NOAA Technical Memorandum ERL PMEL-7

FORMULAS USED TO ANALYZE WIND-DRIVEN CURRENTS AS FIRST-ORDER AUTOREGRESSIVE PROCESSES

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Seattle, Washington
August 1976

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NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
Robert M. White, Administrator

Environmental Research Laboratories
Wilmot N. Hess, Director

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Harold O. Mofjeld
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ABSTRACT. This memorandum justifies formulas used to analyze wind-driven ocean currents as first-order autoregressive processes forced by the wind. A set of numerical experiments with synthetic data shows that the formulas give accurate results with both white and nonwhite noise forcing and with either filtered or nonfiltered data. The formulas are a significant improvement over those traditionally used in autoregression analysis based on covariances.

1. INTRODUCTION

In anticipation of modeling wind-driven surface currents near Kodiak Island, Alaska, as first-order autoregressive processes forced by the wind,[1] a series of numerical experiments have been run to test the analysis procedure and to study the effect of extraneous noise and filtering on the results. As discussed by Mofjeld (1975), a current component υ_i sampled at a uniform sampling interval Δt is related to a wind component V_i by the fundamental equation,

$$\upsilon_i = \underline{a}\upsilon_{i-1} + \underline{b}V_i + Z_i \ , \tag{1}$$

where Z_i is the residual current not related to the wind together with instrumental noise. Equation (1) relates fluctuations in the current component, relative to the mean current, to fluctuations in the wind; υ_i, V_i, and Z_i are all demeaned time series. Given the current and wind series, the first objective of the analysis is the calculation of the coefficients $\underline{a}$ and $\underline{b}$ from the data. In the following discussion the wind component has been replaced by a synthetic series. Because of the extensive theory and simple interpretations applied to white noise forcing (Jenkins and Watts, 1968), this type of forcing was used in the first set of numerical experiments to simulate the wind.

The coefficients $\underline{a}$ and $\underline{b}$ are chosen to minimize the residual variance, which is equal to the zero-lag autocovariance of Z:

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[1]This work is part of the Outer Continental Shelf Environmental Assessment Program sponsored by the Bureau of Land Management.

$$C_{zz}(0) = C_{uu}(0) + \underline{a}^2 C_{uu}(0) + \underline{b}^2 C_{vv}(0)$$
$$-2\underline{a}C_{uu}(1) - 2\underline{b}C_{uv}(0) + 2\underline{ab}C_{uv}(1) \;, \tag{2}$$

where the covariances are given by

$$C_{xy}(k) = \frac{1}{N-k} \sum_{i=1}^{N-k} (x_i y_{i+k}) \tag{3}$$

for N consecutive pairs of x and y. On setting derivatives of equation (2) with respect to $\underline{a}$ and $\underline{b}$ equal to zero, a pair of normal equations is obtained which yield the following formulas $\underline{a}$ and $\underline{b}$ in terms of the covariances:

$$\underline{a} = \frac{C_{uu}(1)\; C_{vv}(0) - C_{uv}(0)\; C_{uv}(1)}{C_{uu}(0)\; C_{vv}(0) - C_{uv}(1)\; C_{uv}(1)} \;; \tag{4}$$

$$\underline{b} = \frac{C_{uu}(0)\; C_{uv}(0) - C_{uu}(1)\; C_{uv}(1)}{C_{uu}(0)\; C_{vv}(0) - C_{uv}(1)\; C_{uv}(1)} \;. \tag{5}$$

In applying the formulas (4) and (5) to actual data, it is not obvious how changes in the covariances due to nontidal currents or low-pass filtering degrade the estimates of the coefficients $\underline{a}$ and $\underline{b}$.

2. IDEAL WHITE NOISE FORCING

The term white noise refers to a series which has the same spectral energy at all frequencies. In terms of covariances, this definition is equivalent to requiring each value in the series to be uncorrelated with preceding or following values. Covariances at nonzero lag are zero, by definition, and the covariance at zero lag is the variance γ^2,

$$C_{vv}(k) = \gamma^2 \delta_{o,k} \;, \tag{6}$$

where the Kronecker $\delta_{o,k}$ is 1 for k=0 and zero for $k \neq 0$. Using white noise for the wind in equation (1), simple expressions for the autocovariance of the current and for the cross covariance between the current and wind are derived in the appendix:

$$C_{uu}(k) = \frac{\underline{b}^2 \gamma^2}{(1 - \underline{a}^2)} \underline{a}^{|k|} \;; \tag{7}$$

$$C_{uv}(k) = \underline{b}\gamma^2 \delta_{o,k} \;, \text{ for } k \geq 0. \tag{8}$$

When $\underline{a}$ and $\underline{b}$ are unknown, they may be computed from the covariances; assuming white noise forcing, $C_{uv}(1) = 0$, formulas (4) and (5) reduce to

$$\underline{a}^* = \frac{C_{uu}(1)}{C_{uu}(0)} \; ; \qquad (9)$$

$$\underline{b}^* = \frac{C_{uv}(0)}{C_{vv}(0)} \; . \qquad (10)$$

The formulas (4) through (10) will be compared with numerical results obtained from computer experiments.

