RELATIONS AMONG SPIN AMPLITUDES FOR 2 $\longrightarrow$ 2 SCATTERING

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Mahlon and Parke, hep-ph/08mm.xxxx (coming Fall 2008).

Seeds of this discussion contained in

Mahlon and Parke, *Phys. Rev.* **D53**, 4886 (1996),

Mahlon and Parke, *Phys. Lett.* **B411**, 173 (1997),

Mahlon and Parke, *Phys. Rev.* **D58**, 054015 (1998),

and

Parke and Shadmi, *Phys. Lett.* **B387**, 199 (1996).

Fermilab Theory Seminar: July 17, 2008.

Outline

0. Introduction and Disclaimers.

I. Why use the Parke-Shadmi spin basis? / Lessons learned from $t\bar{t}$.

II. What is the Parke-Shadmi spin basis?

III. A few calculational tools.

- A. Spinor products and spinor product pitfalls.
- B. Basis for massive fermions.
- C. Basis for massive vector bosons.

IV. Identities to make calculations easier.

V. Examples

- A. $u\bar{d} \to t\bar{b}$ (W^* process).
- B. $e^-_R e^+_L \to W^+ W^-$.

VI. Combining spins.

VII. Zeroing particular spin amplitudes.

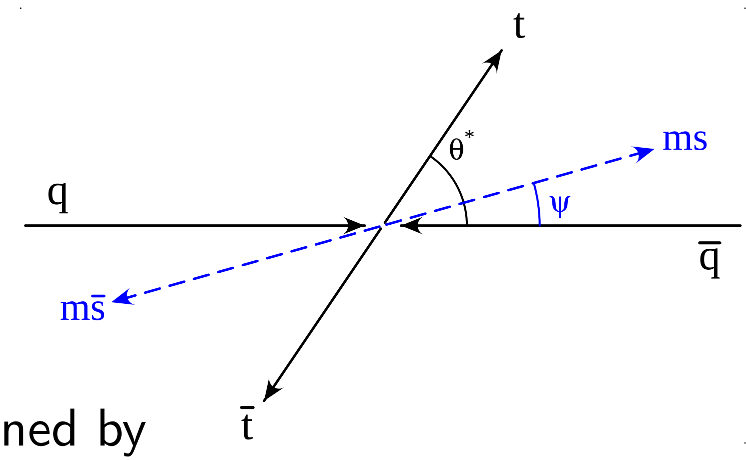
VIII. Current status and future directions.

Introduction and Disclaimers

This is a work in progress:

- “Best” way to package the results still not clear.
- Lots of loose ends!
  - ⋆ Mention these as they crop up.
  - ⋆ Won’t save them all for the end.

Let the ZMF production angle be θ^*, and the ZMF speed be β.



For the spin axis defined by

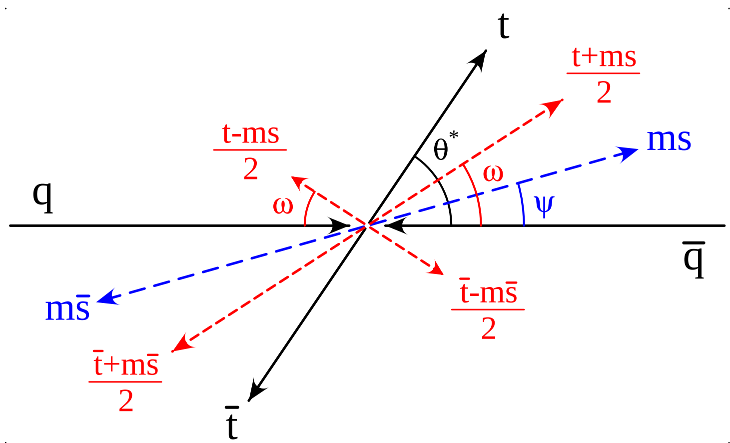
$$\tan\psi = \frac{\beta^2 \cos\theta^* \sin^2\theta^*}{1 - \beta^2 \sin^2\theta^*} \qquad (\textit{off-diagonal basis}),$$

the tops are produced exclusively with unlike spins.

