Cryptanalysis on SHA-1

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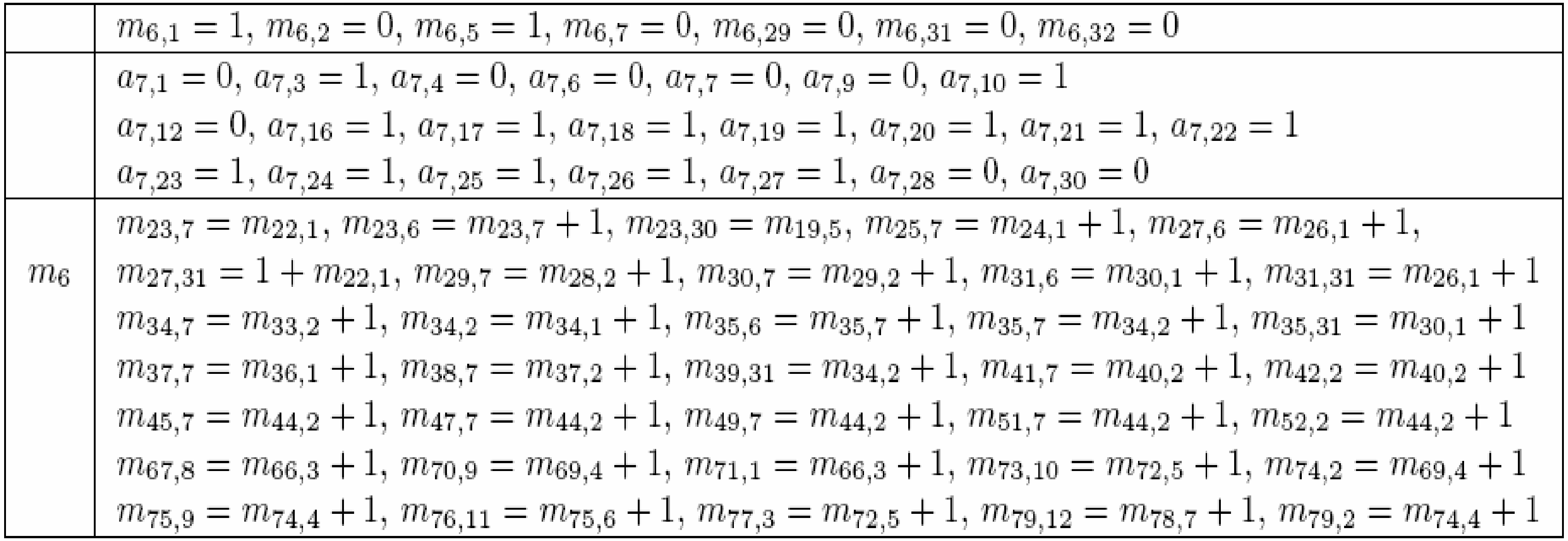
Outline

- Obstacles for further improvement on SHA-1 attack
- New collision path for SHA-1 (First iteration path)
- Comparing new collision path with previous path
- Strategies for message modification
- Details of message modification
- The complexity of searching for collisions

Obstacles for Further Improvement on SHA-1 Attack

- Unlike SHA-0 and MD5, many message conditions and chaining variable conditions must co-exist in each step of differential path



	$m_{6,1}=1,\ m_{6,2}=0,\ m_{6,5}=1,\ m_{6,7}=0,\ m_{6,29}=0,\ m_{6,31}=0,\ m_{6,32}=0$
	$a_{7,1}=0,\ a_{7,3}=1,\ a_{7,4}=0,\ a_{7,6}=0,\ a_{7,7}=0,\ a_{7,9}=0,\ a_{7,10}=1$ $a_{7,12}=0,\ a_{7,16}=1,\ a_{7,17}=1,\ a_{7,18}=1,\ a_{7,19}=1,\ a_{7,20}=1,\ a_{7,21}=1,\ a_{7,22}=1$ $a_{7,23}=1,\ a_{7,24}=1,\ a_{7,25}=1,\ a_{7,26}=1,\ a_{7,27}=1,\ a_{7,28}=0,\ a_{7,30}=0$
m_6	$m_{23,7}=m_{22,1},\ m_{23,6}=m_{23,7}+1,\ m_{23,30}=m_{19,5},\ m_{25,7}=m_{24,1}+1,\ m_{27,6}=m_{26,1}+1,$ $m_{27,31}=1+m_{22,1},\ m_{29,7}=m_{28,2}+1,\ m_{30,7}=m_{29,2}+1,\ m_{31,6}=m_{30,1}+1,\ m_{31,31}=m_{26,1}+1$ $m_{34,7}=m_{33,2}+1,\ m_{34,2}=m_{34,1}+1,\ m_{35,6}=m_{35,7}+1,\ m_{35,7}=m_{34,2}+1,\ m_{35,31}=m_{30,1}+1$ $m_{37,7}=m_{36,1}+1,\ m_{38,7}=m_{37,2}+1,\ m_{39,31}=m_{34,2}+1,\ m_{41,7}=m_{40,2}+1,\ m_{42,2}=m_{40,2}+1$ $m_{45,7}=m_{44,2}+1,\ m_{47,7}=m_{44,2}+1,\ m_{49,7}=m_{44,2}+1,\ m_{51,7}=m_{44,2}+1,\ m_{52,2}=m_{44,2}+1$ $m_{67,8}=m_{66,3}+1,\ m_{70,9}=m_{69,4}+1,\ m_{71,1}=m_{66,3}+1,\ m_{73,10}=m_{72,5}+1,\ m_{74,2}=m_{69,4}+1$ $m_{75,9}=m_{74,4}+1,\ m_{76,11}=m_{75,6}+1,\ m_{77,3}=m_{72,5}+1,\ m_{79,12}=m_{78,7}+1,\ m_{79,2}=m_{74,4}+1$