Deviations of the coefficients $\underline{a}$ and $\underline{b}$ produce an increase in the residual variance. Defining a nondimensional deviation E of the residual variance by

$$E = \frac{C_{zz}(\underline{a}+\underline{a}', \underline{b}+\underline{b}'; 0) - C_{zz}(\underline{a},\underline{b}; 0)}{\underline{b}^2\gamma^2} \; , \qquad (11)$$

which is normalized to the forcing, the following expression for E may be derived for white noise forcing, using equations (2), (6), (7), and (8):

$$E = \frac{(\underline{a}')^2}{(1-\underline{a}^2)} + \frac{(\underline{b}')^2}{\underline{b}^2} \; . \qquad (12)$$

The increase in residual variance is a quadratic function of both the deviations $\underline{a}'$ and $\underline{b}'$. The coefficient $\underline{a}$ ranges from zero, where frictional coupling to the wind completely overwhelms the water's inertia, to 1, where inertia dominates. For small $\underline{a}$, the increase in the nondimensional residual variance is proportional to the square of the deviation $\underline{a}'$. When $\underline{a}$ is close to 1, small deviations in $\underline{a}$ produce relatively large increases in E. Added to the effect of $\underline{a}'$ on the residual variance is the effect of $\underline{b}'$, which is equal to the square of the fractional deviation ($\underline{b}'$/b). The effect of deviations of the forcing coefficient $\underline{b}$ from its optimal value does not depend on the relative strength of inertia, or autoregression, as measured by $\underline{a}$.

3. NUMERICAL EXPERIMENTS

Several problems arise when the coefficients $\underline{a}$ and $\underline{b}$ are computed from data. The covariances are computed from a limited data set which in turn limits the accuracy to which the covariances can be computed. As discussed by Jenkins and Watts (1968), covariances at adjacent leads affect each other when they are computed from data. Hence, equations (9) and (10) are approximate when using a finite set of discreet data. Since the calculations of $\underline{a}$ and $\underline{b}$ use adjacent covariances, the results may be perturbed by the altered covariances. The covariances may be further affected by low-pass filtering of the data, which is done to resample the current data at the relatively long, wind sampling interval.

In the numerical experiments, the white noise forcing V was generated using a random number function which produces random numbers between ± 0.5. The associated probability density function P(V) is one for $-0.5 \leq V \leq 0.5$ and zero outside the interval. The expectation value for the variance is therefore

$$\gamma^2 = \int_{-\infty}^{\infty} V^2 P(V) dV = \int_{-0.5}^{0.5} V^2 dV$$

$$\text{or} \quad \gamma^2 = \frac{1}{12} = 0.083333 \quad .$$

Ideally, the autocovariance at zero lag should equal the variance derived above, and the autocovariances at nonzero lag should all be zero. Figure 1 compares ideal and computed autocovariances obtained from series generated by the random number function. With a series length of N=1000, the autocovariances approach the ideal values. The data-derived covariances deviate significantly from the ideal values for the shorter series, N=100. This example shows the need for current and wind series of sufficient length to give accurate estimates of the covariances.

The autoregressive series υ_i were computed from equation (1) using the random number function to generate the forcing series V . Figure 2 gives a time series of υ_i for a = 0.95 and b = 0.2; the white noise forcing has been renormalized to unit variance. A comparison between ideal and experimental covariances is given in table 1 for series lengths N=500 and N=1000 and plotted in figure 3 for $C_{\upsilon\upsilon}(k)$, N=1000. The agreement between ideal and experimental covariances appears to be relatively good.

By repeating numerical experiments for fixed values of a, b, and N, a set of statistics was obtained for the covariances. In table 2 are given the means and standard deviations for the covariances as obtained from 10 experiments with a = 0.95, b = 0.20 and N = 1000. The numerical and ideal covariances agree in that they are within 1 standard deviation of each other. Variations in the random forcing produce a larger scatter in the covariances of the autoregressive process and the nonzero lead covariances of the forcing than the scatter of the forcing's variance (zero-lead autocovariance). This relatively large scatter produces less scatter in the estimates of a and b, obtained from equations (4) and (5), because the covariances are highly correlated. In estimating error intervals for numerically obtained a and b, this correlation must be considered. Table 3 gives a and b computed from the covariances in table 2 using the full formulas (4) and (5) and the approximate formulas (9) and (10).

Both the full and approximate formulas give coefficients which agree with the ideal values when the mean covariances are used. Only the full formulas (4) and (5) yield coefficients close to ideal values when the