The $\Uparrow\Uparrow$ and $\Downarrow\Downarrow$ spin configurations vanish!

This observation allows us to say where the top decay products prefer to go.

$$\sin\omega = \beta \sin\theta^*$$



$$(t \pm ms)^2 = 0$$

Preferred charged lepton (or d-type quark) emission directions:

$\Uparrow\Downarrow$ spin configuration: $\frac{1}{2}(t + ms)$ (top)
$\frac{1}{2}(\bar{t} + m\bar{s})$ (antitop)

$\Downarrow\Uparrow$ spin configuration: $\frac{1}{2}(t - ms)$ (top)
$\frac{1}{2}(\bar{t} - m\bar{s})$ (antitop)

Why Use the Parke-Shadmi Spin Basis?

Spin-dependent amplitudes with arbitrary spin-axis:

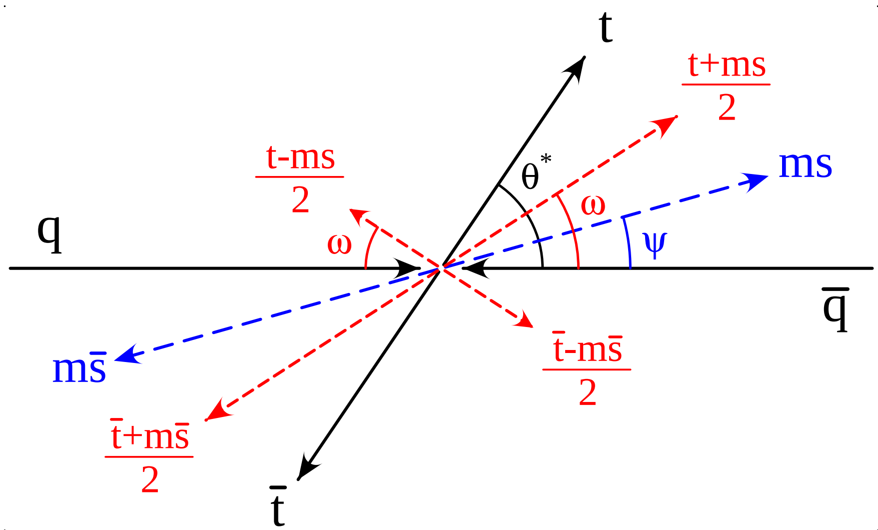
- predict final state particle emission directions:

$$\sin\omega = \beta \sin\theta^*$$

$$\tan\psi = \frac{\beta^2 \cos\theta^* \sin\theta^*}{1 - \beta^2 \sin^2\theta^*}$$

$\Uparrow\Downarrow$: $\frac{1}{2}(t + ms)$ (top)
$\frac{1}{2}(\bar{t} + m\bar{s})$ (antitop)

