Equations Governing Space-Time Variability of Liquid Water Path in Stratus Clouds

K. Ivanova
Pennsylvania State University
University Park, Pennsylvania

T. P. Ackerman
Pacific Northwest National Laboratory
Richland, Washington

M. Ausloos
University of Liège
B-4000 Liège, Belgium

Abstract

We present a method on how to derive an underlying mathematical (statistical or model free) equation for a liquid water path (LWP) signal directly from empirical data. The evolution of the probability density functions (PDFs) from small to large time scales is explicitly derived in the framework of Fokker-Planck equation. A drift and a diffusion term describing the deterministic and stochastic influences on the non-Gaussian fat tails of the liquid water probability distributions are obtained from empirical data.

Introduction

Knowledge of space-time distribution of liquid water is of paramount importance for the global circulation models. However, these models contain no parameterization of the distribution of liquid water in a grid box. We present a method on how to derive an underlying mathematical (statistical or model free) equation – the Fokker-Planck equation that governs time dependent distributions of fluctuations at different time delays starting from empirical data of a LWP. As it is pertinent to natural sequences, the LWP signal is non-stationary with highly irregular and clustered fluctuations due to a set of various influences over different time and space scales. Thus, it is of interest to distinguish and quantify from first principals the deterministic and stochastic influences on the LWP signal in stratus clouds. It is known that two equivalent master equations govern the dynamics of a system, i.e., the Fokker-Planck equation and the Langevin equation, the former for the probability distribution function of time and space signal increments, the latter for the increments themselves. They both are condensates of the huge set of (6N) Hamilton equations that should in practice describe the whole dynamics of the system of N particles by giving the time evolution of both coordinates and momenta of each individual particle. This is a highly unrealistic scheme of work, and therefore conservation laws are used in order to derive the Navier-Stokes equations, from the averaging of basic quantities weighted by the above probability functions. Therefore, starting from LWP observations, we obtain the Langevin equation that represents the change with time of the increments of LWP signal and the Fokker-Planck equation that governs the time dependent distributions of the LWP. These results may contribute to incorporating the space-time distribution of liquid water in global circulation models.

Data and Method of Analysis

Ground based microwave radiometer measurements of the atmosphere at the Southern Great Plains site of Atmospheric Radiation Measurement (ARM) Program of the U.S. Department of Energy during the period of January 9 to January 14, 1998, are considered. The microwave radiometer measures the brightness temperature at two frequency channels, one at 23.8 GHz and the other at 31.4 GHz. Then both brightness temperature data series are used to retrieve the vertical columnar amount of water vapor and vertical columnar amount of liquid water in the cloud, the latter called LWP. The LWP is the most representative quantity of the atmosphere in presence of clouds. Data series x(t) of the LWP in stratus clouds existing during the January 9-14, 1998, period (thus, 25 772 data points) and measured with resolution Δt_0 = 20 sec is shown in Figure 1.