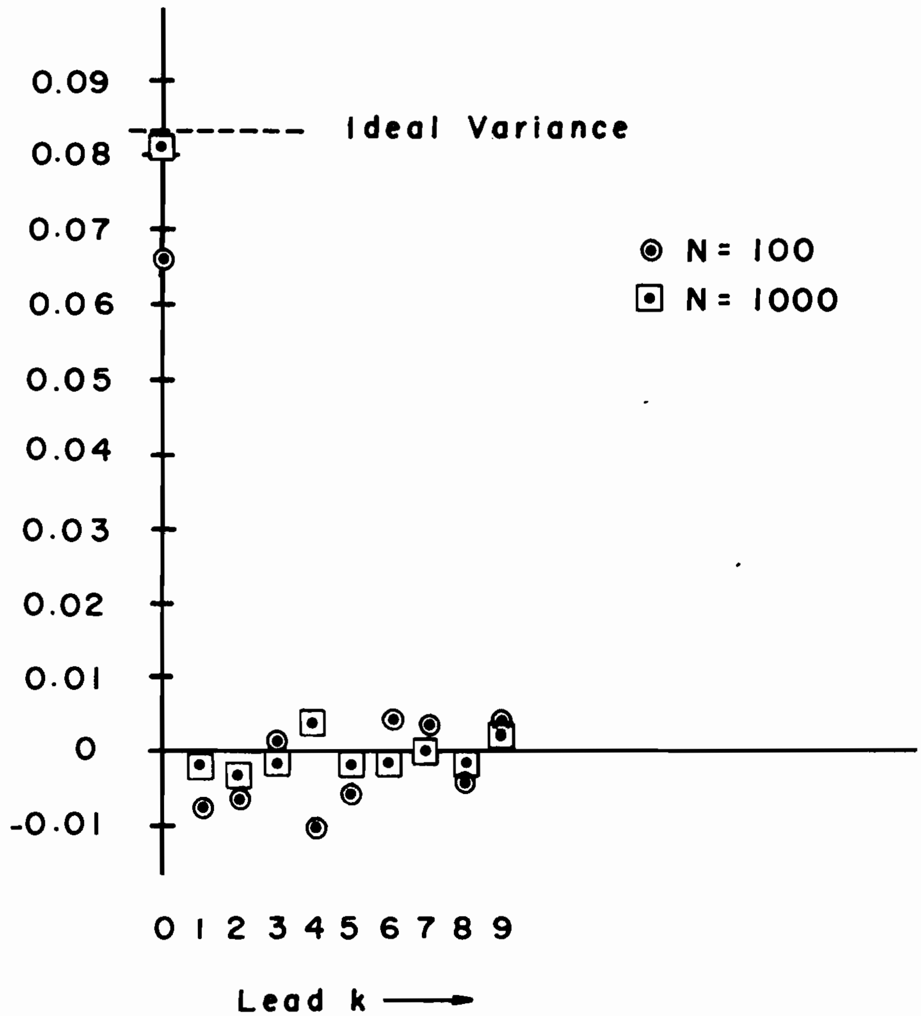

Figure 1. Comparison of the ideal autocovariance for white noise (0.083333 for zero lead and 0.0 for nonzero leads) with covariances computed from white noise time series of length N.

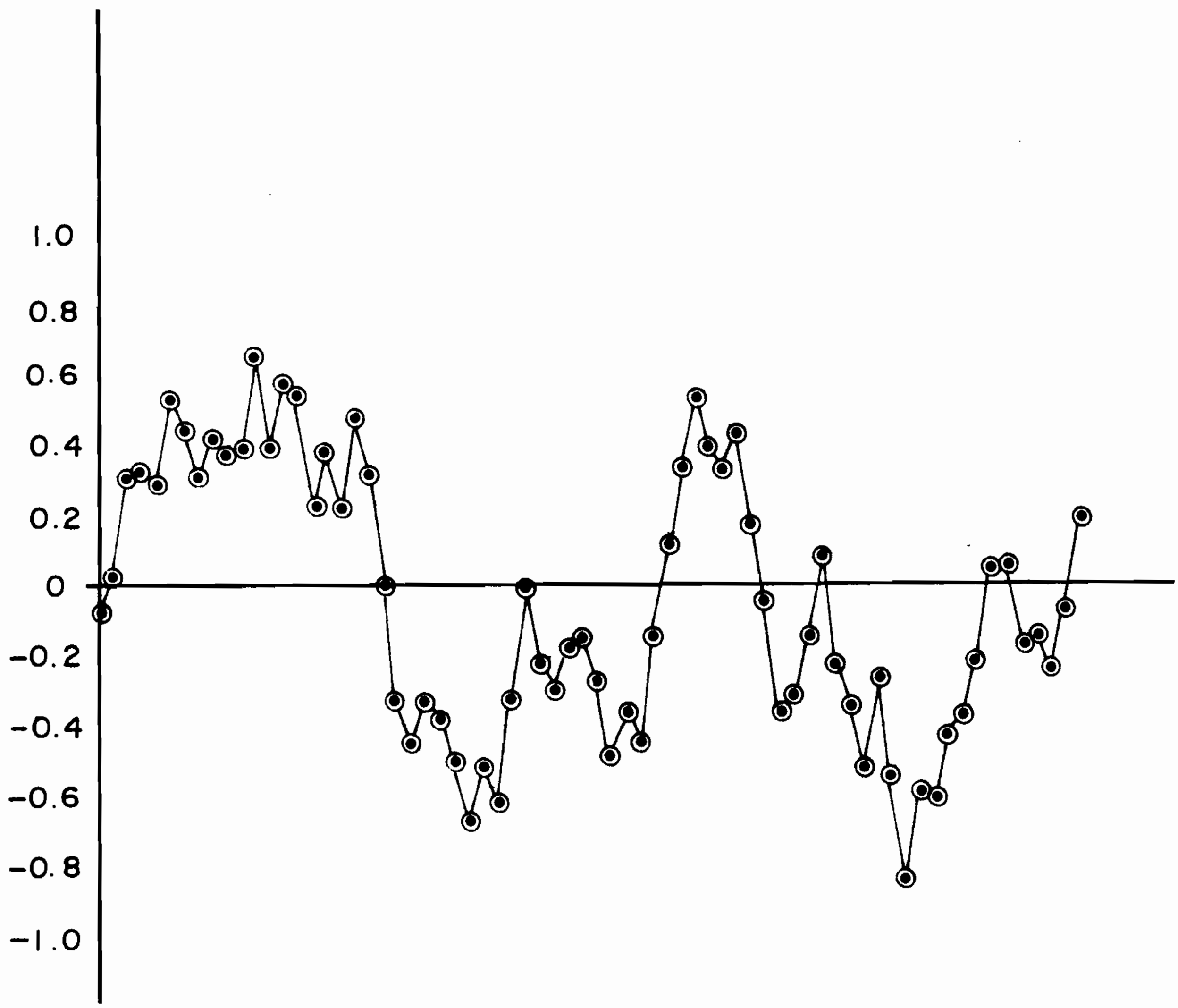

Figure 2. Example of a first-order autoregressive process ($\underline{a}$ = 0.95, $\underline{b}$ = 0.2) forced by white noise of unit variance.

Table 1. Comparison of ideal and experimental covariances (N = series length) a = 0.70, b = 0.20.

