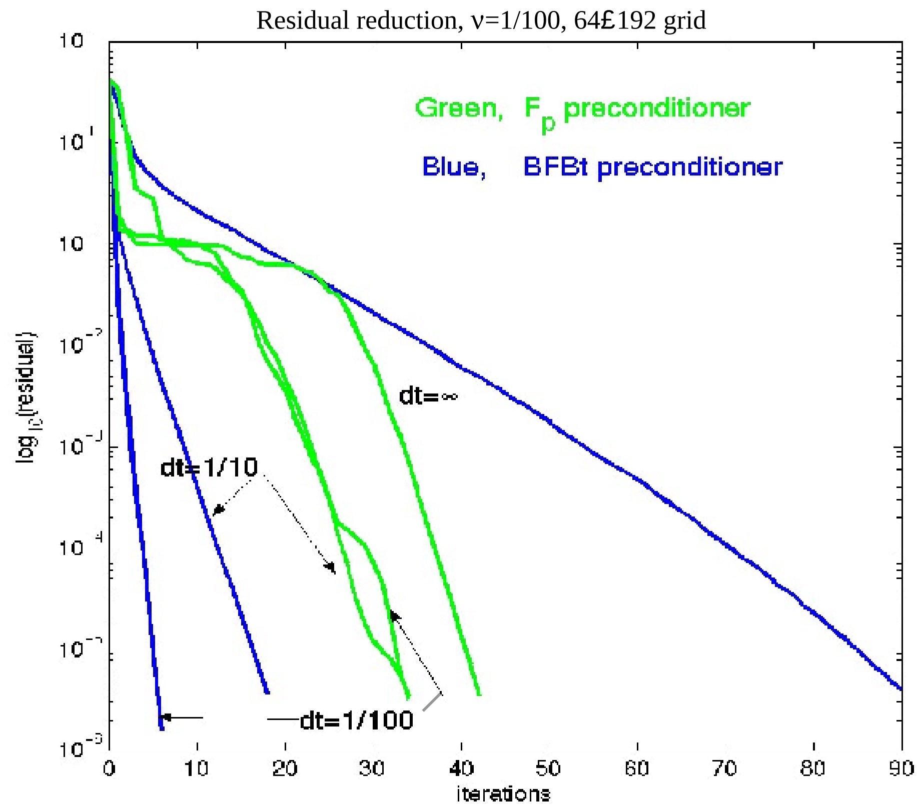




Experiment 3: backward facing step, Re=200

Q_2-Q_1 fem spatial discretization
Backward Euler time discretization
Iterations of GMRES at sample time / Picard step