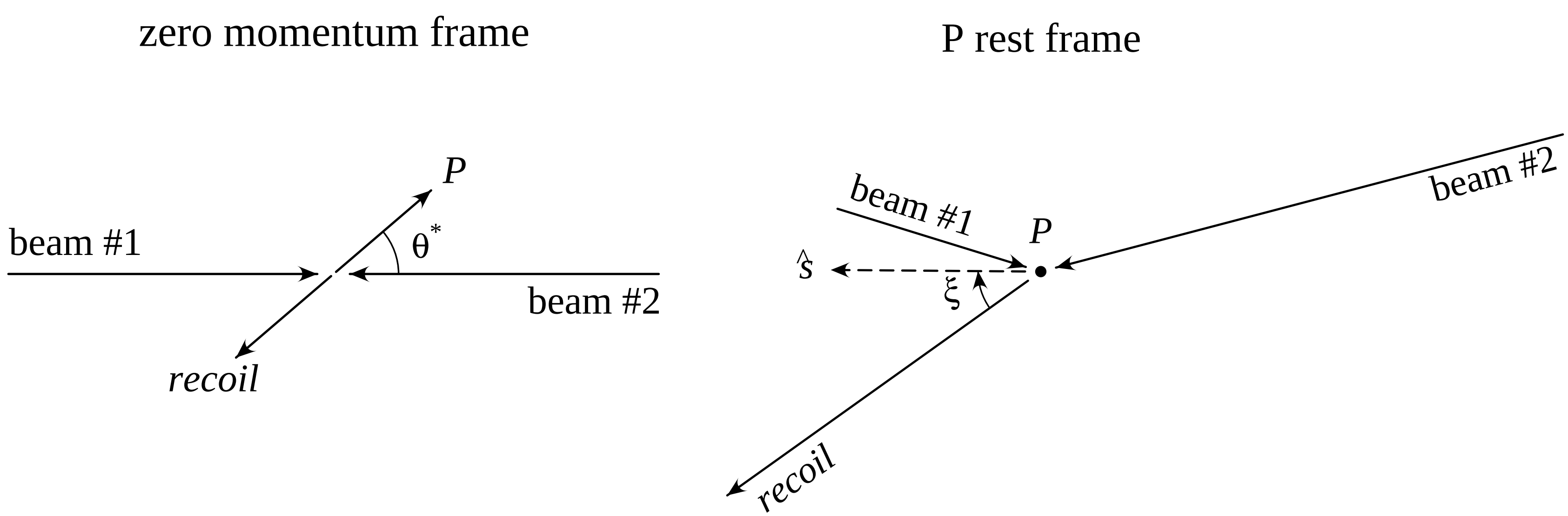
$\Downarrow\Uparrow$: $\frac{1}{2}(t - ms)$ (top)
$\frac{1}{2}(\bar{t} - m\bar{s})$ (antitop)



- maximize angular correlations for easier observation: no $\Uparrow\Uparrow$ or $\Downarrow\Downarrow$ pairs.
- provide a deeper understanding of the structure of the collision.

What is the Parke-Shadmi Spin Basis?

adapted from Parke and Shadmi, *Phys. Lett.* **B387** (1996), 199.



Can recover (traditional) helicity basis by letting $\xi \longrightarrow \pi$.

Spinor Products

Write (massless) 4-momenta as $p \equiv (p_0, p_1, p_2, p_3)$; $q \equiv (q_0, q_1, q_2, q_3)$

- $(1, 2, 3)$ need not be (x, y, z)
- 3-axis $=$ singular axis

Define

$$p_+ \equiv p_0 + p_3$$
$$q_+ \equiv q_0 + q_3 .$$

Then, for positive energy spinors:

$$\begin{aligned}\langle p+|q-\rangle &\equiv \bar{u}(p)\tfrac{1}{2}(1-\gamma^5)u(q) \\ &= \frac{(q_1 p_+ - p_1 q_+) + i(p_2 q_+ - q_2 p_+)}{\sqrt{p_+ q_+}}\end{aligned}$$

Spinor Products

Properties:

$$\langle p+|q-\rangle = -\langle q+|p-\rangle$$

$$\langle p+|q-\rangle = \sqrt{2p\cdot q}\, e^{i\Phi}$$

Pitfalls:

- Phase depends on choice of singular axis.
- Phase is not always simple to write down.

  Typical example: $e^{i\Phi} = \dfrac{\beta + c_\xi - i\gamma^{-1} s_\xi}{1 + \beta c_\xi}$
- Phase is frame-dependent!

Prudent strategy: choose singular axis orientation to make all spinor products real, if possible.