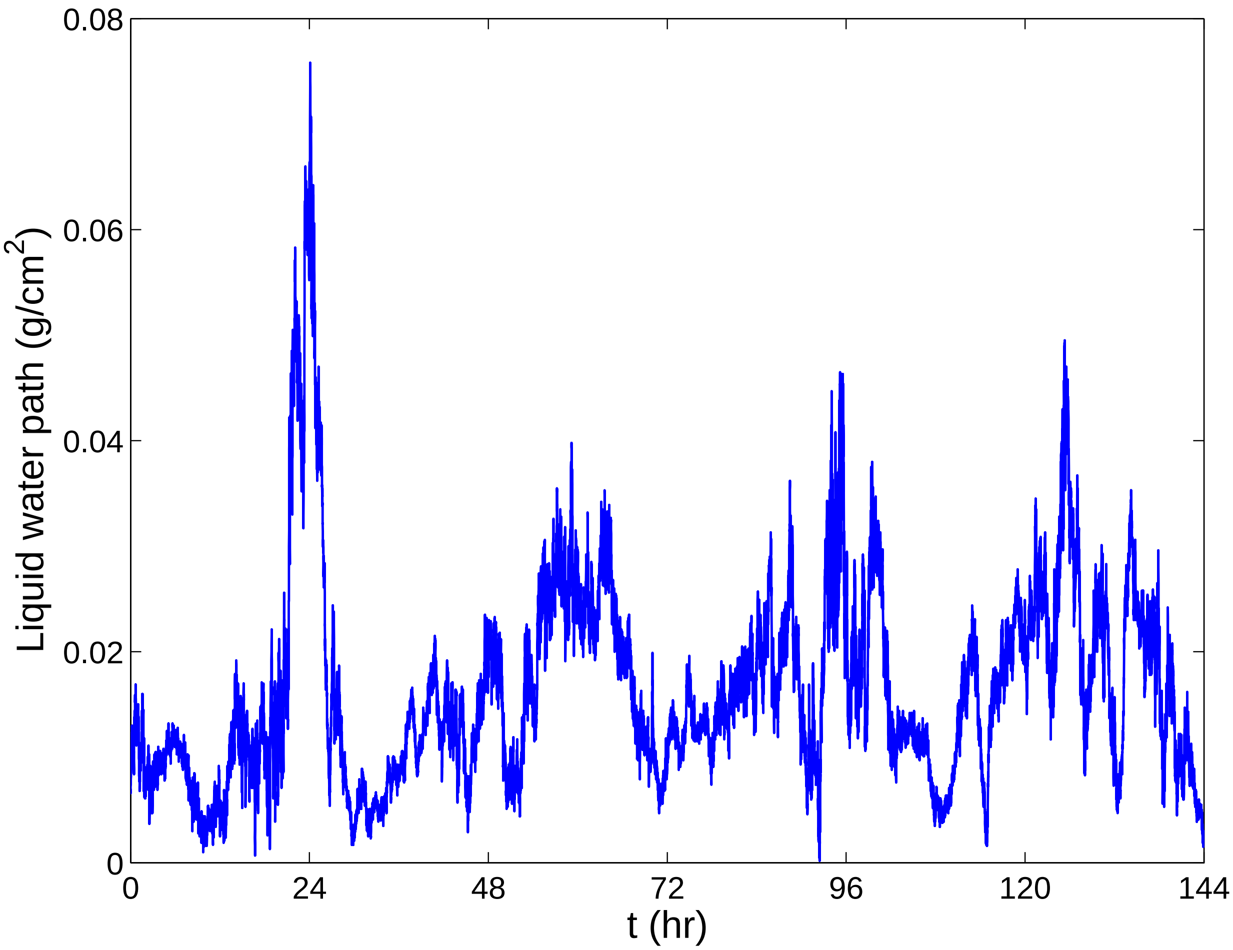


Figure 1. Time dependence of a stratus cloud LWP from January 9 to 14, 1998.

In previous studies, we have shown that scaling and long range correlations exist between the fluctuations of the LWP signal in broad range of time scales, e.g., between about 5 min and 3 hrs (Ivanova et al. 2000; Ivanova et al. 2002). These scaling properties are a signature of the time cross-correlations between the fluctuations of the signal. The fluctuations of the signal can be measured by returns $r(t) = x(t+\Delta t)/x(t)$, increments $\Delta x = x(t+\Delta t) - x(t)$ or normalized increments. Recently was suggested that the manner of calculating the increments, e.g., left-justified $\Delta x = x(t+\Delta t) - x(t)$ or centered $\Delta x = x(t+\Delta t/2) - x(t-\Delta t/2)$ can influence the results from scale-dependent analysis of stochastic data (Friedrich et al. 2000). We will use the centered definition of increments in order to avoid possible spurious correlations that the left-justified definition may introduce (Friedrich et al. 2000). In this study we are concerned with the statistics and evolution of the increments of the signal Δx at different time lags Δt (Waechter et al. XXXX; Ivanova and Ausloos 2002). Applying the Fokker-Planck equation approach the statistics and evolution of the normalized increments of LWP signal is presented in (Ivanova and Ausloos 2002). Probability distribution functions of the LWP signal for different time delays Δt = 40s, 8 min, 80 min, are shown in Figure 2.