Obstacles for Further Improvement on SHA-1 Attack (continued)

- Difficult, because message space available is tight:

-- 50 message conditions in steps 17-80

-- hence 50 message conditions in steps 12-16

-- resulting in 50 message bit equations

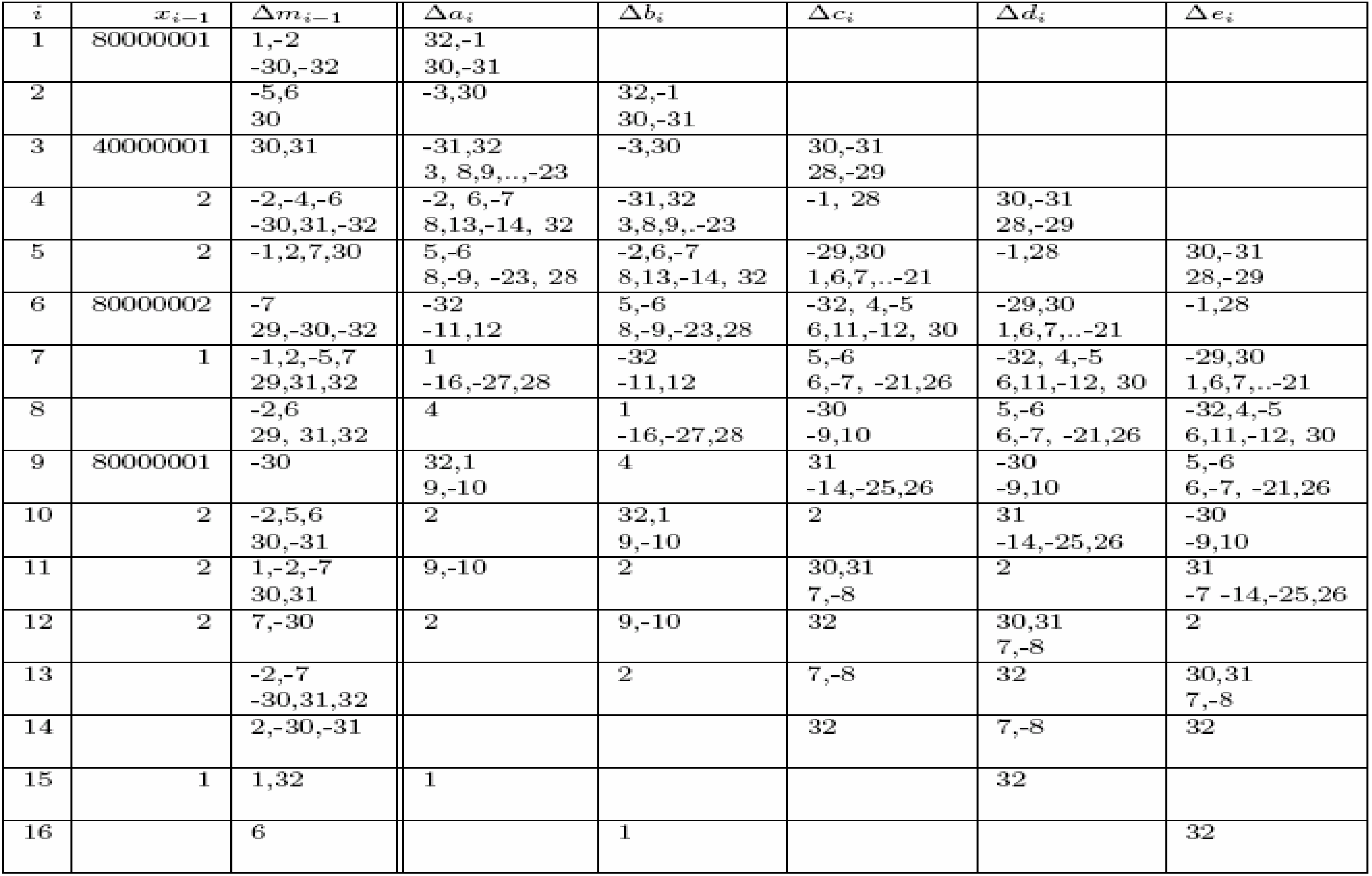
-- most message bits are involved

$$
\begin{aligned}
m_{13,29} = {} & m_{0,2} + m_{0,24} + m_{0,25} + m_{0,28} + m_{0,29} + m_{0,30} + m_{1,0} + m_{1,3} + m_{1,26} + m_{1,27} + m_{1,28} + m_{1,29} \\
& + m_{1,30} + m_{2,0} + m_{2,2} + m_{2,3} + m_{2,24} + m_{2,25} + m_{2,29} + m_{2,30} + m_{2,31} + m_{3,2} + m_{3,3} + m_{3,4} + m_{3,25} + m_{3,27} \\
& + m_{3,28} + m_{3,31} + m_{4,2} + m_{4,3} + m_{4,4} + m_{4,28} + m_{4,30} + m_{4,31} + m_{5,0} + m_{5,3} + m_{5,25} + m_{5,26} + m_{5,29} + m_{5,31} + m_{6,0} \\
& + m_{6,3} + m_{6,26} + m_{6,27} + m_{7,1} + m_{7,4} + m_{7,28} + m_{7,29} + m_{8,2} + m_{8,3} + m_{8,24} + m_{8,25} + m_{8,26} + m_{8,27} + m_{8,28} + m_{8,29} \\
& + m_{8,31} + m_{9,0} + m_{9,1} + m_{9,2} + m_{9,3} + m_{9,4} + m_{9,26} + m_{9,28} + m_{9,31} + m_{10,1} + m_{10,2} + m_{10,3} + m_{10,5} + m_{10,28} + m_{10,29} \\
& + m_{11,0} + m_{11,2} + m_{11,3} + m_{11,25} + m_{11,26} + m_{11,27} \\
& + m_{11,28} + m_{11,29} + m_{11,30} + m_{11,31} + m_{12,1} + m_{12,2} + m_{12,5} + m_{12,28} + m_{12,30} + m_{13,0} + m_{13,1} + m_{13,3} + m_{13,24} + m_{13,25} + m_{13,}
\end{aligned}
$$