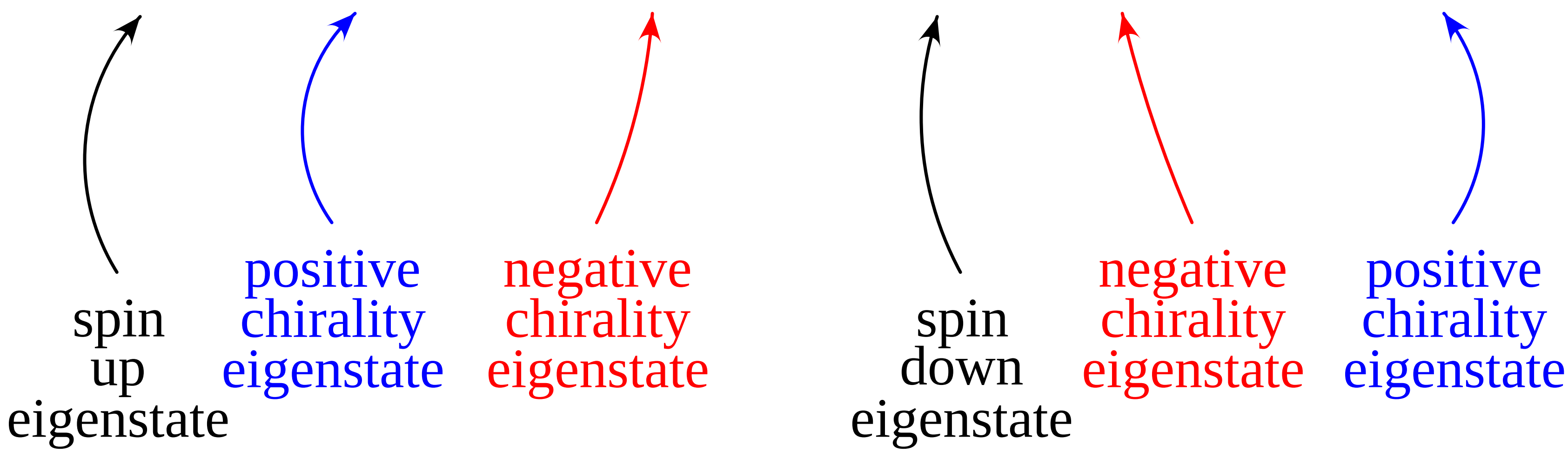
Basis Spinors for Massive Fermions

Decompose fermion 4-momentum into two pieces:

$$P \equiv p_1 + p_2; \quad P^2 = M^2; \quad p_1, p_2 \leftrightarrow \tfrac{1}{2}(p \pm M s); \quad p_1^2 = p_2^2 = 0.$$

Fermion spin states are then

$$u_\uparrow(P) = |p_1 +\rangle - e^{i\Phi}|p_2 -\rangle \quad \textbf{and} \quad u_\downarrow(P) = |p_1 -\rangle + e^{-i\Phi}|p_2 +\rangle$$



$$e^{i\Phi} = \frac{\langle p_2 \; - | p_1 \; + \rangle}{M}$$

Basis for Massive Vector Bosons

The three polarization states for a massive vector boson may be represented in spinor notation by

$$\tfrac{1}{2}(1-\gamma^5)\ \not\epsilon(p_\uparrow) = \sqrt{2}\,\frac{|p_1\ -\rangle\langle p_2\ -|}{m},$$

$$\tfrac{1}{2}(1-\gamma^5)\ \not\epsilon(p_0) = \frac{|p_1\ -\rangle\langle p_1\ -| - |p_2\ -\rangle\langle p_2\ -|}{m},$$

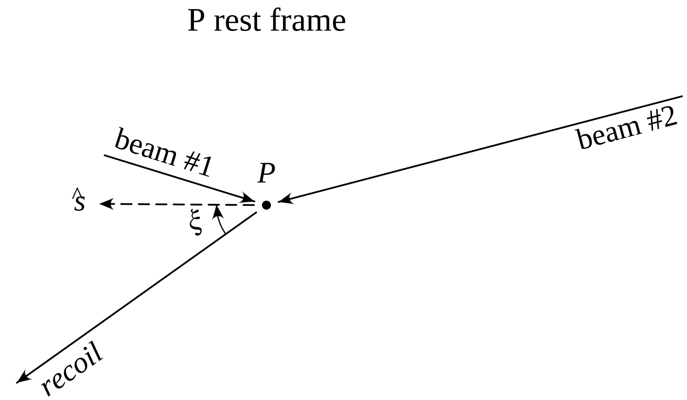
and

$$\tfrac{1}{2}(1-\gamma^5)\ \not\epsilon(p_\downarrow) = -\sqrt{2}\,\frac{|p_2\ -\rangle\langle p_1\ -|}{m}.$$

These polarizations satisfy the usual properties:

1. **Transversality:** $p \cdot \epsilon(p_\lambda) = 0$.
2. **Orthonormality:** $\epsilon(p_\lambda) \cdot \epsilon^*(p_{\lambda'}) = -\delta_{\lambda\lambda'}$.
3. **Completeness:** $\sum_\lambda \epsilon_\mu(p_\lambda)\epsilon^*_\nu(p_\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$

Rotate spin axis from ξ to $\xi + \Delta\xi$ using rotation operator:



$$|\, j\ m\rangle_{\xi+\Delta\xi} = \exp(i\Delta\xi J_y)|\, j\ m\rangle_\xi .$$

Suppose that $\Delta\xi \to 0$. Then $\exp(i\Delta\xi J_y) \to 1 + i\Delta\xi J_y$, and

$$\frac{|\, j\ m\rangle_{\xi+\Delta\xi} - |\, j\ m\rangle_\xi}{\Delta\xi} = iJ_y|j\ m\rangle_\xi$$

Left-hand side is just a derivative. Also, recall that $iJ_y = \frac{1}{2}(J_+ - J_-)$:

$$\frac{\partial}{\partial\xi}|j\ \ m\rangle_\xi = \frac{1}{2}(J_+ - J_-)|j\ \ m\rangle_\xi .$$

Spin Identities

General:

$$\frac{\partial}{\partial \xi}|j \;\; m\rangle_\xi = \frac{1}{2}(J_+ - J_-)|j \;\; m\rangle_\xi .$$

Spin-$\frac{1}{2}$:

$$\frac{\partial \mathcal{A}_\uparrow}{\partial \xi} = -\frac{1}{2}\mathcal{A}_\downarrow$$
$$\frac{\partial \mathcal{A}_\downarrow}{\partial \xi} = \frac{1}{2}\mathcal{A}_\uparrow$$

Spin-1:

$$\mathcal{A}_0 = -\sqrt{2}\,\frac{\partial \mathcal{A}_+}{\partial \xi}$$
$$\mathcal{A}_- = \left(1 + 2\,\frac{\partial^2}{\partial \xi^2}\right)\mathcal{A}_+$$

Spin Identities

Some consequences:

- Starting at the end of the chain, one can obtain **all** other spin-amplitudes by (simple) differentiation!

  (Is this starting point necessary, or merely sufficient?)

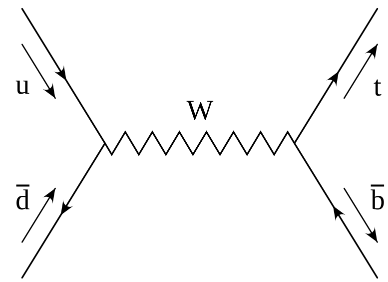
- If helicity amplitudes are desired, set $\xi = \pi$.

- The Wigner d-functions satisfy the *same* differential equations.

  $\Rightarrow$ All spin amplitudes are linear combinations of these functions.

  $\Rightarrow$ Explains why we see the same functions over and over in apparently unrelated processes.

Example: $u\bar{d} \to t\bar{b}$ (W^* process)



Retain b-quark mass for the sake of illustration.

Start by calculating

$$\mathcal{N}_{\uparrow\uparrow} \sim \langle t_2 \; -|d \; +\rangle\langle u \; +|b_1 \; -\rangle$$
$$\sim \left[\sqrt{1-c_\theta}\sqrt{1-\beta_t}\sin\frac{\xi_t}{2} + \sqrt{1+c_\theta}\sqrt{1+\beta_t}\cos\frac{\xi_t}{2}\right]$$
$$\times \left[\sqrt{1+c_\theta}\sqrt{1+\beta_b}\sin\frac{\xi_b}{2} + \sqrt{1-c_\theta}\sqrt{1-\beta_b}\cos\frac{\xi_b}{2}\right]$$

Remaining spins by differentiation:

$$\mathcal{N}_{\downarrow\uparrow} = -2\frac{\partial\mathcal{N}_{\uparrow\uparrow}}{\partial\xi_t}; \qquad \mathcal{N}_{\uparrow\downarrow} = -2\frac{\partial\mathcal{N}_{\uparrow\uparrow}}{\partial\xi_b}; \qquad \mathcal{N}_{\downarrow\downarrow} = 4\frac{\partial^2\mathcal{N}_{\uparrow\uparrow}}{\partial\xi_t\partial\xi_b}.$$

Set $\xi_t = \xi_b = \pi$ for helicity basis.