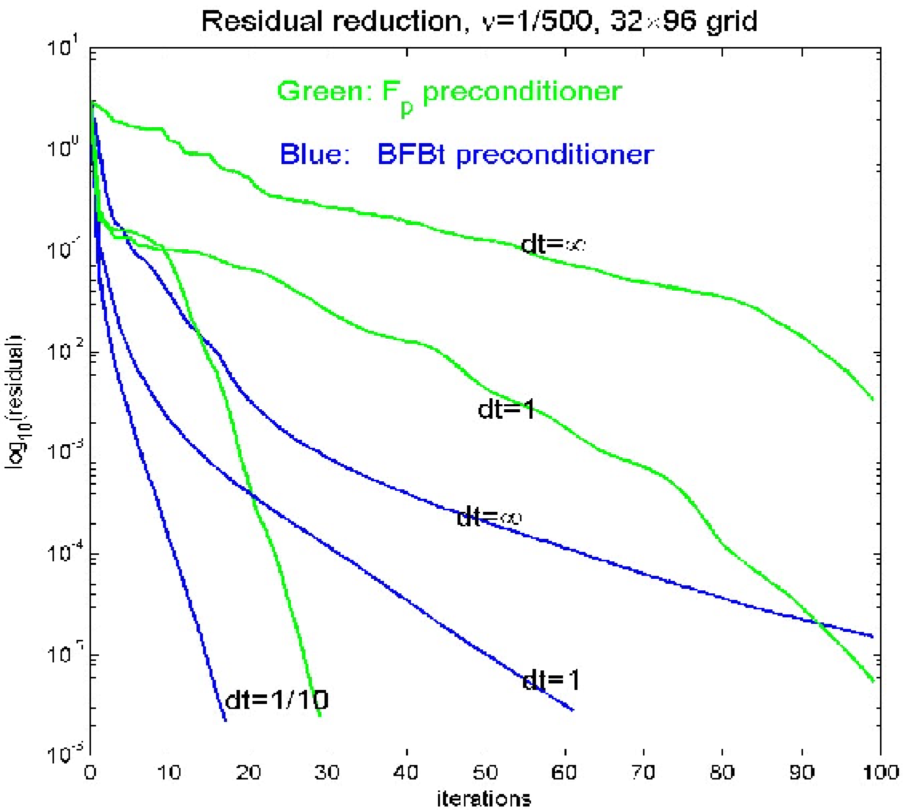


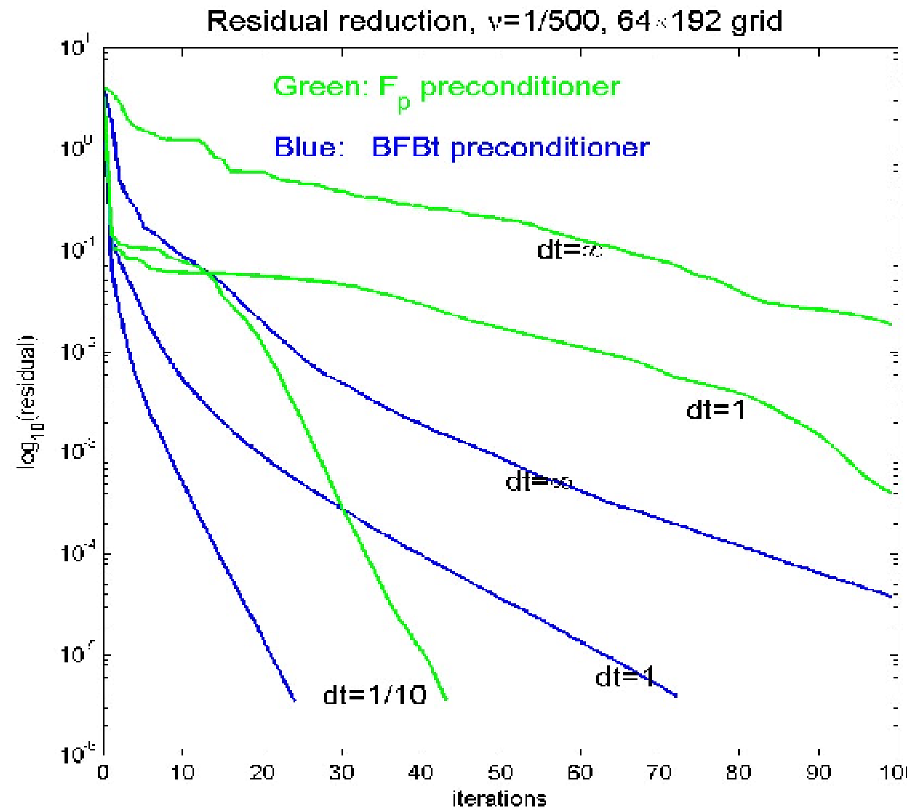
For CFL=||u|| dt/h, ||u||¼26)

dt=.1 ! CFL~ 41.6

dt=.1 ! CFL~ 83.2

Experiment 3, continued: backward facing step, Re=1000





For CFL = ||u||dt/h, ||u||¼29)

dt=1 ! CFL~464
dt=.1 ! CFL~ 46.4

dt=1 ! CFL~928
dt=.1 ! CFL~ 92.8

Experiment 4: 3D driven cavity problem on $[0,1]^3$

Marker-and-cell finite differences

Pseudo-transient iteration:

ten time steps at various CFL nos. and Re, h=1/64

Average iteration counts with F_p preconditioning to satisfy *mild* stopping criterion $\| r_k \| \cdot \quad 10^{-2} \|f\|$, f=nonlinear residual

Re	CFL #	Iterations
500	.1, .5, 1, 10, 50, 100	2
	5000	5
	10,000	6
	50,000	9
1000	5000	5
	10,000	6
	50,000	10

Key aspect of computations:

