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Construction of planar diagrams*

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We give a simple prescription for explicitly constructing a class of planar diagrams and we discuss their physical relevance. All diagrams are generated from what we call the maximal ones, which are constructed iteratively. The prescription is proved in detail and examples are given.

I. INTRODUCTION

Asymptotic properties of non-Abelian gauge theories in perturbation theory have received much attention recently. The analysis is more complicated than the Abelian case (QED) because each Feynman graph involves products of the usual momentum-space factor and a (new) group-space factor. Higher-order graphs involve higher-order Casimir invariants. Delicate cancellations must take place if the higher-order graphs are to be related to the lower-order graphs by some iteration. Explicit calculations in the first few orders show that this may occur in a way that does not depend on the particular gauge group.[1,2] Such cancellations may well indicate the inappropriateness of perturbation theory in factorizing the momentum space and group space. However, in the absence of any nonperturbative approach, one can only hope to go beyond low orders by some convenient choice of the group and/or the representation.

One method of drastically simplifying such calculations is well known.[3] One considers the fundamental (quark) representation of U_N and keeps only the leading power of N in each order of the coupling constant. Then, only a well-defined subset of graphs contribute and the group factor is determined by the number of loops, so that the momentum integrals may be directly compared (as in QED). Such situations provide a much simplified context in which to check results which are expected to apply to any gauge group.

As an explicit example, we may consider the elastic form factor in the limit of very large momentum transfer, keeping only the leading terms in a given order of the coupling constant.[2] In QED the leading graphs are generated from rainbow (ladder) graphs by permuting vertices on one side of the current insertion. For a non-Abelian group all single-boson corrections to these graphs involving the three-point self-interactions must be included, excluding boson self-energies. In the particular case of the U_N or SU_N scalar quark form factor, only the planar subset of these graphs will contribute to the leading power of N.[4] It is this class of planar graphs on which we concentrate in this paper.

In Sec. II we give a prescription for constructing a set of planar diagrams which we call simple diagrams. We also generalize to include planar diagrams with both three- and four-line vertices. Sec. III gives the construction of the form-factor diagrams from the simple diagrams and concludes the paper. Finally, technical aspects of a proof of the prescription (given in Sec. II) are relegated to three appendixes.

II. CONSTRUCTION OF DIAGRAMS[5-8]

This section is divided into seven parts. In part A we define two classes of planar diagrams. In part B we define two maps connecting the two classes; one map turns out to be the inverse of the other. In part C we use these maps to give a prescription for constructing all diagrams of either class. Part D is devoted to examples. Part E is a proof that we get all the simple diagrams from the prescription. Part F discusses a technique for determining when two simple diagrams are topologically distinct. Part G gives the prescription for including self-energies, and comments on the construction of all planar diagrams.

A. Properties of the diagrams

Here we define simple diagrams and simple duals to be diagrams with the properties listed below:

simple diagrams	simple duals
(1) planar (2) one-particle irreducible[9] (1PI) (3) three lines[9] per vertex[9] (4) no internal self-energies and no more than three boundary self-energies	(1) planar (2) connected (3) all interiors[9] are loop[9] interiors; three vertices per loop (4) two vertices are directly connected[9] at most once; no more than three lobes[9]

Neither simple diagrams nor simple duals contain a vertex directly connected to itself.[10] The reason for introducing simple duals is that it is easier to work with n vertices of a simple dual than to work with n loops of a simple diagram. The reason for "no more than three boundary self-energies" is that one can destroy up to three by the attachment of three incident lines or three legs (see Appendix C, paragraph 5, line 15).

In the next section we introduce the dual operation and the antidual operation. Appendixes A and B show that the dual is a bijective map (1-to-1 and onto) from the set of simple diagrams onto the set of simple duals, and the antidual is its inverse.