-- in addition, 51 chaining variable conditions in steps 10-16

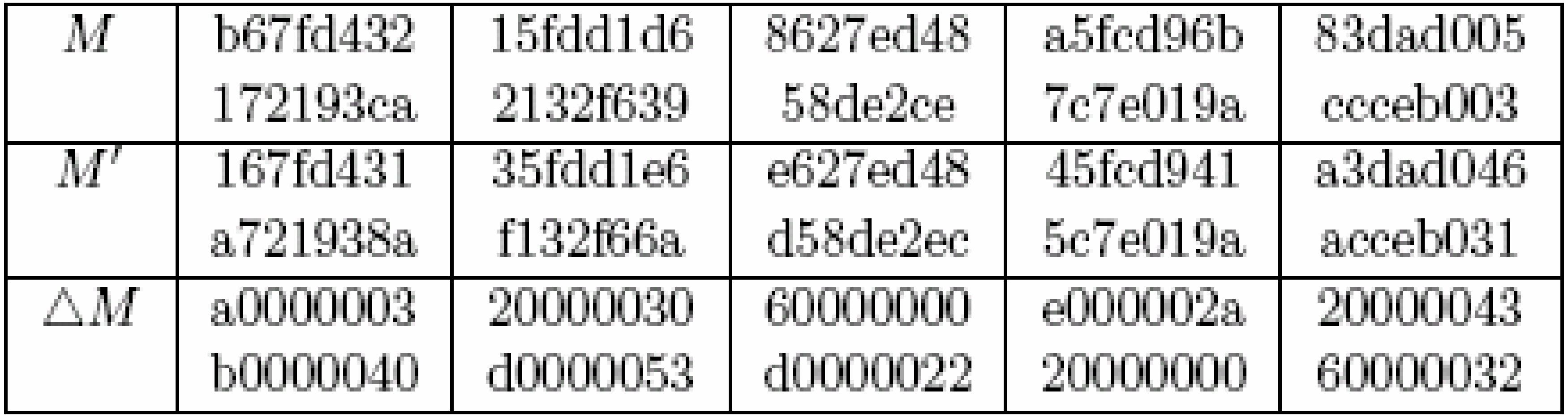
-- extra chaining variable conditions and message conditions coming from the message modification

Table 1 New Collision Path for SHA-1 (First Iteration)



i	x_{i-1}	Δm_{i-1}	Δa_i	Δb_i	Δc_i	Δd_i	Δe_i
1	80000001	1,-2 -30,-32	32,-1 30,-31				
2		-5,6 30	-3,30	32,-1 30,-31			
3	40000001	30,31	-31,32 3, 8,9,..,-23	-3,30	30,-31 28,-29		
4	2	-2,-4,-6 -30,31,-32	-2, 6,-7 8,13,-14, 32	-31,32 3,8,9,.-23	-1, 28	30,-31 28,-29	
5	2	-1,2,7,30	5,-6 8,-9, -23, 28	-2,6,-7 8,13,-14, 32	-29,30 1,6,7,..-21	-1,28	30,-31 28,-29
6	80000002	-7 29,-30,-32	-32 -11,12	5,-6 8,-9,-23,28	-32, 4,-5 6,11,-12, 30	-29,30 1,6,7,..-21	-1,28
7	1	-1,2,-5,7 29,31,32	1 -16,-27,28	-32 -11,12	5,-6 6,-7, -21,26	-32, 4,-5 6,11,-12, 30	-29,30 1,6,7,..-21
8		-2,6 29, 31,32	4	1 -16,-27,28	-30 -9,10	5,-6 6,-7, -21,26	-32,4,-5 6,11,-12, 30
9	80000001	-30	32,1 9,-10	4	31 -14,-25,26	-30 -9,10	5,-6 6,-7, -21,26
10	2	-2,5,6 30,-31	2	32,1 9,-10	2	31 -14,-25,26	-30 -9,10
11	2	1,-2,-7 30,31	9,-10	2	30,31 7,-8	2	31 -7 -14,-25,26
12	2	7,-30	2	9,-10	32	30,31 7,-8	2
13		-2,-7 -30,31,32		2	7,-8	32	30,31 7,-8
14		2,-30,-31			32	7,-8	32
15	1	1,32	1			32	
16		6		1			32

Table 2 An Sample Solution to 1-10 Steps Differential



M	b67fd432 172193ca	15fdd1d6 2132f639	8627ed48 58de2ce	a5fcd96b 7c7e019a	83dad005 ccceb003
M'	167fd431 a721938a	35fdd1e6 f132f66a	e627ed48 d58de2ec	45fcd941 5c7e019a	a3dad046 acceb031
$\triangle M$	a0000003 b0000040	20000030 d0000053	60000000 d0000022	e000002a 20000000	20000043 60000032

Comparison between New Collision Path and Previous Collision Path

Comparison:	Old	New
1. Message conditions	50	42
2. Chaining variable conditions in steps 10-16	51	30
3. Message space in steps 10-16 available for direct modification	2^{47}	2^{55}
4. Message space in steps 10-16 available for searching collision before advanced message modification	2^{123}	2^{151}