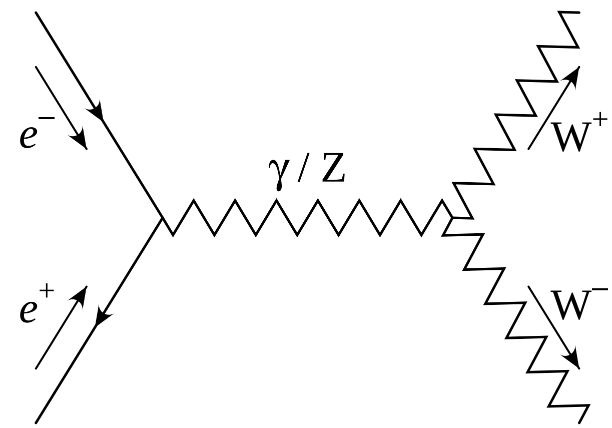
Example: $e^-_R e^+_L \to W^+ W^-$

Starting point:

$$\begin{aligned}\mathcal{N}_{\uparrow\downarrow} \sim & -\langle k'+|W_2'-\rangle\langle W_1'-|k+\rangle\langle k+|W_1-\rangle\langle W_2-|k+\rangle \\ & + \langle k'+|W_2'-\rangle\langle W_1'-|W_2+\rangle\langle W_2+|W_1-\rangle\langle W_2-|k+\rangle \\ & + \langle k'+|W_1-\rangle\langle W_2-|k+\rangle\langle k+|W_2'-\rangle\langle W_1'-|k+\rangle \\ & - \langle k'+|W_1-\rangle\langle W_2-|W_1'+\rangle\langle W_1'+|W_2'-\rangle\langle W_1'-|k+\rangle\end{aligned}$$

Insert spinor products and simplify:

$$\begin{aligned}\mathcal{N}_{\uparrow\downarrow} \sim \quad & 2\beta\gamma^{-1}(s_{\xi_+} - s_{\xi_-}) \\ & - \beta\gamma^{-2}s_\theta(1 + c_{\xi_+}c_{\xi_-}) \\ & + (3-\beta^2)\beta s_\theta s_{\xi_+}s_{\xi_-} \\ & + 2\beta\gamma_Z^{-1}c_\theta(s_{\xi_+}c_{\xi_-} + c_{\xi_+}s_{\xi_-})\end{aligned}$$



Example: $e^-_R e^+_L \to W^+ W^-$

$$\begin{aligned}\mathcal{N}_{\uparrow\downarrow} \sim \quad & 2\beta\gamma^{-1}(s_{\xi_+} - s_{\xi_-}) \\ & - \beta\gamma^{-2} s_\theta (1 + c_{\xi_+} c_{\xi_-}) \\ & + (3-\beta^2)\beta s_\theta s_{\xi_+} s_{\xi_-} \\ & + 2\beta\gamma_Z^{-1} c_\theta (s_{\xi_+} c_{\xi_-} + c_{\xi_+} s_{\xi_-})\end{aligned}$$

Apply spin-1 relations to obtain

$$\mathcal{N}_{0\downarrow} = -\sqrt{2}\frac{\partial}{\partial \xi_+}\mathcal{N}_{\uparrow\downarrow}; \qquad \mathcal{N}_{\uparrow 0} = \sqrt{2}\frac{\partial}{\partial \xi_-}\mathcal{N}_{\uparrow\downarrow}; \qquad \mathcal{N}_{00} = -2\frac{\partial}{\partial \xi_+}\frac{\partial}{\partial \xi_-}\mathcal{N}_{\uparrow\downarrow};$$

$$\mathcal{N}_{\uparrow\uparrow} = \left(1 + 2\frac{\partial^2}{\partial \xi_-^2}\right)\mathcal{N}_{\uparrow\downarrow}; \qquad \mathcal{N}_{\downarrow\downarrow} = \left(1 + 2\frac{\partial^2}{\partial \xi_+^2}\right)\mathcal{N}_{\uparrow\downarrow}; \qquad \text{etc.}$$

Extract helicity basis by setting $\xi_+ = \xi_- = \pi$.