B. Dual and antidual operations

Definition. The *dual* of a simple diagram is a diagram constructed from the simple diagram by

(1) *placing a vertex* in the interior of each loop of the simple diagram,

(2) *connecting* the vertices of (1) whose corresponding loops share a line, and

(3) *erasing* the simple diagram.

For examples see Fig. 1.

Definition. The *antidual* of a simple dual is a diagram constructed from the simple dual by

(1) replacing the simple dual by its *maze*,[11] that is

(i) placing the simple dual inside a loop,

(ii) inserting a loop inside each loop of the simple dual, and

(iii) erasing the simple dual;

(2) *blocking* the roads in the maze (a road has replaced each line of the simple dual in the maze construction); and

(3) *shrinking* the inserted loops of (1) (ii) to vertices.

For examples see Fig. 2.

C. Prescriptions

The purpose of this section is to give the prescription for generating all n-loop simple diagrams, prescription B. First we introduce two definitions and prescription A.

Definition. Maximal simple dual: a simple dual which is no longer a simple dual if we attach the

TABLE I. Dictionary of graphical terminology.

boundary line: a line whose every point touches points of the exterior of a diagram.

boundary vertex: a vertex which touches points of the exterior of a diagram.

diagram (planar): a finite set of lines and/or vertices in a plane. Also, a loop with no vertices is a diagram.

directly connected: two vertices are directly connected if there is a single line connecting them. A vertex is directly connected to itself if the end points of some line coincide at that vertex.

exterior (of a diagram): the connected region in the plane containing no lines or vertices which extends arbitrarily far away from the diagram.

interior: a connected region in a plane containing no vertices or lines, bounded by lines (and vertices) of a planar diagram. The topological interior of a loop with no vertices is also an interior; the exterior of a diagram is not an interior.

line: any topological distortion of a closed line segment in a plane, nonzero and finite in length and whose end points are defined to be vertices.

lobe: a portion of a diagram which is connected to another portion (not necessarily the remainder of the diagram) by only a vertex. The other portion is called a body (with respect to that lobe). We say a diagram has n lobes if there are n distinct portions connected to the same body, each by a single vertex. (The same vertex of attachment may be used by many or all of the lobes, and the body may be *only* this vertex.) The form of Fig. 14(a) shows four lobes.

loop: an interior plus its boundary is called a loop if it is topologically equivalent to a circle. (The boundary may contain vertices.)

1PI (one particle irreducible): a diagram is 1PI if it is connected and cannot be made disconnected by cutting a line (not at a vertex) in the diagram. For example, the hourglass diagram in Fig. 8 is 1PI.

topologically distinct: two planar diagrams are topologically distinct if one cannot be elastically deformed (in the plane) into the other without passing through or identifying lines and/or vertices. This definition reduces to the definitions given in the text for topological distinction of simple duals and simple diagrams. Two diagrams are said to be topologically equivalent if they are not topologically distinct.

vertex: (i) any point of incidence of three or more lines; (ii) the end points of any line are vertices; (iii) vertices of zero or two lines are defined where indicated by a dot on the diagram.

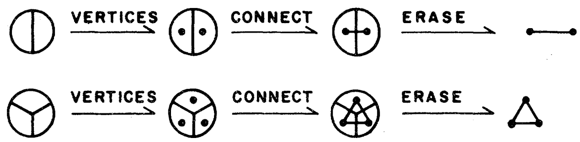

FIG. 1. Examples of the dual operation.

ends of a new line to two of its vertices.

Definition. Maximal simple diagram: a simple diagram whose dual is a maximal simple dual; equivalently (see Appendix B), the antidual of a maximal simple dual.

Prescription A. Construction of all n-vertex maximal simple duals:

(1) For $n=1$, 2, and 3 they are easily constructed directly from their definition. We show them in Fig. 3.