Poisson solves: $q = A_p^{-1} p$

Convection-diffusion solves: $w = F^{-1} v$

} required at each step

Each can be approximated using existing technology

- multigrid
- domain decomposition
- fast direct methods
- other iterative methods

Experiment 4, continued:

Replace convection-diffusion solve and Poisson solve with multigrid approximations

		Iterations	
Re	CFL #	Exact	Inexact
500	50,000	9	12 3 Poisson 8 Conv-diff
1000	50,000	10	13 3 Poisson 8 Conv-diff

Boundary conditions for preconditioners

To define operators F_p and A_p:
need to "specify" boundary conditions on pressure space

Derivation of preconditioners does not offer guidance

Formulation of problem does: have convection-diffusion operator

$$(-\nu\nabla^2+(w\cdot\nabla))$$

defined on pressure space, w=current velocity iterate

No specific b.c. on pressures suggests "natural" condition

$$\partial p / \partial n = 0$$

But: flow problems require Dirichlet conditions on inflow boundary,
where $w\cdot n < 0$

Therefore: formulate F_p using

Dirichlet conditions $p=0$ on $\partial\,\Omega_-$ (inflow, w¢n<0)

Neumann conditions $\partial p/\partial n = 0$ on $\partial\,\Omega_0$ (characteristic, w¢n=0)

$\partial\,\Omega_+$ (outflow, w¢n>0)

Comments:

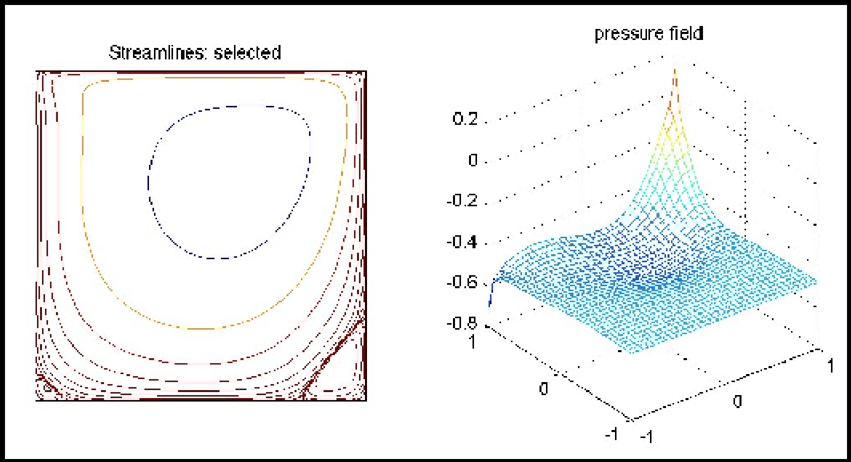
1. Not really specifying values, just defining matrix F_p
2. Formulate A_p in compatible manner
3. This issue is important

but

it only affects performance of solvers, not accuracy

For benchmark problems

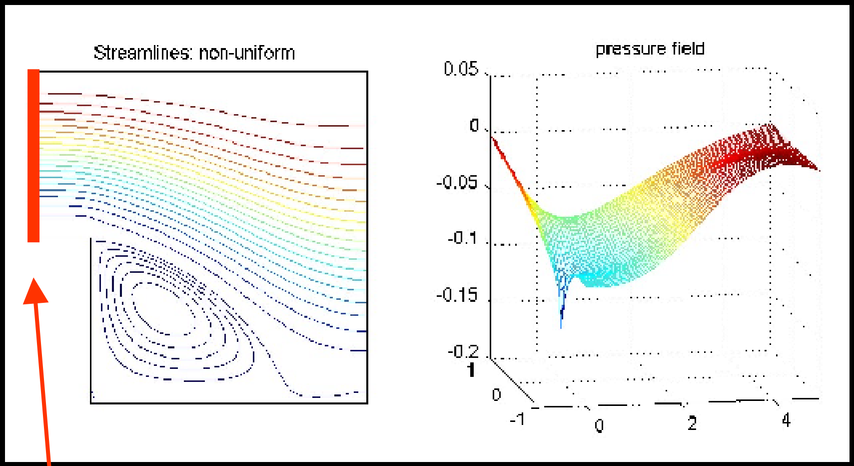
1. 2D Driven Cavity Problem



$u_1=u_2=0$, except $u_1=1$ at top

u¢n´ 0) F_p defined using Neumann b.c

2. 2D Backward Facing Step



$u_1=u_2=0$, except

$u_1=1-y^2$ at inflow

$\nu\, \partial u_1/\partial x=p$, $\partial u_2/\partial x=0$ } at outflow

u¢n < 0 at inflow)