Can obtain additional instances of the identities by setting $\xi_b \equiv \xi_t$ and forming the appropriate linear combinations of amplitudes. For example,

$$
\begin{aligned}
|1\ \ 1\rangle \sim \mathcal{N}_{\uparrow\uparrow} & \\
\sim \quad & \left[(1-c_\theta)\sqrt{1-\beta_b}\sqrt{1-\beta_t} - (1+c_\theta)\sqrt{1+\beta_b}\sqrt{1+\beta_t}\right]\cos\xi \\
& + \left[-|s_\theta|\sqrt{1+\beta_b}\sqrt{1-\beta_t} - |s_\theta|\sqrt{1-\beta_b}\sqrt{1+\beta_t}\right]\sin\xi \\
& + \left[(1-c_\theta)\sqrt{1-\beta_b}\sqrt{1-\beta_t} + (1+c_\theta)\sqrt{1+\beta_b}\sqrt{1+\beta_t}\right]
\end{aligned}
$$

$$
\begin{aligned}
|1\ \ {-1}\rangle \sim \mathcal{N}_{\downarrow\downarrow} & \\
\sim \quad & \left[(1+c_\theta)\sqrt{1+\beta_b}\sqrt{1+\beta_t} - (1-c_\theta)\sqrt{1-\beta_b}\sqrt{1-\beta_t}\right]\cos\xi \\
& + \left[|s_\theta|\sqrt{1-\beta_b}\sqrt{1+\beta_t} + |s_\theta|\sqrt{1+\beta_b}\sqrt{1-\beta_t}\right]\sin\xi \\
& + \left[(1+c_\theta)\sqrt{1+\beta_b}\sqrt{1+\beta_t} + (1-c_\theta)\sqrt{1-\beta_b}\sqrt{1-\beta_t}\right]
\end{aligned}
$$

Combining Spins

$$
\begin{aligned}
|1\ \ 0\rangle \sim & \frac{1}{\sqrt{2}}\Big[\mathcal{N}_{\uparrow\downarrow} + \mathcal{N}_{\downarrow\uparrow}\Big] \\
\sim & \sqrt{2}\Big[|s_\theta|\sqrt{1+\beta_b}\sqrt{1-\beta_t} + |s_\theta|\sqrt{1-\beta_b}\sqrt{1+\beta_t}\Big]\cos\xi \\
& + \Big[(1-c_\theta)\sqrt{1-\beta_b}\sqrt{1-\beta_t} - (1+c_\theta)\sqrt{1+\beta_b}\sqrt{1+\beta_t}\Big]\sin\xi
\end{aligned}
$$

$$
\begin{aligned}
|0\ \ 0\rangle \sim & \frac{1}{\sqrt{2}}\Big[\mathcal{N}_{\uparrow\downarrow} - \mathcal{N}_{\downarrow\uparrow}\Big] \\
\sim & \sqrt{2}\Big[\quad |s_\theta|\sqrt{1+\beta_b}\sqrt{1-\beta_t} - |s_\theta|\sqrt{1-\beta_b}\sqrt{1+\beta_t}\Big]
\end{aligned}
$$

- $|1\ \ m\rangle$ states satisfy the spin-1 relations.
- $|0\ \ 0\rangle$ is independent of ξ (*c.f.* the spin-0 relation).

Let $H_\lambda \equiv$ helicity amplitude for spin projection λ.

$M_\lambda \equiv \xi$-basis amplitude for spin projection λ.

Then, we know that

$$M_\lambda(\xi) = \sum_{\lambda'} d^j_{\lambda\lambda'}(\xi) H_{\lambda'}$$

Viewing the H_λ's as a bunch of coefficients, we can try to construct a spin basis (*i.e.* choice of ξ) where a particular M_λ vanishes.

- This may not always be possible.