(2) For $n>3$, we first construct all n-loop maximal simple diagrams; that is, draw all diagrams of the form of Fig. 4, in which the hatched circle is filled with all $(n-3)$-loop simple diagrams in all topologically distinct[9] ways which

(i) maintain three lines per vertex, that is, do not allow any of the three lines incident on the hatched circle to hit a vertex, and

(ii) avoid self-energies, that is, if the $(n-3)$-loop simple diagram has boundary self-energies, they must be destroyed by attaching incident lines to them. See Fig. 5 for examples.

Taking the dual gives the n-vertex maximal simple duals.

Prescription B. Generation of all n-loop simple diagrams:

(1) Construct all n-vertex maximal simple duals by prescription A.

(2) From each n-vertex maximal simple dual, remove lines from the consecutive boundaries[12] in all topologically distinct ways which leave the resulting diagrams connected. In removing a line, its end-point vertices remain with the diagram.

(3) Discard diagrams with more than three lobes.

(4) Take the antidual of the diagrams resulting from (1), (2), and (3).

The result of (4) is the set of all n-loop simple diagrams. Note that the same diagram may be generated from two different maximal simple duals.[13]

D. Construction of the 3-, 4-, and 5-loop simple diagrams

$n=3$. The maximal simple dual is given in Fig. 3. The removal process generates the duals of Fig. 6(a). We were allowed to remove at most one line in preserving connectedness, and there is only one way to remove one line because of the symmetry. The corresponding antiduals are in Fig. 6(b).

$n=4$. We must fill in the hatched circle with all one-loop simple diagrams; there is only one (the single loop with no vertices, which comes from the maximal simple dual composed of a single vertex) and there is only one topological way to do it [see Fig. 7(a)]. The duals of the removal process and their antiduals are shown in Figs. 7(b) and (c).

$n=5$. We must fill in the hatched circle with all two-loop simple diagrams [in general, prescription B requires us to know all $(n-3)$-loop simple diagrams in order to construct all of the n-loop maximal simple diagrams]. It is obvious from the two-vertex maximal simple dual that there is only one two-loop simple diagram, and there is only one topological way to attach the three incident lines if we are to avoid internal self-energies and four lines per vertex. This is shown in Fig. 8 along with the removal process, the duals generated, and their corresponding antiduals.

E. Proof that prescription B generates all and only n-loop simple diagrams

Structure of the proof. Show that steps (1), (2), and (3) of prescription B generate only the set of all n-vertex simple duals (lemma 1), and show the antidual operation on this set is the set of all n-loop simple diagrams (lemma 2).

Lemma 1. Steps (1), (2), and (3) of prescription B generate only the set of all n-vertex simple duals.

Only n-vertex simple duals are generated: step (1) begins the prescription with simple duals, step (3) maintains the lobes property, and step (2) maintains the remaining properties. Removing a

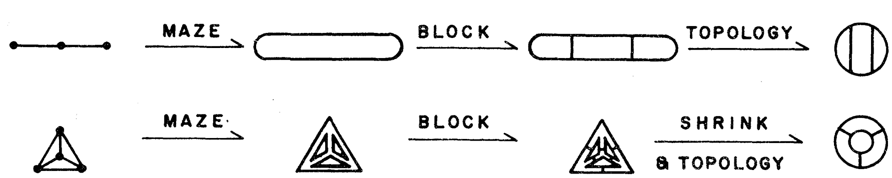

FIG. 2. Examples of the antidual operation.

FIG. 3. Maximal simple duals for $n=1,2,3$.

line from the boundary neither creates nor alters interiors (it can only destroy them) or the number of vertices on their boundaries; also, it does not create new direct connections, it maintains planarity, and connectedness is specifically required. When one removes a line its end-point vertices remain with the diagram, so the number of vertices is unchanged.

All n-vertex simple duals are generated: We prove here that all n-vertex simple duals are subdiagrams of some n-vertex maximal simple dual, and are generated by the removal process. Appendix C shows that prescription A generates all and only n-vertex maximal simple duals.