$C_{uu}(k)$

k	N = 500	N = 1000	Ideal
0	0.076308	0.075321	0.078431
1	0.053788	0.051760	0.054902
2	0.039529	0.035518	0.038431
3	0.030248	0.026168	0.026902

$C_{uv}(k)$

k	N = 500	N = 1000	Ideal
0	0.19328	0.19545	0.20000
1	0.00227	-0.00373	0.00000
2	0.00929	-0.00280	0.00000
3	0.01290	0.00716	0.00000

$C_{vv}(k)$

k	N = 500	N = 1000	Ideal
0	0.95985	0.99099	1.00000
1	-0.01864	-0.00760	0.00000
2	0.00368	-0.03787	0.00000
3	0.00338	-0.00539	0.00000

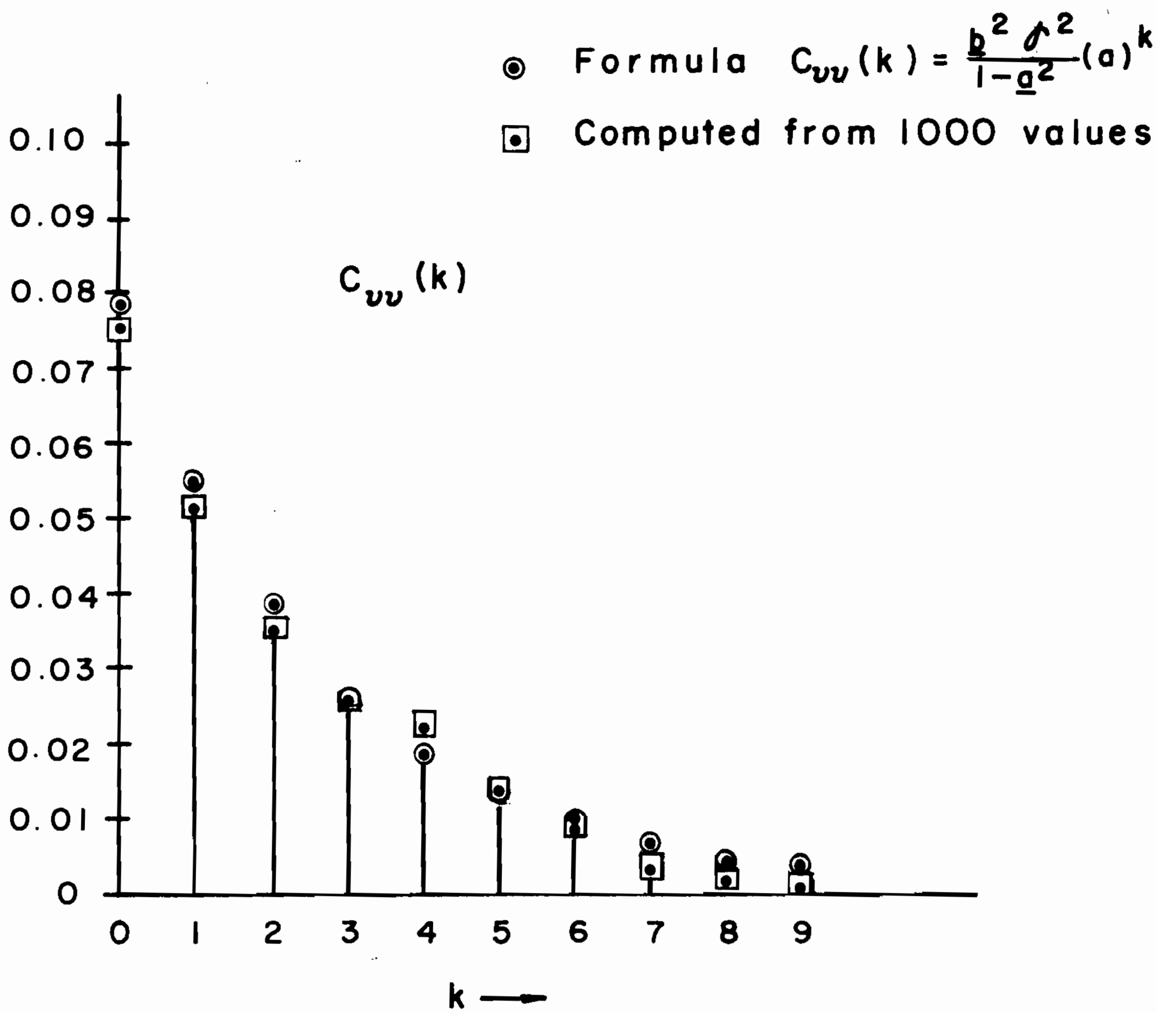

Figure 3. Comparison of the ideal autocovariance of an autoregressive process ($\underline{a}$ = 0.70, $\underline{b}$ = 0.2) with the autocovariance computed from a time series of length N = 1000.

Table 2. Mean covariances and their standard deviations obtained from 10 numerical experiments; a = 0.95, b = 0.20, N = 1000.

$\bar{C}_{\upsilon\upsilon}(k)$

k	N = 1000	Ideal
0	0.40381 ± 0.02566	0.41026
1	0.38311 ± 0.02569	0.38974

$\bar{C}_{\upsilon v}(k)$

k	N = 1000	Ideal
0	0.19929 ± 0.00634	0.20000
1	-0.00016 ± 0.00621	0.00000

$\bar{C}_{vv}(k)$

k	N = 1000	Ideal
0	0.99832 ± 0.00869	1.00000
1	0.00350 ± 0.02858	0.00000

Table 3. Coefficients a and b from covariances.