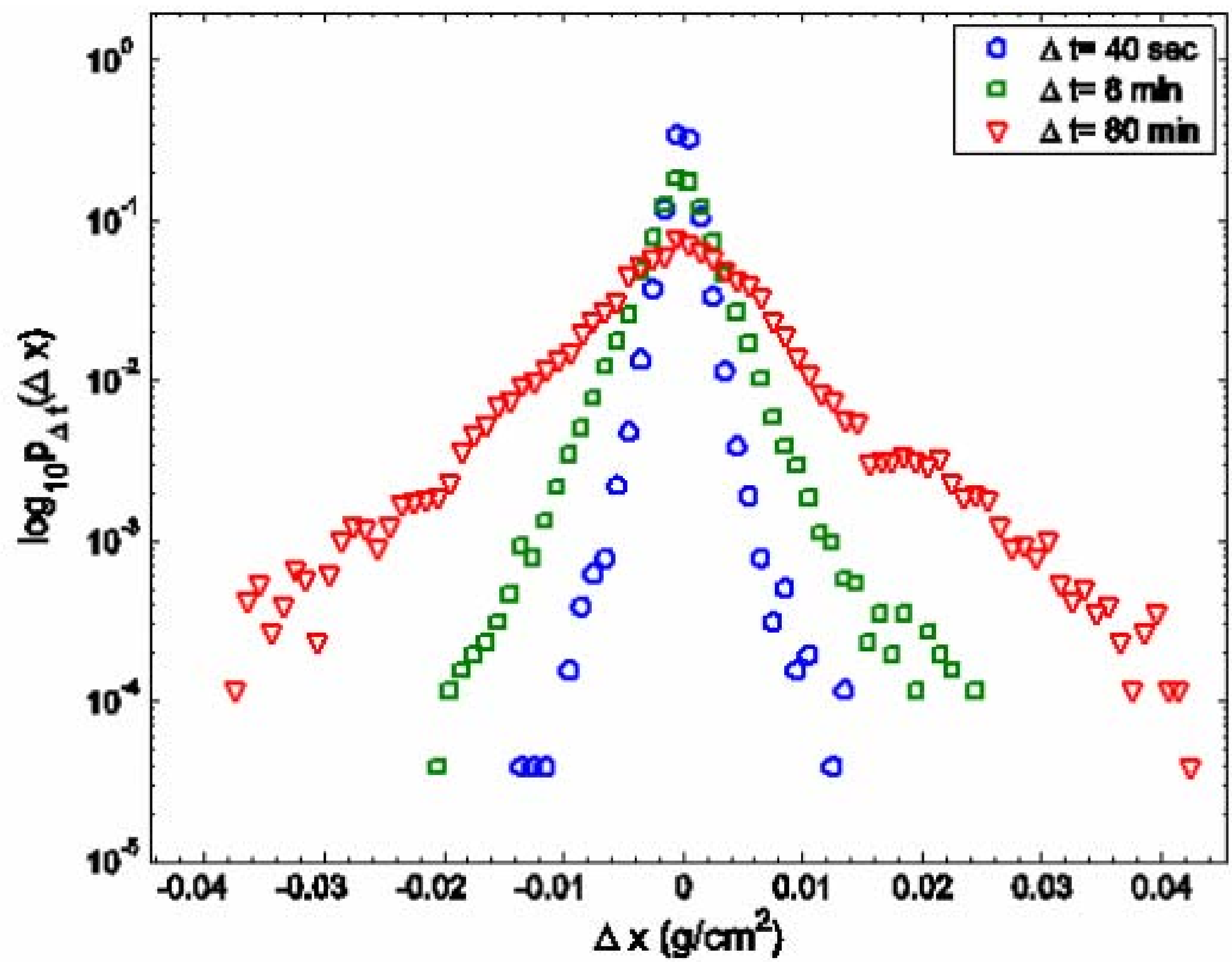


Figure 2. Frequency $p(\Delta x, \Delta t)$ of the LWP increments Δx for different time delays Δt.

In order to characterize the statistics of the LWP changes, LWP increments Δx1, Δx2 for delay times Δt1, Δt2 at the same time t are considered. The corresponding joint PDFs are evaluated for various time delays Δtn <Δtn-1 <… < Δt2 <Δt1 directly from the given data set.

A contour plot of the joint PDF for $\Delta t_1 = 4\Delta t_0$ and $\Delta t_2 = 2\ \Delta t_0$ is shown in Figure 3. If two LWP changes, i.e., Δx_1 and Δx_2 are statistically independent, the joint PDF should factorize into a product of two PDFs:

$$P(\Delta x_2, \Delta t_2; \Delta x_1, \Delta t_1) = P(\Delta x_2, \Delta t_2)p(\Delta x_1, \Delta t_1)$$

However, the tilted anisotropic form of the joint PDF clearly shows that such a factorization does not hold for small values of $|\log(\Delta t_2/\ \Delta t_1)|$, whence the two LWP changes are statistically dependent. The same is found to be true for other $(\Delta t_i/\Delta t_j)$ ratios. To analyze these correlations in more detail, the question on what kind of statistical process underlies the LWP changes over a series of nested time delays Δt_i of decreasing duration should be raised.