Given an n-vertex simple dual, add lines so as to maintain the n-vertex simple-dual properties until this is no longer possible. The number of lines added will be finite because no vertex is directly connected to itself, two vertices are directly connected at most once, and the number of vertices is finite. The diagram thus reached is by definition an n-vertex maximal simple dual. All of the lines must have been added to consecutive boundaries.[14] Thus the reverse of this addition of lines must be an element of the removal process, that is, a removing of lines from consecutive boundaries, which maintains connectedness.

Lemma 2. The antidual operation with domain being the set of all n-vertex simple duals has the set of all n-loop simple diagrams as its range.

Given an n-loop simple diagram not in the range, we know its dual is an n-vertex simple dual by Appendix A. The antidual of this n-vertex simple dual is the original n-loop simple diagram, as the antidual-of-the-dual operation is the identity operation when the domain is the set of simple diagrams (Appendix B); thus we are led to a contradiction.

We know there can be *only* n-loop simple diagrams in the range by Appendix A.

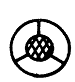

FIG. 4. The form of the maximal simple diagrams ($n>3$); the hatched circle is to be filled with all $(n-3)$-loop simple diagrams.

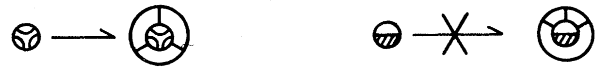

FIG. 5. Examples of constructing maximal simple diagrams; in particular, avoiding self-energies.

F. Distinguishing the diagrams

When one is not able to immediately tell if two simple diagrams are topologically distinct, it is often easier to look at the corresponding simple duals. Before giving some simple checks, we develop a picture of the topology of simple duals and we define topological equivalence of simple duals. Of course, if two simple duals are topologically equivalent (or distinct), the corresponding simple diagrams are topologically equivalent (distinct), and vice versa.

From Ref. 14 we know that an internal vertex of a simple dual is the center of a *wheel*. One can easily see that a boundary vertex of a simple dual is the center of a wheel which has lost some or all of its rim connections (where we allow wheels with any non-negative number of spokes). Where a rim connection is missing, we say the boundary vertex is open to the exterior on that side. We define a wheel which has lost some or all of its rim connections to be an *almost wheel*. Thus a simple dual is made up of wheels and almost wheels much as a puzzle is made of its pieces (the big difference being that you have some freedom to decide the shape of the pieces of a simple dual).

Topological equivalence of two simple duals requires there exist a 1-to-1 correspondence between the vertices of one and the vertices of the other, with the following properties:

(i) Given a vertex in one simple dual, write down in clockwise order the vertices it is directly connected to. The corresponding vertex of the second simple dual must have direct connections with the corresponding vertices in the same clockwise order. For example, if vertex 1 is directly connected to vertices 2, 3, and 4 in clockwise order, then vertex 1′ must be directly connected to 2′, 3′, and 4′ in clockwise order (where vertex m of the first simple dual corresponds to vertex m' of the second simple dual).

(ii) In the case of boundary vertices, the openings to the exterior must also correspond. For

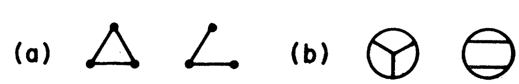

FIG. 6. (a) All three-vertex simple duals, (b) all three-loop simple diagrams [the antiduals of (a)].

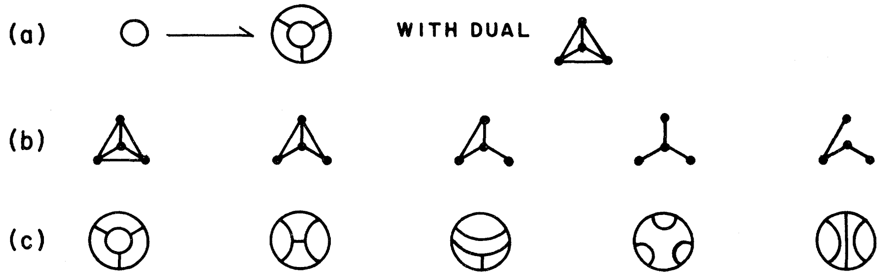

FIG. 7. (a) Construction of the three-loop maximal simple diagram, and its dual, (b) all four-vertex simple duals, (c) all four-loop simple diagrams [the antiduals of (b)].