Strategies for Message Modification

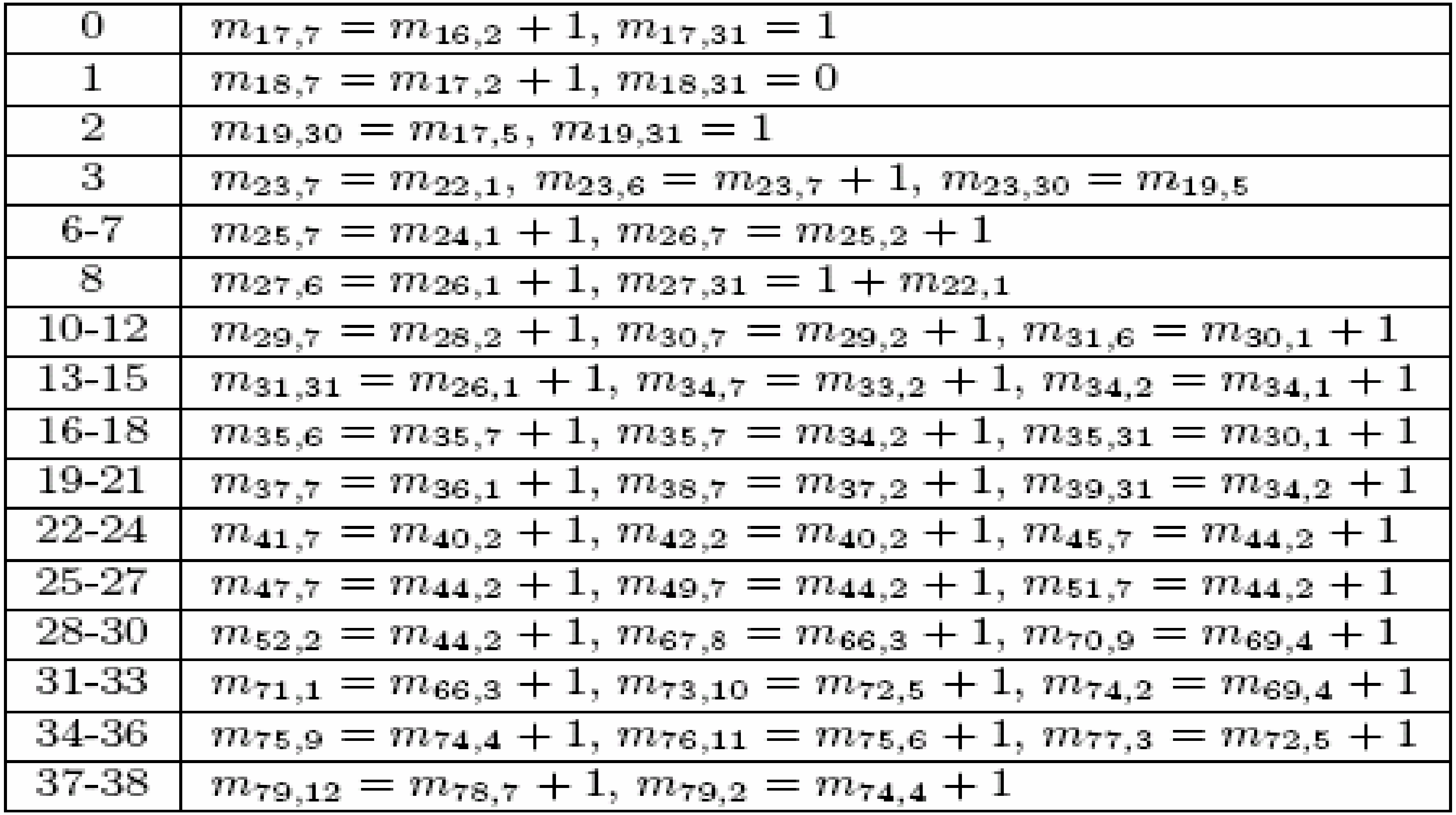
- Determine which message bits are *possible candidates (control bits)* for modification (Table 3).

- The message modification process *must respect* all chaining variable conditions and message conditions.

  --require adding *extra chaining variable* conditions in steps 1-16 and message conditions.
  Especially Consider the carry effect.

  -- message modification follow certain *topological order* coming from correlations among chaining variable conditions.