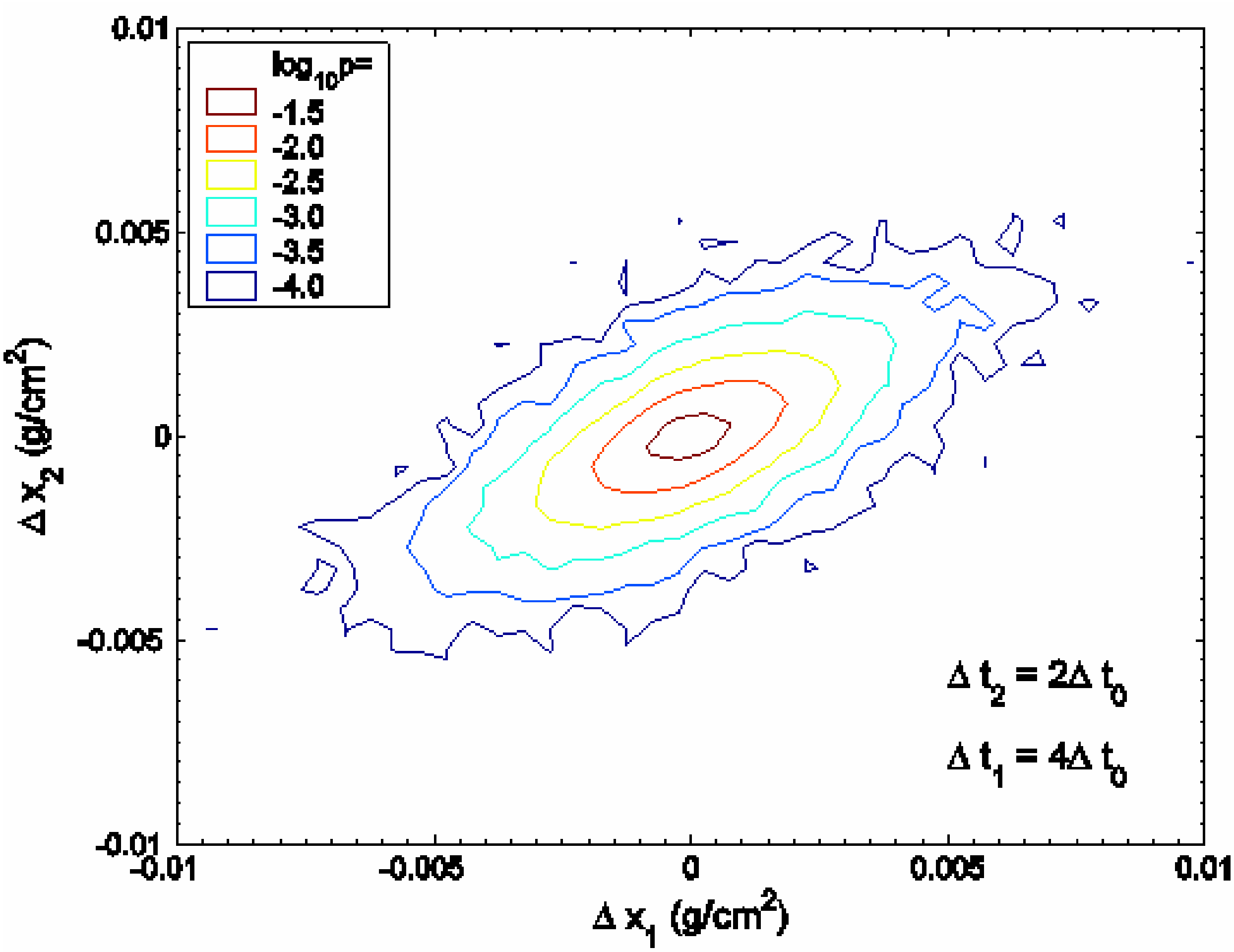


Figure 3. Contour plot of the joint LWP increments $p(\Delta x_2, \Delta t_{2;}\ \Delta x_1, \Delta t_1)$ for $\Delta t_2 = 40$ s and $\Delta t_1 = 80$s.