Zero Hunting: Spin-1 Example

Transverse amplitude:

$$\begin{aligned} M_+ &= d^1_{++}H_+ + d^1_{+0}H_0 + d^1_{+-}H_- \\ &= (1+c_\xi)H_+ - \sqrt{2}s_\xi H_0 + (1-c_\xi)H_- \end{aligned}$$

Basis with $M_+ = 0$ would satisfy

$$0 = (H_+ + H_-) + c_\xi(H_+ - H_-) - s_\xi H_0\sqrt{2}$$

Solution for ξ may or may not exist, depending on values of $H_\pm$, H_0.

Longitudinal amplitude:

$$\begin{aligned} M_0 &= d^1_{0+}H_+ + d^1_{00}H_0 + d^1_{0-}H_- \\ &= \frac{s_\xi}{\sqrt{2}}H_+ + c_\xi H_0 - \frac{s_\xi}{\sqrt{2}}H_- \end{aligned}$$

Basis with $M_0 = 0$ satisfies

$$0 = s_\xi H_+ + c_\xi(H_0\sqrt{2}) - s_\xi H_- \quad \Longrightarrow \quad \tan\xi = \frac{H_0\sqrt{2}}{H_- - H_+}$$

Zero Hunting: Spin-1 Example

Helicity amplitudes for $q_L \bar{q}_R \to t\bar{t}$ are

$$H_+ \sim \mathcal{A}_L^{\uparrow\downarrow} = 1 - c_\theta$$
$$H_- \sim \mathcal{A}_L^{\downarrow\uparrow} = 1 + c_\theta$$
$$H_0 \sim \frac{\mathcal{A}_L^{\uparrow\uparrow} - \mathcal{A}_L^{\downarrow\downarrow}}{\sqrt{2}} = \gamma^{-1} s_\theta \sqrt{2}$$

Basis with $M_0 = 0$ satisfies

$$\tan\xi = \frac{H_0\sqrt{2}}{H_- - H_+} \quad \Longrightarrow \quad \text{or} \quad \tan\xi = \frac{(\gamma^{-1} s_\theta \sqrt{2})\sqrt{2}}{1 + c_\theta - (1 - c_\theta)}$$
$$\tan\xi = \gamma^{-1}\tan\theta.$$

This is precisely the off-diagonal basis!

i.e. $t\bar{t}$ spins purely $\Uparrow\Downarrow$ and $\Downarrow\Uparrow$: no $\Uparrow\Uparrow$ or $\Downarrow\Downarrow$ present.

Zero Hunting: Spin-1 Example

Helicity amplitudes for $e_L \bar{e}_R \to Zh$ are

$$H_+ \sim \frac{\gamma^{-1}}{\sqrt{2}}(1 - c_\theta)$$
$$H_- \sim \frac{\gamma^{-1}}{\sqrt{2}}(1 + c_\theta)$$
$$H_0 \sim s_\theta$$

Basis with $M_0 = 0$ satisfies

$$\tan\xi = \frac{H_0\sqrt{2}}{H_- - H_+} \quad \Longrightarrow \quad \text{or} \quad \tan\xi = \frac{s_\theta\sqrt{2}}{\sqrt{2}\gamma^{-1}c_\theta}$$
$$\tan\xi = \gamma\tan\theta.$$

This is precisely the Zh-transverse basis.

$i.e.$ Z's are all $(+)$ or $(-)$ spin projection; (0) projection is absent.

Current Status & Future Directions

- We have a method to easily generate *all* of the helicity amplitudes for a $2 \to 2$ scattering from a calculation of just **one**!
  - ⋆ Use independent spin axis orientation for each particle.
  - ⋆ Start at end of spin chain.
  - ⋆ Obtain remaining spin amplitudes by (simple) differentiation.
  - ⋆ Set $\xi = \pi$ for helicity amplitudes, if desired.
- Cases explicitly investigated:
  - ⋆ $u\bar{d} \to t\bar{b}$ $(J = 0, \frac{1}{2}, 1)$
  - ⋆ $gb \to W^- t$ $(J = \frac{1}{2}, 1, \frac{3}{2})$
  - ⋆ $e^+e^- \to W^+W^-$ $(J = 0, 1, 2)$
- Generalize to $2 \to n$ processes?