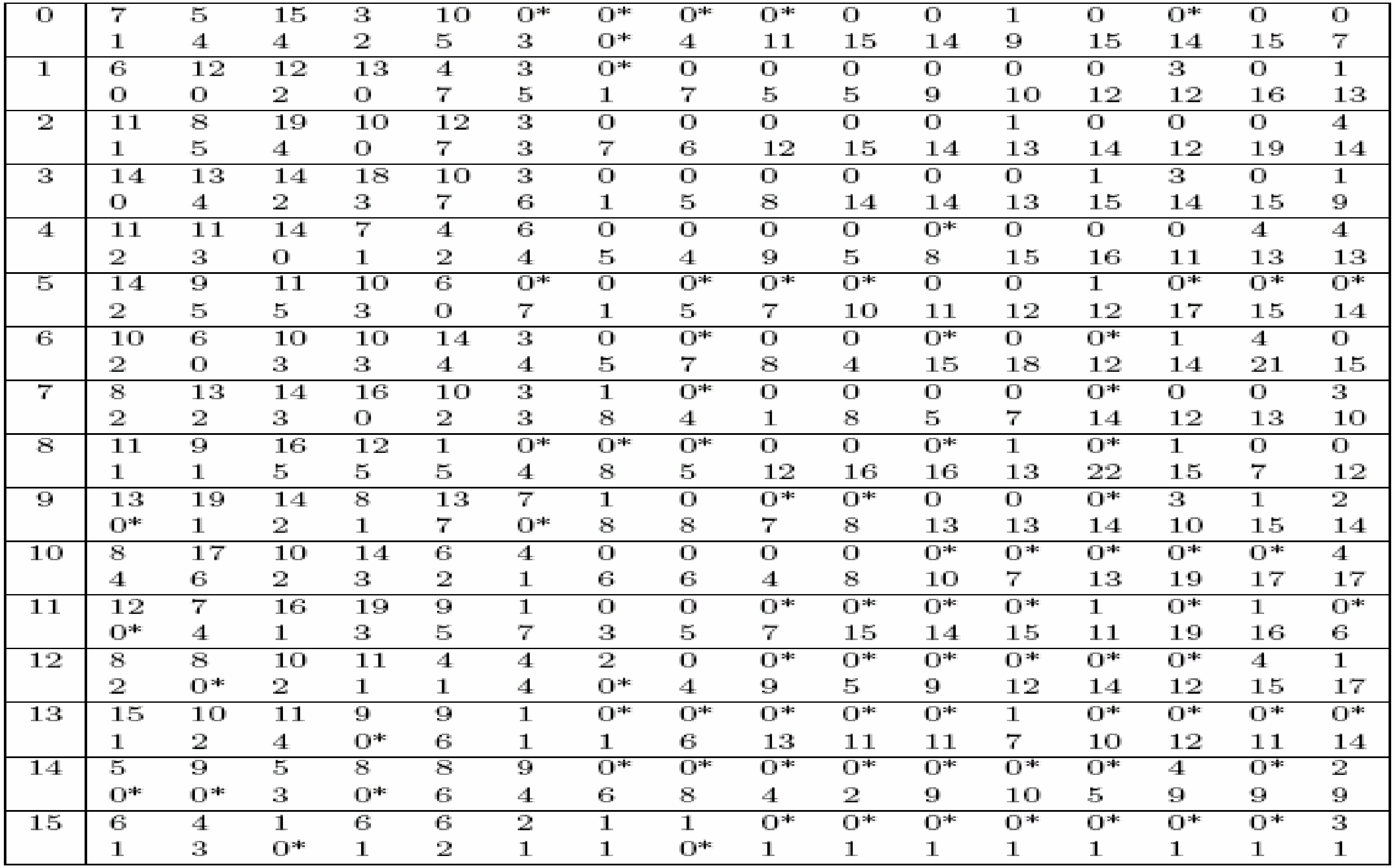
42 Message Conditions in Steps 17-80 for SHA-1 First Iteration



0	$m_{17,7} = m_{16,2} + 1,\ m_{17,31} = 1$
1	$m_{18,7} = m_{17,2} + 1,\ m_{18,31} = 0$
2	$m_{19,30} = m_{17,5},\ m_{19,31} = 1$
3	$m_{23,7} = m_{22,1},\ m_{23,6} = m_{23,7} + 1,\ m_{23,30} = m_{19,5}$
6-7	$m_{25,7} = m_{24,1} + 1,\ m_{26,7} = m_{25,2} + 1$
8	$m_{27,6} = m_{26,1} + 1,\ m_{27,31} = 1 + m_{22,1}$
10-12	$m_{29,7} = m_{28,2} + 1,\ m_{30,7} = m_{29,2} + 1,\ m_{31,6} = m_{30,1} + 1$
13-15	$m_{31,31} = m_{26,1} + 1,\ m_{34,7} = m_{33,2} + 1,\ m_{34,2} = m_{34,1} + 1$
16-18	$m_{35,6} = m_{35,7} + 1,\ m_{35,7} = m_{34,2} + 1,\ m_{35,31} = m_{30,1} + 1$
19-21	$m_{37,7} = m_{36,1} + 1,\ m_{38,7} = m_{37,2} + 1,\ m_{39,31} = m_{34,2} + 1$
22-24	$m_{41,7} = m_{40,2} + 1,\ m_{42,2} = m_{40,2} + 1,\ m_{45,7} = m_{44,2} + 1$
25-27	$m_{47,7} = m_{44,2} + 1,\ m_{49,7} = m_{44,2} + 1,\ m_{51,7} = m_{44,2} + 1$
28-30	$m_{52,2} = m_{44,2} + 1,\ m_{67,8} = m_{66,3} + 1,\ m_{70,9} = m_{69,4} + 1$
31-33	$m_{71,1} = m_{66,3} + 1,\ m_{73,10} = m_{72,5} + 1,\ m_{74,2} = m_{69,4} + 1$
34-36	$m_{75,9} = m_{74,4} + 1,\ m_{76,11} = m_{75,6} + 1,\ m_{77,3} = m_{72,5} + 1$
37-38	$m_{79,12} = m_{78,7} + 1,\ m_{79,2} = m_{74,4} + 1$

Details for Message Modification —

Available Message Bits to Correct Sufficient Conditions (Table 3)