Coefficients computed from mean covariances obtained from 10 experiments (N = 1000)

	Full formulas (4) and (5)	Approximate formulas (9) and (10)	Ideal values
a	0.94882 ± 0.00392	0.94874 ± 0.00392	0.95
b	0.19978 ± 0.00229	0.19962 ± 0.00229	0.20

Coefficients computed from covariances obtained from one experiment (N = 1000)

	Full formulas (4) and (5)	Approximate formulas (9) and (10)	Ideal values
a	0.94769	0.93904	0.95
b	0.19983	0.18570	0.20

covariances from a single experiment (N=1000) are used. Since these results were obtained from synthetic series uncontaminated by extraneous noise, the differences between the results using the full and approximate formulas must be due to deviations in the covariances for a single experiment. These deviations are correlated such that the full formulas (4) and (5) compensate for these deviations.

To study the effects of extraneous noise on the coefficients a and b, white noise was added to the autoregressive series υ after the series was generated using equation (1) with Z_i=0. Table 4 shows that the computed autoregression coefficient a decreases systematically with increasing noise. The coupling coefficient b is relatively unaffected, except that it is smaller than the ideal value. The extraneous noise in these experiments corresponds to instrumental or high frequency noise not directly related to the autoregressive process. This type of noise is seen to degrade the estimates of the autoregressive coefficient a.

If estimating the coefficients a and b, another source of error occurs when the forcing series V_i is poorly known. A set of experiments was run in which the extraneous noise Z_i was allowed to enter the autoregressive series υ_i through equation (1). Table 5 shows the effects of progressively smaller fractions of the forcing used to compute the coefficients. The computed autoregressive coefficient a is seen to decrease slowly as the additional forcing, not used in V_i, increases. The coupling coefficient b increases more rapidly than a decreases, having a 5.6% error when the additional forcing is equal in amplitude to the forcing V_i.

4. EFFECTS OF LOW-PASS FILTERING

In modeling wind-driven currents as autoregressive processes, the data must often be low-pass filtered. For example, the sampling interval for the current data is 0.5 hr or less while the interval for the wind data is 3.0 hr. The current data must be low-pass filtered and resampled at 3.0 hr to compute cross-covariances. If the cutoff period of the filter is small compared with the characteristic time constant τ (defined in previous lab notes), the filtered data would be expected to yield accurate coefficients. If the cutoff period exceeds the characteristic time constant of the currents, the computed coefficients could differ significantly from the correct values.

In table 6 are shown the coefficients computed from low-passed data where both the forcing and the autoregressive process have been low-passed. For the examples shown, the filtered data is seen to produce coefficients comparable in accuracy with the unfiltered data, even when the cutoff period of the low-pass filter exceeds the characteristic time constant of the autoregressive process. The coefficients were computed using the full formulas (4) and (5). When the approximate formulas (9) and (10) are used, the computed coefficients err significantly from the ideal values. For the second example in table 6, the approximate formulas give a = 0.9953 and b = 11.6314. It is clear that the full formulas must be used with filtered data.

Table 4. Coefficients a and b computed from white noise contaminated autoregressive series υ (N = 1000)

Variance of Extraneous noise	a (Ideal=0.95)	Error (%)	b (Ideal=0.20)	Error (%)
0.00	0.9477	0.2	0.1998	0.1
0.01	0.9200	3.2	0.1943	2.9
0.02	0.8949	5.8	0.1981	1.0
0.03	0.8644	9.0	0.1912	4.4
0.04*	0.8353	12.1	0.1974	1.3
0.05	0.8077	15.0	0.1947	2.6

*Amplitude of the extraneous white noise equals the amplitude of the noise forcing ($\underline{b}^2\gamma^2 = 0.04$) the autoregressive process.

Table 5. Coefficients a and b obtained from fraction of the forcing

$$v_i = \underline{a} v_{i-1} + \underline{b} V_i + \underline{c} z_i$$

where the last term is allowed to affect the autoregressive series

($N = 1000$, $\underline{a} = 0.95$, $\underline{b} = 0.2$, $\gamma^2 = 1$)

c	a	Error (%)	b	Error (%)
0.0	0.9477	0.2	0.1998	0.1
0.1	0.9424	0.8	0.2055	2.8
0.2*	0.9381	1.3	0.2113	5.6
0.5	0.9344	1.7	0.2287	14.4
1.0	0.9341	1.7	0.2578	28.9

*Amplitude of the additional forcing equals the amplitude of the forcing series V_i.

Table 6. Coefficients a and b from low-passed data (both υ and V filtered after υ was generated from the white noise forcing V); N = 875.

Filter's 6 dB Period	AR process* time constant	a		b	
		Data	Ideal	Data	Ideal
6.0 Δt	19.0 Δt	0.9478	0.95	0.1972	0.20
24.0 Δt	19.0 Δt	0.9480	0.95	0.2014	0.20

* $\tau = \frac{a \Delta t}{1-a}$, from H. Mofjeld (1975).

5. NONWHITE NOISE FORCING

In general, the wind cannot be represented realistically as white noise. While the discussion in previous sections is essential background for modeling wind-driven currents, other numerical experiments are needed to study the procedure for computing coefficients from data where the forcing is not white noise. The wind typically has a red spectrum with decreasing spectral energy with increasing frequency. In the numerical experiments described below, the wind is modeled by low-pass filtering white noise.