A complete characterization of the statistical properties of the data set in general requires the evaluation of joint PDF's $p^N(\Delta x_1,\Delta t_1;...;\Delta x_N,\Delta t_N)$ depending on N variables (for arbitrarily large N). In the case of a Markov process, an important simplification arises: The N-point PDF p^N is generated by a product of the conditional probabilities $p(\Delta x_{i+1},\Delta t_{i+1}|\Delta x_i,\Delta t_i) = p(\Delta x_{i+1},\Delta t_{i+1};\Delta x_i,\Delta t_i)/p(\Delta x_i, \Delta t_i)$ for i = 1,..., N-1. The conditional probability is given by the probability of finding Δx_{i+1} values for fixed Δt_{i+1}.

As a necessary condition, the Chapman-Kolmogorov equation

$$\mathrm{p}(\Delta x_2,\Delta t_2\,|\Delta x_1,\Delta t_1) = \int d(\Delta x_i)\mathrm{p}(\Delta x_2,\Delta t_2\,|\Delta x_i,\Delta t_i)\mathrm{p}(\Delta x_i,\Delta t_i\,|\Delta x_1,\Delta t_1) \tag{1}$$

should hold for any value of Δt_i, with $\Delta t_2 < \Delta t_i < \Delta t_1$. We checked the validity of the Chapman-Kolmogorov equation for different Δt_i triplets by comparing the directly evaluated conditional probability distributions $p(\Delta x_{i+1},\Delta t_{i+1}|\Delta x_i,\Delta t_i)$ with the ones calculated according to Eq.1. Results confirming that the LWP PDF's satisfy Eq. 1 are plotted in Figure 4 (Ivanova and Ausloos 2002).

The Chapman-Kolmogorov equation formulated in differential form yields a master equation, which can take the form of a Fokker-Planck Eq. (1) and (2)

$$\frac{d}{d\tau}\mathrm{p}(\Delta x,\tau) = \left[-\frac{\partial}{\partial\Delta x}D^{(1)}(\Delta x,\tau) + \frac{\partial^2}{\partial^2\Delta x}D^{(2)}(\Delta x,\tau)\right]\mathrm{p}(\Delta x,\tau) \tag{2}$$

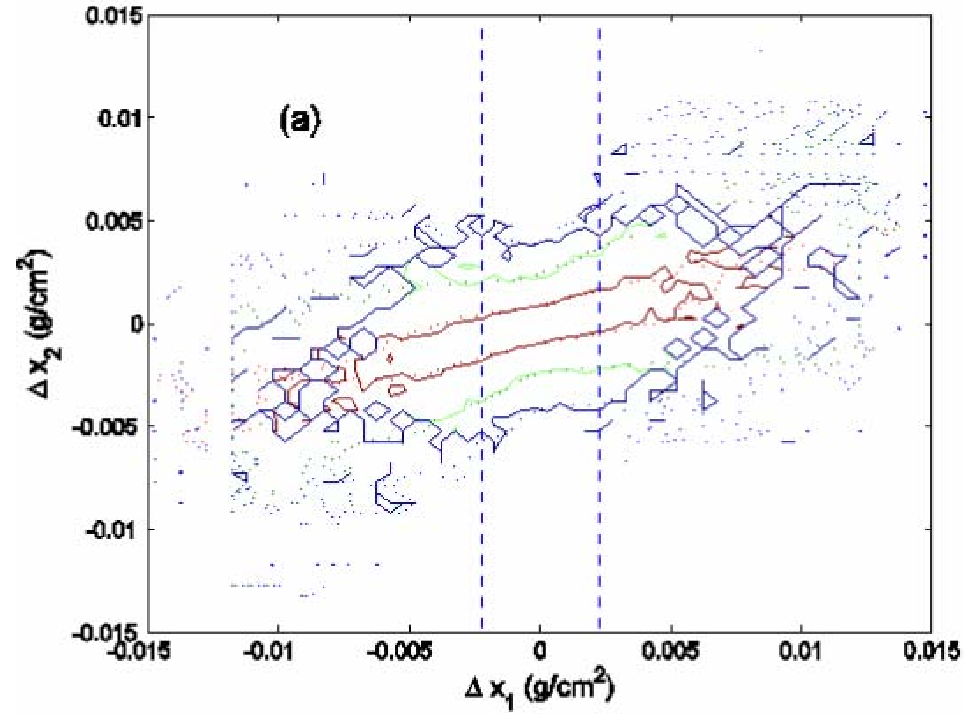
in terms of a drift $D^{(1)}(\Delta x,\tau)$ and a diffusion $D^{(2)}(\Delta x,\tau)$ term. The functional dependence of the drift and diffusion terms can be estimated directly from the moments $M^{(k)}$ of the conditional probability distributions