0	7 1	5 4	15 4	3 2	10 5	0* 3	0* 0*	0* 4	0* 11	0 15	0 14	1 9	0 15	0* 14	0 15	0 7
1	6 0	12 0	12 2	13 0	4 7	3 5	0* 1	0 7	0 5	0 5	0 9	0 10	0 12	3 12	0 16	1 13
2	11 1	8 5	19 4	10 0	12 7	3 3	0 7	0 6	0 12	0 15	0 14	1 13	0 14	0 12	0 19	4 14
3	14 0	13 4	14 2	18 3	10 7	3 6	0 1	0 5	0 8	0 14	0 14	0 13	1 15	3 14	0 15	1 9
4	11 2	11 3	14 0	7 1	4 2	6 4	0 5	0 4	0 9	0 5	0* 8	0 15	0 16	0 11	4 13	4 13
5	14 2	9 5	11 5	10 3	6 0	0* 7	0 1	0* 5	0* 7	0* 10	0 11	0 12	1 12	0* 17	0* 15	0* 14
6	10 2	6 0	10 3	10 3	14 4	3 4	0 5	0* 7	0 8	0 4	0* 15	0 18	0* 12	1 14	4 21	0 15
7	8 2	13 2	14 3	16 0	10 2	3 3	1 8	0* 4	0 1	0 8	0 5	0 7	0* 14	0 12	0 13	3 10
8	11 1	9 1	16 5	12 5	1 5	0* 4	0* 8	0* 5	0 12	0 16	0* 16	1 13	0* 22	1 15	0 7	0 12
9	13 0*	19 1	14 2	8 1	13 7	7 0*	1 8	0 8	0* 7	0* 8	0 13	0 13	0* 14	3 10	1 15	2 14
10	8 4	17 6	10 2	14 3	6 2	4 1	0 6	0 6	0 4	0 8	0* 10	0* 7	0* 13	0* 19	0* 17	4 17
11	12 0*	7 4	16 1	19 3	9 5	1 7	0 3	0 5	0* 7	0* 15	0* 14	0* 15	1 11	0* 19	1 16	0* 6
12	8 2	8 0*	10 2	11 1	4 1	4 4	2 0*	0 4	0* 9	0* 5	0* 9	0* 12	0* 14	0* 12	4 15	1 17
13	15 1	10 2	11 4	9 0*	9 6	1 1	0* 1	0* 6	0* 13	0* 11	0* 11	1 7	0* 10	0* 12	0* 11	0* 14
14	5 0*	9 0*	5 3	8 0*	8 6	9 4	0* 6	0* 8	0* 4	0* 2	0* 9	0* 10	0* 5	4 9	0* 9	2 9
15	6 1	4 3	1 0*	6 1	6 2	2 1	1 1	1 0*	0* 1	0* 1	0* 1	0* 1	0* 1	0* 1	0* 1	3 1

Details for Message Modification — Control bit and Control path

- Choices for control bit: a message bit m_{i',j'} (i'<16) which does not appear explicitly in 42 message conditions or chaining variable conditions. (marked by 0* and 0 in Table 3)

0*: No appearance in 42 message bit equations and no chaining variable condition in the same bit position

0 : No appearance in 42 message bit equation, but a chaining variable condition in the same bit position

- Control Path: A chain of intermediate variable bits which can transmit a bit change from control bit m_{i',j'} to the target bit a_{i,j}.
- An example for Control Path:

$$m_{14,10} \longrightarrow a_{18,11} \longrightarrow a_{20,11} \longrightarrow a_{21,16} \longrightarrow a_{22,21} \longrightarrow a_{23,26} \longrightarrow a_{24,31}$$

Details for Message Modification — Topological Order

- A preferred order for processing a set of conditions a_{i,j} so as to minimize the chance that a previously enforced condition may later get undone.
- An example of topological order

$$a_{18,2} \longrightarrow a_{17,31} \longrightarrow a_{17,32} \longrightarrow a_{17,2} \longrightarrow a_{16,31} \longrightarrow a_{17,4} \longrightarrow a_{20,4}$$

$$\longrightarrow a_{19,32} \longrightarrow a_{19,2} \longrightarrow a_{18,30} \longrightarrow a_{18,32} \longrightarrow a_{20,30} \longrightarrow a_{21,30} \longrightarrow a_{21,2} \longrightarrow a_{22,3}$$

$$\longrightarrow a_{24,4} \longrightarrow a_{23,1} \longrightarrow a_{24,31} \longrightarrow a_{25,31}$$