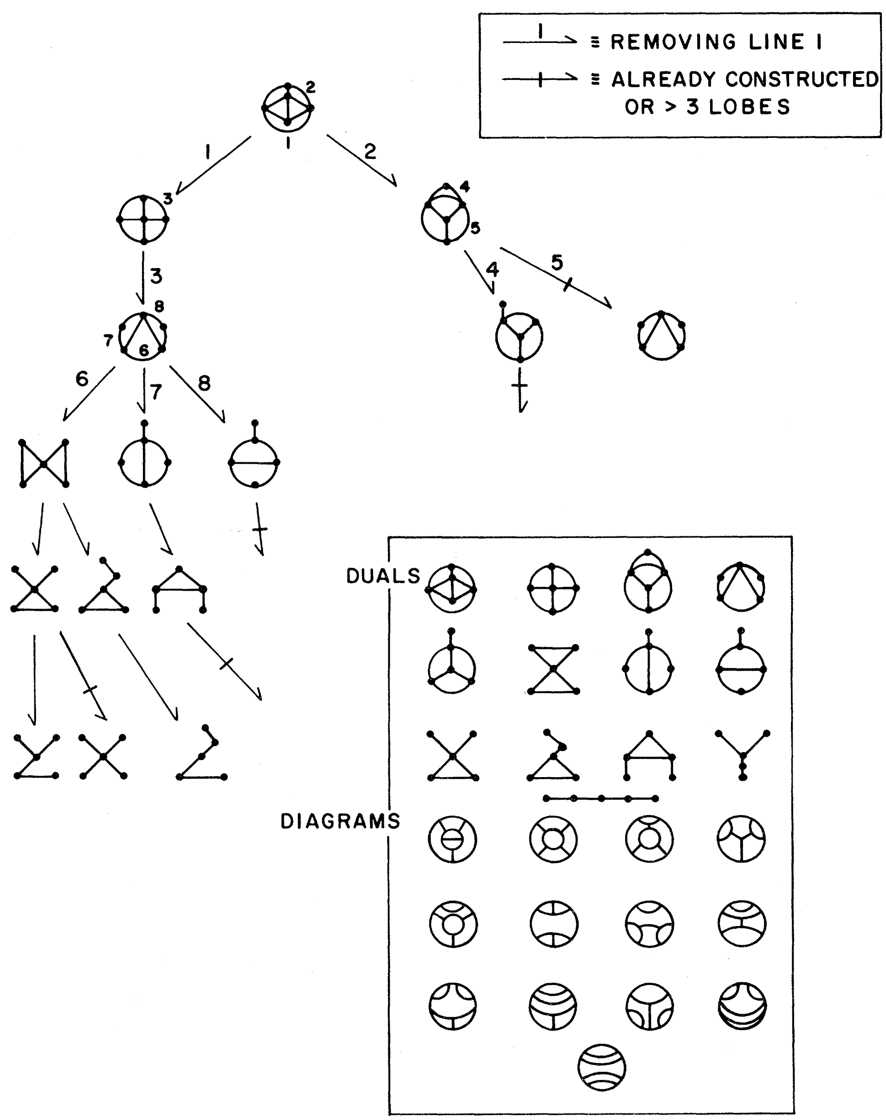

FIG. 8. Schematic of the removal process on the five-vertex maximal simple dual; the box contains the five-vertex simple duals and their antiduals, the five-loop simple diagrams.