Figure 4 shows the autocovariance of the forcing before and after low-pass filtering where the 6 dB period of the filter is 6.0 Δt. The low-pass filter has a Lanczos-squared form. The covariance takes on the form of the filter since it is convolved with a delta function (covariance of white noise). The corresponding autoregressive process' autocovariance and the cross-covariance are shown in figures 5 and 6, respectively. Using formulas (4) and (5), the covariances yield the coefficients $\underline{a} = 0.9500$ and $\underline{b} = 0.1993$, which compare well with the ideal values, $\underline{a} = 0.95$ and $\underline{b} = 0.20$. These formulas are therefore able to produce accurate coefficients from data in which the forcing is not white noise.

6. CONCLUDING REMARKS

Jenkins and Watts (1968) dissuade investigators from using the approach used above to compute the coefficients $\underline{a}$ and $\underline{b}$; they advocate a spectral approach rather than one using covariances. Their argument is essentially based on the inaccurate results obtained from the approximate formulas (9) and (10), the errors resulting from the correlation between adjacent covariances. The numerical experiments above show that the approach using covariances does yield accurate results when the full formulas (4) and (5) are used. This approach also avoids a myriad of problems arising from the spectral approach.

7. REFERENCES

Jenkins, G. M., and Watts, D. G., 1968. Spectral analysis and its applications. Holden-Day, San Francisco, 525 pp.

Mofjeld, H. O., 1975. Wind-driven currents; generalized approach. (unpublished manuscript)

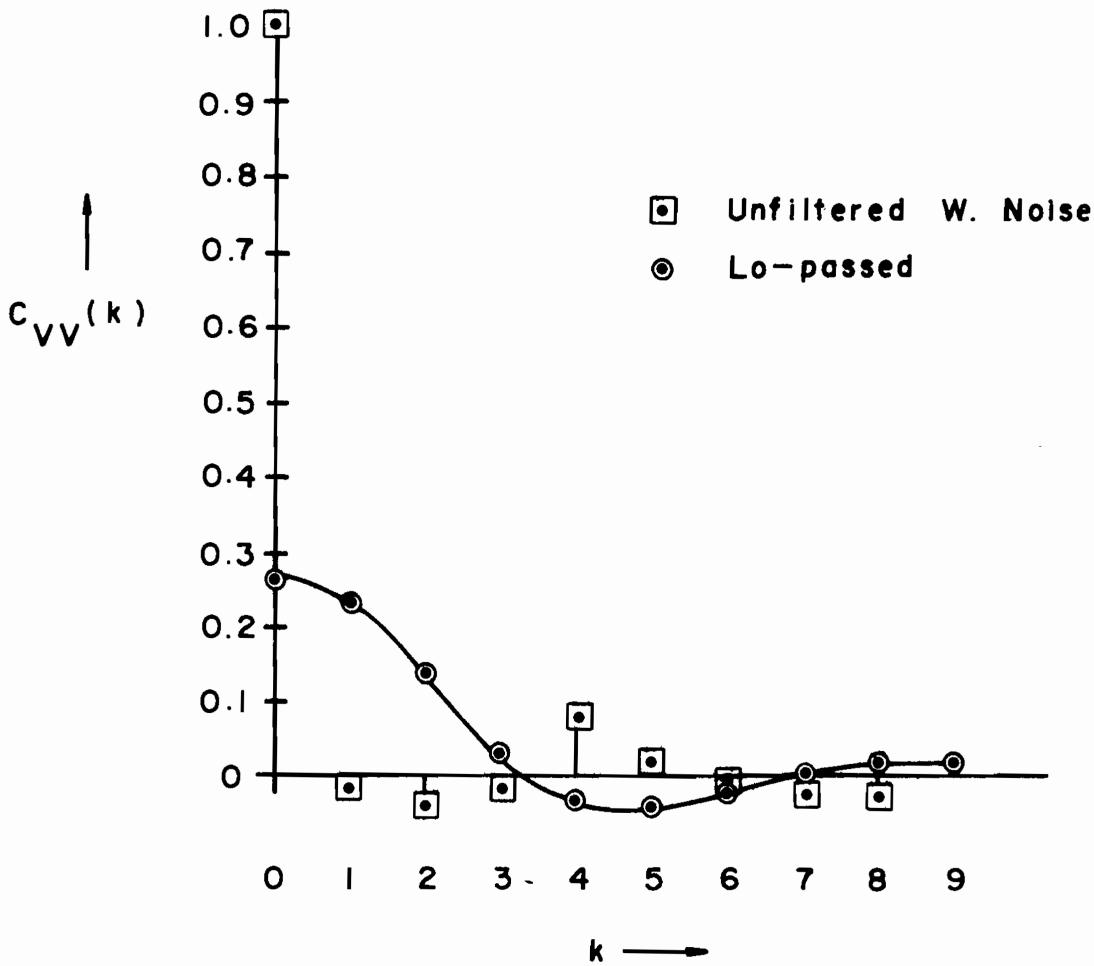

Figure 4. Autocovariances of white noise left unfiltered and low-pass filtered; each autocovariance was computed from a time series of length N = 1000.