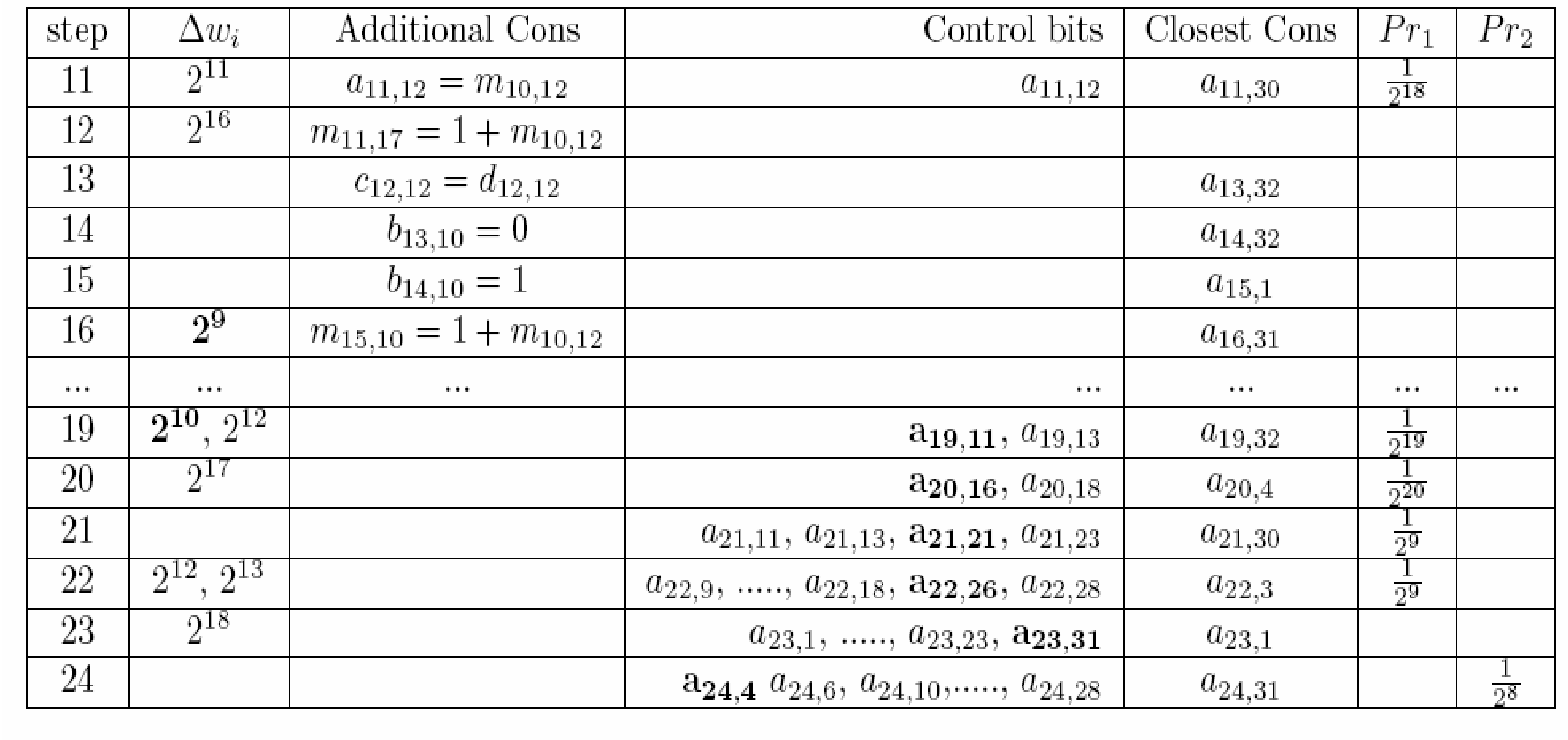
$$a_{18,29} \longrightarrow (a_{19,2}, a_{18,30})$$

Details for Message Modification
------Error Probability

- Error probability In spite of topological order, there is some probability that at the end of the message modification process, not all conditions are satisfied . We refer to this probability as error probability.

- Calculation of error probability (See Table 4)

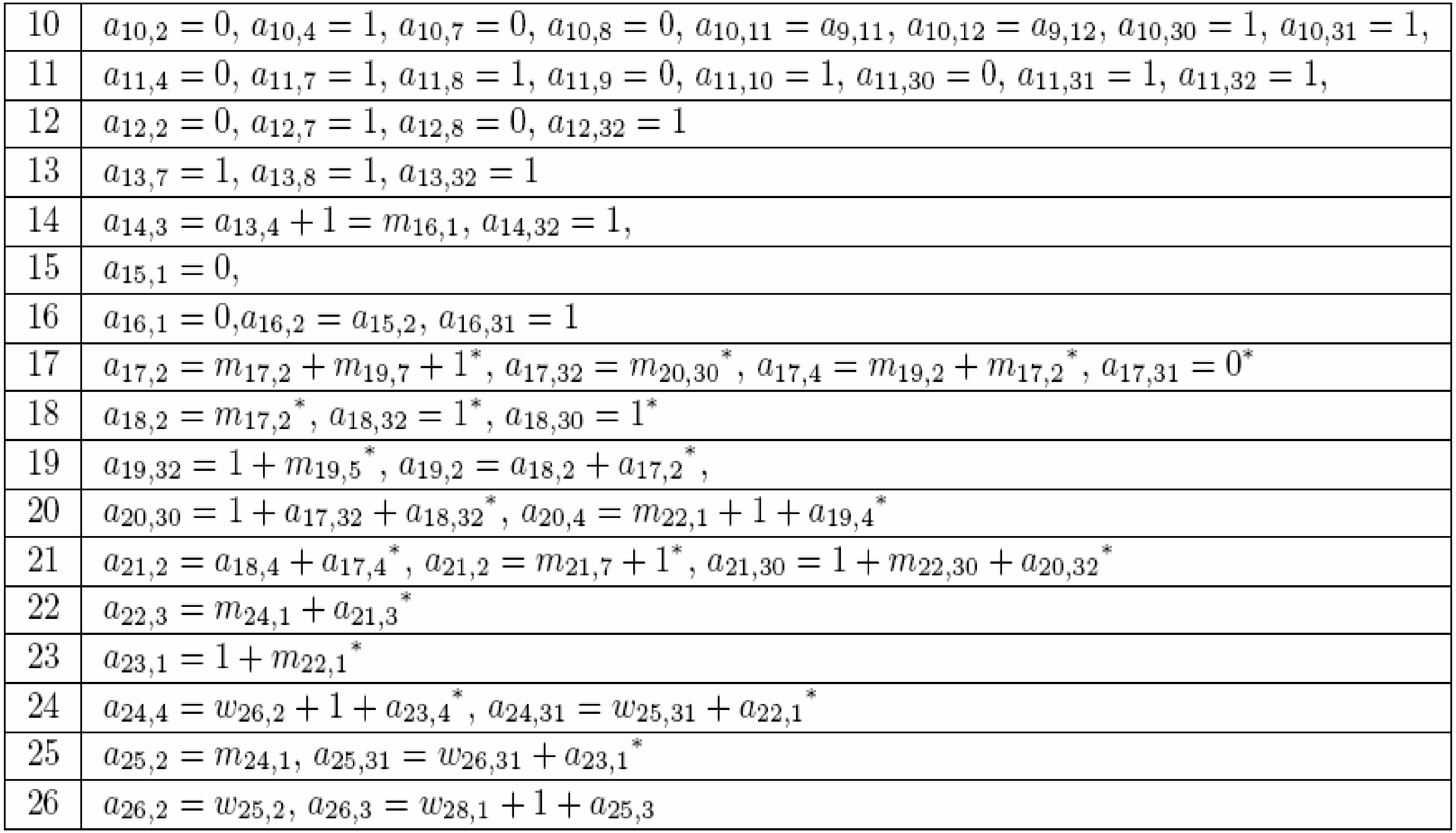
Table 4 An Example for One Condition Correction