FIG. 9. The form of the maximal semisimple diagrams ($n>2$); the hatched circle is to be filled with all $(n-2)$-loop chains.

example, in obvious notation, where O indicates an opening, vertex 1 $(2, 3, O, 4, O, 5)$ must correspond to vertex 1′ $(2', 3', O, 4', O, 5')$.

Some simple things that can be quickly checked between two simple duals are:

(1) Is the number of lines the same?

(2) Is the number of boundary vertices the same? Is the number of internal vertices the same?

(3) Since internal vertices are at the center of wheels and boundary vertices are at the center of almost wheels, we can pull wheels and almost wheels apart as units, like taking apart a puzzle. Try to pull both simple duals apart in the same way. It is often best to do this check first.

G. Construction of a larger set of diagrams

We briefly mention how to construct a larger set of diagrams, a subset of which we will call semisimple diagrams. Semisimple diagrams are defined to be diagrams having all the properties of simple diagrams except that property (4) is not a requirement, that is, self-energies *are* allowed (thus the simple diagrams are a subset of the set of semisimple diagrams). The maximal n-loop semisimple diagrams are constructed by filling the hatched circle of Fig. 9 (which certainly looks like a self-energy) with all topologically distinct "chains" of semisimple diagrams, where each chain contains exactly $(n-2)$-loops. We show all three-loop chains in Fig. 10. The two lines incident on the hatched circle in Fig. 9 must attach one to each end of the chain (to maintain 1PI) in all topologically distinct ways which maintain three lines per vertex (for example we can construct 14 different five-loop maximal semisimple diagrams from the three-loop chains, if one includes reflections). Then one takes the duals to get the maximal semisimple duals, and the rules of removal are as before. Ignore the discarding of duals with lobes, and construct all antiduals to get all of the n-loop semisimple diagrams. (We have not yet proved this rigorously.)

Finally, it is easy to further enlarge the set of planar diagrams to include n-line vertices. By generalizing the removal process to include removal of lines from the *interior* of the duals, the antiduals will contain diagrams with internal vertices which have more than three lines per vertex. If no *adjacent* interior lines are removed, we obtain only three- or four-line vertices, which are the vertices of renormalizable field theories. The antidual operation always produces diagrams whose *boundary* vertices have exactly three lines. The simplest way to get around this is to construct diagrams with only one boundary vertex [as in Fig. 11(c)], and then remove that vertex and the lines attached to it. Thus we are led to the third method of generating diagrams, this time by attaching only one line to a hatched circle.

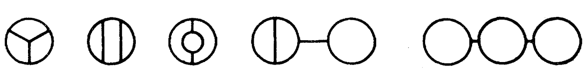

FIG. 10. The three-loop chains.

IV. CONCLUSION

Having in hand the iterative method for explicitly constructing the n-loop simple diagrams, it is now straightforward to construct the leading scalar form-factor graphs, mentioned in the Introduction, by attaching the three external legs. One attaches the three legs in all topologically distinct ways (exactly as in the construction of the $(n+3)$-loop maximal simple diagrams, Sec. II C). If one leg is then chosen to be the photon leg (in all topologically distinct ways) then the other two are the quark legs, which are part of the continuous quark line running from the incoming quark leg along the boundary through the photon vertex and out the other quark leg. All other lines are gluon lines.

These form-factor graphs represent momentum integrals which may be expressed in parametric form by functions directly determined from the simple diagrams.[15] What is needed to proceed beyond low orders is a way of associating the asymptotic values of such integrals with the topology of the graphs. So far this problem is unsolved.

In summary, we have solved the problem of construction of a general class of planar diagrams. We feel that such constructions are of interest apart from the form-factor problem. For example, graphical analysis appears to be useful in reducing products of generators of the gauge group. In addition, much work has been done to associate particular dynamical behaviors with given classes of Feynman diagrams. We feel the closer one is to understanding the topology of the diagrams, the more physics one can draw from them.