$$D^{(k)}(\Delta x,\tau) = \frac{1}{k!}\lim_{\Delta\tau\to 0} M^{(k)}(\Delta x,\tau,\Delta\tau) \tag{3}$$

$$M^{(k)}(\Delta x,\tau,\Delta\tau) = \frac{1}{\Delta\tau}\int d(\Delta x')(\Delta x' - \Delta x)^k \mathrm{p}(\Delta x',\tau+\Delta\tau\,|\,\Delta x,\tau) \tag{4}$$

From a careful analysis of the data based on the functional dependences it is found that the drift term $D^{(1)}$ is well approximated by a linear function of Δx, whereas the diffusion term $D^{(2)}$ follows a polynomial of degree two in Δx (Figure 5).

$$\begin{aligned} D^{(1)} &= -\,0.11\Delta x \\ D^{(2)} &= \;\; 0.17(\Delta x)^2 - 0.0001\Delta x \end{aligned} \tag{5}$$



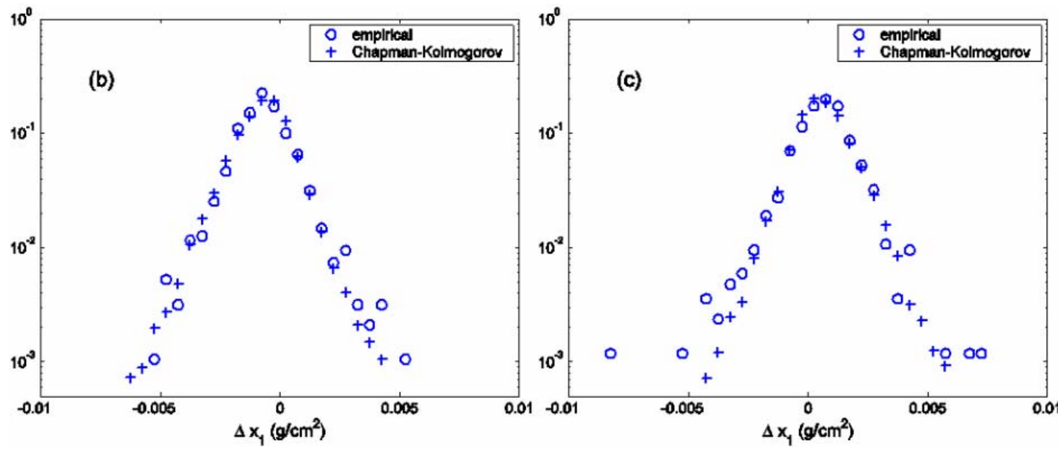


Figure 4. Conditional PDF $p(\Delta x_2, \Delta t_2|\Delta x_1, \Delta_1)$ for $\Delta t_2 = 6\ \Delta t_0$ and $\Delta t_1 = 2\ \Delta t_0$ from direct calculations and from Eq. 1. Slices are taken at $\Delta x_1 = 0.00225$ g/cm^2.

Finally, the Fokker-Planck equation for the distribution function is known to be equivalent to a Langevin equation for the variable, i.e., Δx here, (within the Ito interpretation [Reichl 1980; Risken 1984]):

$$\frac{d}{d\tau}\Delta x(\tau) = D^{(1)}(\Delta x(\tau),\tau) + \eta(\tau)\sqrt{D^{(2)}(\Delta x(\tau),\tau)} \tag{6}$$

where $\eta(\tau)$ is a fluctuating δ-correlated force with Gaussian statistics $<\eta(\tau)\eta(\tau')> = 2\delta(\tau-\tau')$.