Dirichlet b.c. there

Otherwise Neumann

Analysis: For solving $AQ_A^{-1}x = b$ using GMRES, assuming

$$AQ_A^{-1} = V\Lambda V^{-1}$$

is diagonalizable:

$$\| r_k \| \leq \min_{p_k(0)=1} \| p_k(AQ_A^{-1}) r_0 \|$$

$$\leq \kappa(V) \min_{p_k(0)=1} \max_{\lambda \in \sigma(AQ_A^{-1})} | p_k(\lambda) | \| r_0 \|$$

Here: $A = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}, \quad Q_A = \begin{pmatrix} F & B^T \\ 0 & -Q_S \end{pmatrix}$

For F_p preconditioning: $c_1 \frac{\nu\left(\nu + \frac{1}{\Delta t}\right)}{\|w\|^2} \leq |\lambda(AQ_A^{-1})| \leq c_2 \frac{1}{\nu}\left(\frac{1}{\Delta t} + \| w \|\right)$

(Loghin)

) asymptotic convergence rate $(\|r_k\|/\|r_0\|)^{1/k}$, k!1,
independent of (small) h, Δt
pessimistic wrt ν

Generalizations:

Boussinesq equations:

$$\alpha \mathbf{u}_t - \nabla \cdot (\nu_u \nabla \mathbf{u}) + (\mathbf{u} \cdot \mathsf{grad})\mathbf{u} + \mathsf{grad} \mathsf{p} = \mathsf{f}(T)$$
$$\alpha T_t - \nabla \cdot (\nu_T \nabla T) + (\mathbf{u} \cdot \mathsf{grad}) T = g(T)$$
$$-\mathsf{div}\, \mathbf{u} = 0$$

! coefficient matrix

$$\left(\begin{array}{cc:c} F_u & G & B^T \\ H & F_T & 0 \\ \hdashline B & 0 & 0 \end{array}\right) = \begin{pmatrix} \hat{F} & \hat{B}^T \\ \hat{B} & 0 \end{pmatrix}$$

"Ideal" preconditioner is $\hat{Q} = \begin{pmatrix} \hat{F} & \hat{B}^T \\ 0 & -\hat{S} \end{pmatrix}, \quad \hat{S} = \hat{B}\hat{F}^{-1}\hat{B}$

For Picard iteration, H=0 and Schur complement is

$$\hat{S} = BF_u^{-1}B^T = S,$$

the same as for the Navier-Stokes equations

Add chemistry: molecular species with concentration Y

Add equation of form $\alpha Y_t - \nabla \cdot (D_Y \nabla Y) + (\mathbf{u} \cdot \mathsf{grad}) Y = 0$

Coupled with

$$
\begin{aligned}
&\alpha \mathbf{u}_t - \nabla \cdot (\nu_u \nabla \mathbf{u}) + (\mathbf{u} \cdot \mathsf{grad})\mathbf{u} + \mathsf{grad} \mathsf{p} = \mathsf{f}(T) \\
&\alpha T_t - \nabla \cdot (\nu_T \nabla T) + (\mathbf{u} \cdot \mathsf{grad}) T = g(T) \\
&-\mathsf{div}\, \mathbf{u} = 0
\end{aligned}
$$

! coefficient matrix

$$
\begin{pmatrix} F_u & G & B^T \\ H & F_{T,Y} & 0 \\ B & 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{F} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix}
$$

Concluding remarks

Goal: develop strategies to handle linearized Navier-Stokes equations in a flexible manner

- Allow large time steps if stiffness is not critical
- Respect coupling of velocities and pressures
- Automatically adapt to handle different scenarios (creeping flow, stiff systems, steady problems)

Technical approach:

- Take advantage of saddle point structure of problem
- Develop preconditioners for Schur complement and accompanying systems