step	Δw_i	Additional Cons	Control bits	Closest Cons	Pr_1	Pr_2
11	2^{11}	$a_{11,12} = m_{10,12}$	$a_{11,12}$	$a_{11,30}$	$\frac{1}{2^{18}}$	
12	2^{16}	$m_{11,17} = 1 + m_{10,12}$				
13		$c_{12,12} = d_{12,12}$		$a_{13,32}$		
14		$b_{13,10} = 0$		$a_{14,32}$		
15		$b_{14,10} = 1$		$a_{15,1}$		
16	$\mathbf{2^9}$	$m_{15,10} = 1 + m_{10,12}$		$a_{16,31}$		
...	...	...	...	...	...	...
19	$\mathbf{2^{10}}$, 2^{12}		$\mathbf{a_{19,11}}$, $a_{19,13}$	$a_{19,32}$	$\frac{1}{2^{19}}$	
20	2^{17}		$\mathbf{a_{20,16}}$, $a_{20,18}$	$a_{20,4}$	$\frac{1}{2^{20}}$	
21			$a_{21,11}$, $a_{21,13}$, $\mathbf{a_{21,21}}$, $a_{21,23}$	$a_{21,30}$	$\frac{1}{2^9}$	
22	2^{12}, 2^{13}		$a_{22,9}$, ..., $a_{22,18}$, $\mathbf{a_{22,26}}$, $a_{22,28}$	$a_{22,3}$	$\frac{1}{2^9}$	
23	2^{18}		$a_{23,1}$, ..., $a_{23,23}$, $\mathbf{a_{23,31}}$	$a_{23,1}$		
24			$\mathbf{a_{24,4}}$ $a_{24,6}$, $a_{24,10}$,....., $a_{24,28}$	$a_{24,31}$		$\frac{1}{2^8}$

$$a_{23,23} \longrightarrow a_{23,28} \longrightarrow a_{24,1} \longrightarrow a_{24,4}$$

Table 5 Conditions can be Corrected by Advanced Message Modification (with Star)



10	$a_{10,2} = 0,\ a_{10,4} = 1,\ a_{10,7} = 0,\ a_{10,8} = 0,\ a_{10,11} = a_{9,11},\ a_{10,12} = a_{9,12},\ a_{10,30} = 1,\ a_{10,31} = 1,$
11	$a_{11,4} = 0,\ a_{11,7} = 1,\ a_{11,8} = 1,\ a_{11,9} = 0,\ a_{11,10} = 1,\ a_{11,30} = 0,\ a_{11,31} = 1,\ a_{11,32} = 1,$
12	$a_{12,2} = 0,\ a_{12,7} = 1,\ a_{12,8} = 0,\ a_{12,32} = 1$
13	$a_{13,7} = 1,\ a_{13,8} = 1,\ a_{13,32} = 1$
14	$a_{14,3} = a_{13,4} + 1 = m_{16,1},\ a_{14,32} = 1,$
15	$a_{15,1} = 0,$
16	$a_{16,1} = 0, a_{16,2} = a_{15,2},\ a_{16,31} = 1$
17	$a_{17,2} = m_{17,2} + m_{19,7} + 1^*,\ a_{17,32} = m_{20,30}{}^*,\ a_{17,4} = m_{19,2} + m_{17,2}{}^*,\ a_{17,31} = 0^*$
18	$a_{18,2} = m_{17,2}{}^*,\ a_{18,32} = 1^*,\ a_{18,30} = 1^*$
19	$a_{19,32} = 1 + m_{19,5}{}^*,\ a_{19,2} = a_{18,2} + a_{17,2}{}^*,$
20	$a_{20,30} = 1 + a_{17,32} + a_{18,32}{}^*,\ a_{20,4} = m_{22,1} + 1 + a_{19,4}{}^*$
21	$a_{21,2} = a_{18,4} + a_{17,4}{}^*,\ a_{21,2} = m_{21,7} + 1^*,\ a_{21,30} = 1 + m_{22,30} + a_{20,32}{}^*$
22	$a_{22,3} = m_{24,1} + a_{21,3}{}^*$
23	$a_{23,1} = 1 + m_{22,1}{}^*$
24	$a_{24,4} = w_{26,2} + 1 + a_{23,4}{}^*,\ a_{24,31} = w_{25,31} + a_{22,1}{}^*$
25	$a_{25,2} = m_{24,1},\ a_{25,31} = w_{26,31} + a_{23,1}{}^*$
26	$a_{26,2} = w_{25,2},\ a_{26,3} = w_{28,1} + 1 + a_{25,3}$

Complexity Estimation
----Complexity for Second Iteration

- There are 83 conditions in steps 17-80
- After advanced message modification, there are 65 conditions left in 17-80 steps
- Searching for two conditions in steps 25-26 by one computation
- Relax one condition in the final step
- 62 conditions left
- Error probability for correcting 17-25 conditions amounts to one failed condition.
- The complexity is about 2^{63} computations.

Complexity Estimation ---Total Complexity

- Complexity for first iteration: further relax 3 conditions in the final 2 steps.

  The complexity is about 2^{60} computations

- Complexity for the second iteration

  2^{63} computations

- Total complexity

  $2^{63} + 2^{60} = 1.125 \times 2^{63} \sim 2^{63}$

Thanks!