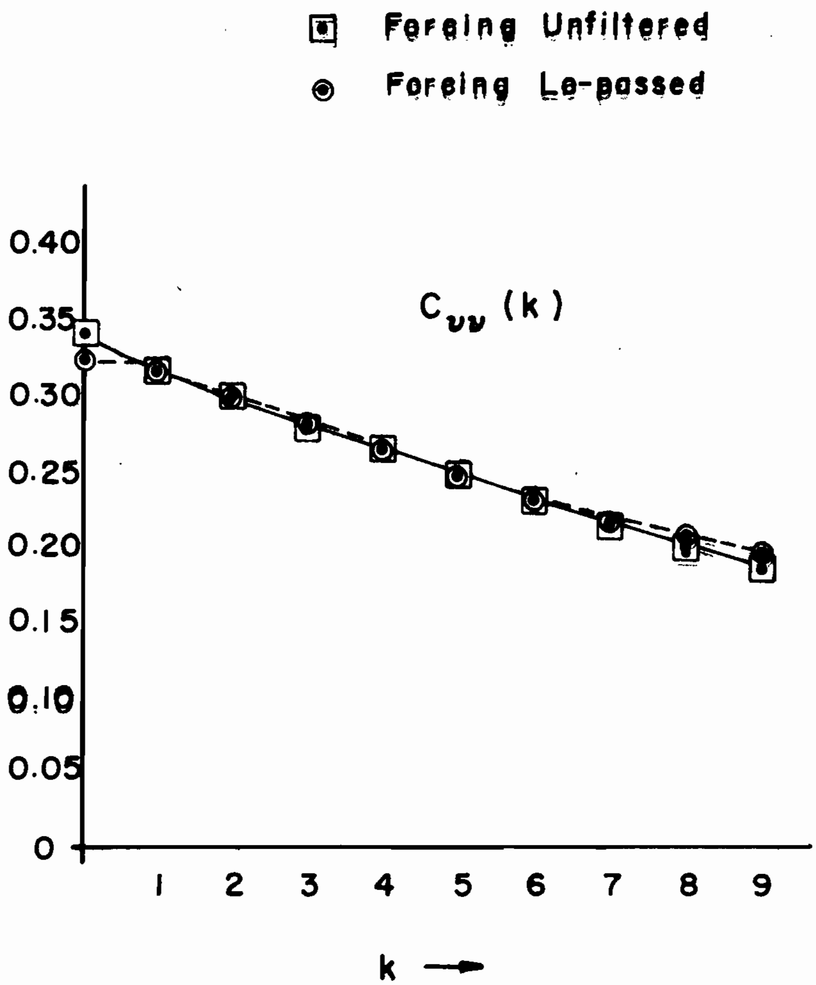

Figure 5. Autocovariance of a first-order autoregressive process (a = 0.95, b = 0.2) when the white noise forcing is unfiltered and when it is low-pass filtered. The autocovariances were computed from a time series of length N = 1000.

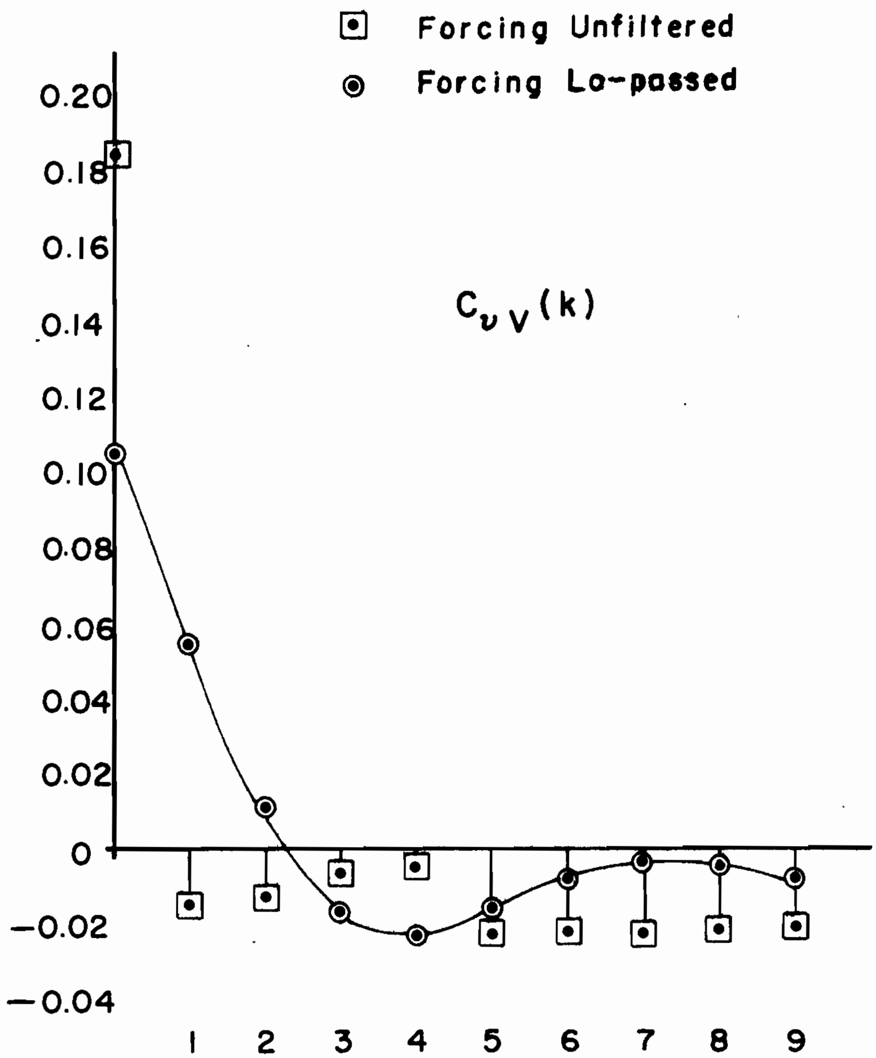

Figure 6. Cross-covariances of a first-order autoregressive process ($\underline{a}$ = 0.95, $\underline{b}$ = 0.2) and its white noise forcing where the latter is unfiltered and when it has been low-pass filtered.