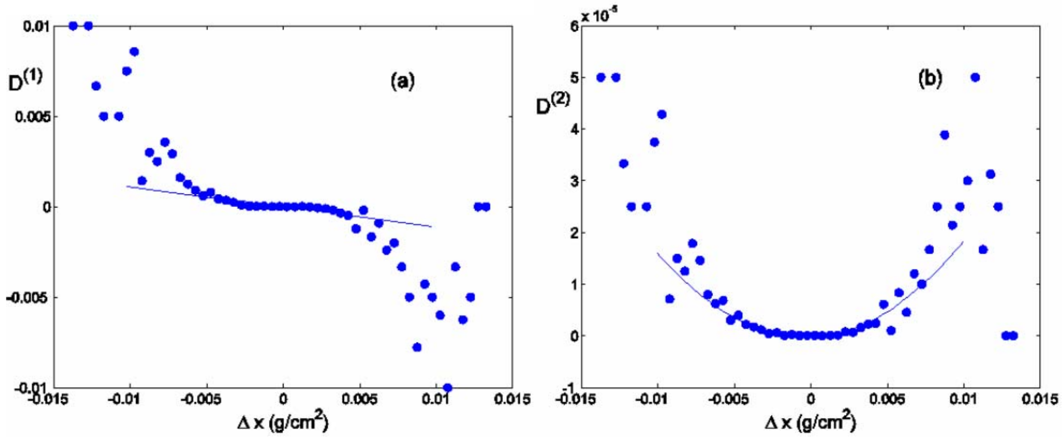


Figure 5. Kramers-Moyal coefficients (a) $D^{(1)}$ and (b) $D^{(2)}$ estimated from conditional PDF $p(\Delta x_{2,}\ \Delta t_2 | \Delta x_{1,}\ \Delta_1)$; Δt_2=40 s and Δt_1=80 s. The solid curves present a linear and a quadratic fit, respectively, for Δx in the interval (-0.005, 0.005) g/cm^2.

Conclusions

At least since the pioneering work of Lorenz (1963) stochastic problems in turbulence are commonly treated as processes running in time t with long time correlations. Inspired by the idea of an existing energy cascade process in clouds (Cahalan 1994) we present here a new approach, namely, we investigate how LWP changes are correlated on different time steps Δt. The PDF shape expresses an unexpected high probability (compared to a Gaussian PDF) of large LWP changes. Furthermore, in contrast to the use of phenomenological fitting functions, the above method provides the evolution process of PDF's from small time delays to larger ones. This is through an analogy with two physically meaningful coefficients, a drift term $D^{(1)}$ and a diffusion term $D^{(2)}$. The first one behaves linearly, thus looks like a "restoring force," the second behaving quadratically in Δx, is obviously like an autocorrelation function as for diffusion. Finally, we present a method on how to derive an underlying mathematical (statistical or model free) equation for a LWP cascade from empirical data. The method yields an effective stochastic equation in the form of a Fokker-Planck equation for the PDF of the LWP signal. Further studies would focus on the implementation of these findings in cloud modeling.

Acknowledgements

This research was supported by the Office of Science (BER), U.S. Department of Energy, Grants No. DE-FG02-04ER63773 and DE-FG02-90ER61071.

Corresponding Author

K. Ivanova, kristy@essc.psu.edu, (814) 863-4722

References

Cahalan, R. F., 1994: *Nonlinear Proc. Geophys*., **1**, 156.

Friedrich, R., J. Peinke, and Ch. Renner, 2000: *Phys. Rev. Lett*., **84**, 5224.

Ivanova, K., and M. Ausloos, 2002: *J. Geophys. Res. - Atmospheres*, 107, D23, 4708.

Ivanova, K., M. Ausloos, E. E. Clothiaux, and T. P. Ackerman, 2000: *Europhys. Lett*., 52, 40.

Ivanova, K., E. E. Clothiaux, H. N. Shirer, T. P. Ackerman, J. C. Liljegren, and M. Ausloos, 2002: *J. Appl. Meteorology*, **41**, 56.

Kolmogorov, A. N., 1941: *Doklady Akad. Nauk SSSR*, **30**, 299.

Lorenz, E. N., 1963: *J. Atmos. Sci*., 20, 130.

Parisi, G., and U. Frisch, 1985: In *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, eds., M. Ghil, R. Benzi, and G. Parisi. North Holland, New York.

Reichl, L. E., 1980: *A Modern Course in Statistical Physics*, Univ. Texas Press, Austin, Texas.

Risken, H., 1984: *The Fokker-Planck Equation*, Springer-Verlag, Berlin.

Waechter, M., A. Kouzmitchev, and J. Peinke, XXXX: A note on increment definitions for scale dependent analysis of stochastic data, *arXiv:physics/0404021 v1*