APPENDIX

ANALYTIC EXPRESSIONS FOR THE COVARIANCES

$C_{\upsilon\upsilon}(k)$ and $C_{\upsilon V}(k)$ WHEN V IS WHITE NOISE

For the first-order autoregressive process υ uncontaminated by extraneous noise Z and forced by white noise V, the individual values υ_i in a uniformly sampled time series satisfy the equation

$$\upsilon_i = \underline{a}\upsilon_{i-1} + \underline{b}V_i \ . \qquad \text{(A1)}$$

Since $\upsilon_{i+k} = \underline{a}\upsilon_{i-1+k} + \underline{b}V_{i+k}$,

$$(\upsilon_i - \underline{a}\upsilon_{i-1})(\upsilon_{i+k} - \underline{a}\upsilon_{i-1+k}) = \underline{b}^2 V_i V_{i+k} \ ,$$

or

$$-\underline{a}C_{\upsilon\upsilon}(k-1) + (1+\underline{a}^2)C_{\upsilon\upsilon}(k) - \underline{a}C_{\upsilon\upsilon}(k+1) = \underline{b}^2 C_{VV}(k) \ . \qquad \text{(A2)}$$

For white noise, $C_{VV}(k) = \gamma^2\delta_{o,k}$. Equations (A2) may be put into matrix form

$$M\,\vec{C}_{\upsilon\upsilon} = \vec{C}_{VV} \ , \qquad \text{(A3)}$$

where

$$M = \begin{pmatrix} 1+\underline{a}^2 & -2\underline{a} & 0 & 0 & \cdots \\ -\underline{a} & 1+\underline{a}^2 & -\underline{a} & 0 & \cdots \\ 0 & -\underline{a} & 1+a^2 & -\underline{a} & \cdots \\ 0 & 0 & -\underline{a} & 1+\underline{a}^2 & \cdots \\ 0 & 0 & 0 & -\underline{a} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix} ,$$

$$\vec{C}_{\upsilon\upsilon} = \begin{pmatrix} C_{\upsilon\upsilon}(0) \\ C_{\upsilon\upsilon}(1) \\ C_{\upsilon\upsilon}(2) \\ C_{\upsilon\upsilon}(3) \\ \cdots \end{pmatrix} , \text{ and } C_{VV} = \begin{pmatrix} \underline{b}^2\gamma^2 \\ 0 \\ 0 \\ 0 \\ \cdots \end{pmatrix} .$$

Equation (A3) can be readily solved by Cramer's rule where the determinants are expanded by minors. The determinant, $|M|$, which appears in the denominator of the solution, is expanded along the first row,

$$|M| = (1+\underline{a}^2)\,D_o - 2\underline{a}^2 D_o = (1-\underline{a}^2)D_o \ ,$$

where

$$D_o = \begin{vmatrix} 1+\underline{a}^2 & -\underline{a} & 0 & \dots \\ -\underline{a} & 1+\underline{a}^2 & -\underline{a} & \dots \\ 0 & -\underline{a} & 1+\underline{a}^2 & \dots \\ 0 & 0 & -\underline{a} & \dots \\ \dots & \dots & \dots & \end{vmatrix} .$$

By induction, it can be shown that D_o converges ($|\underline{a}| < 1$) to $(1-\underline{a}^2)^{-1}$, implying that $|M| = 1$. For $C_{\upsilon\upsilon}(0)$, an expansion by minors along the first column

$$C_{\upsilon\upsilon}(0) = \frac{1}{|M|}\begin{vmatrix} b^2\gamma^2 & -2\underline{a} & 0 & \dots \\ 0 & 1+\underline{a}^2 & -\underline{a} & \dots \\ 0 & -\underline{a} & 1+\underline{a}^2 & \dots \\ 0 & 0 & -\underline{a} & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \end{vmatrix} ,$$

gives $C_{\upsilon\upsilon}(0) = \underline{b}^2\gamma^2 D_o$

or $$C_{\upsilon\upsilon}(0) = \frac{\underline{b}^2\gamma^2}{(1-\underline{a}^2)} . \qquad \text{(A4)}$$

Since $\upsilon_{i+k} = \underline{a}\upsilon_{i-1+k} + \underline{b}V_{i+k}$,

$$\upsilon_i\upsilon_{i+k} = \underline{a}\upsilon_i\upsilon_{i-1+k} + \underline{b}\upsilon_i V_{i+k} ,$$

which when summed over i yields

$$C_{\upsilon\upsilon}(k) = \underline{a}C_{\upsilon\upsilon}(k-1) + \underline{b}C_{\upsilon V}(k) .$$

Since V is white noise, its value is uncorrelated with its past values and the past values of υ. Hence,

$$C_{\upsilon\upsilon}(k) = \underline{a}C_{\upsilon\upsilon}(k-1)$$

or $$C_{\upsilon\upsilon}(k) = \frac{\underline{b}^2\gamma^2}{(1-\underline{a}^2)}\underline{a}^{|k|} . \qquad \text{(A5)}$$

In like manner,

$$\upsilon_i V_i = \underline{a}\upsilon_{i-1}V_i + \underline{b}V_i V_i$$

or $$C_{\upsilon V}(0) = \underline{b}\gamma^2 . \qquad \text{(A6)}$$

Because V is white noise,

$$C_{\upsilon V}(k) = 0 \text{ for } k > 0 . \qquad \text{(A7)}$$

Further,

$$\upsilon_i V_{i+k} = \underline{a}\upsilon_{i-1} V_{i+k} + \underline{b} V_i V_{i+k}$$

or $$C_{\upsilon V}(k) = \underline{a} C_{\upsilon V}(k+1) \text{ for } k \neq 0 \text{ .}$$

Hence,

$$C_{\upsilon V} = \begin{bmatrix} \underline{b}\gamma^2 \underline{a}^{|k|} \text{ , } & k \leq 0 \\ 0, & k > 0 \end{bmatrix} \text{ .} \tag{A8}$$

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