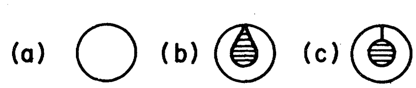

FIG. 11. An interior of a diagram can be circular, as in (a), or have any combination of the appendages shown in (b) and (c).

APPENDIX A

Here we show

(1) the set of duals of simple diagrams is (a subset of) the set of simple duals, and

(2) the set of antiduals of simple duals is (a subset of) the set of simple diagrams.

The results of Appendix B allow us to ignore "(a subset of)". See lemma 2 for this argument.

We prove (1) by showing that simple-diagram properties are converted into simple-dual properties by the dual operation; we prove (2) by showing that simple-dual properties are converted into simple-diagram properties by the antidual operation. We leave the proof that in both cases the result is a planar diagram to the reader.

To facilitate the proof of (1), we now show that for simple diagrams

(i) every vertex and every line is part of a loop (therefore the loop is a basis element), and

(ii) a loop never shares isolated points with another loop or the exterior,[9] that is, every point shared by two loops (or a loop and the exterior) belongs to some line which they also share.

Thus we can picture simple diagrams as loops shoved together, always sharing lines with other loops or the exterior, wherever sharing occurs.

(i) We know that *every* line must be part of the boundary of some interior; otherwise we could cut the line and separate the result into two pieces, that is, the diagram would not be 1PI. Thus we can start on an appropriate side of a given line and expand a topological circle until it meets the boundary of the interior. In this way we see that the only topological possibilities for this boundary are that it is topologically circular, or, that it comes back on itself at a point or along a line (or combinations of points and/or lines). See Fig. 11. The latter two are ruled out by three lines per vertex and 1PI, respectively. Thus every line and every vertex (because if the diagram is connected, every vertex is the endpoint of some line except in the case of the single-vertex diagram) is part of a loop. Now it is clear the boundary vertices must be shared by exactly two loops (by boundary vertex, three lines per vertex, all interiors are loops, and 1PI) and all internal vertices are shared by exactly three loops (by internal vertex, three lines per vertex, and all interiors are loops). Thus all vertices can be thought of as being created in the shoving of the loops together, and these basis loops need have no vertices before being shoved together. With this picture of simple diagrams it is clear that the one-loop simple diagram (which did not really conform to our definition of a planar diagram because it is not made of a finite number of lines and/or vertices) fits in.

(ii) There can never be a sharing between two loops, or a loop and the exterior, of an isolated point (which would have to be a vertex) by three lines per vertex. Keeping this picture in mind, we move to the proof of (1).

The dual of a simple diagram is connected: If there are at least two pieces in the dual, none of the loops of the simple diagram that correspond to one piece share a line with any of the loops that correspond to the second piece. Since this is the only possible type of connection, this contradicts connectedness (1PI) of the simple diagram.

All interiors in the dual of a simple diagram are loop interiors, and the loops have exactly three vertices: Given any interior in the dual, there is a vertex and two adjacent lines emanating from it that are part of the boundary of that interior. This is because there must be at least one line in the boundary, and at least one of its end-point vertices must have at least two lines emanating from it if the dual is to be connected as we have just proven (of course no line in the dual of a simple diagram can have its end points coincide); thus by moving around the vertex with at least two lines, we find two adjacent lines that are in the boundary of the given interior. The loop of the simple diagram which leads to this dual vertex must have had a vertex on its boundary separating two shared lines that lead to the two dual lines leaving the dual vertex (see Fig. 12). The separating vertex, on the boundary of the loop in the simple diagram, must be an internal vertex, and thus gives rise to a three-vertex loop in the dual, and the interior of this loop must be the very interior we are investigating.

Two vertices in the dual of a simple diagram are directly connected at most once: If two vertices are directly connected more than once, there must have been at least two lines shared between the corresponding loops of the simple diagram (see Fig. 13). This requires an internal self-energy in the simple diagram, which is a contradiction.

There are not more than three lobes (attached to the same body) in the dual of a simple diagram: We show that more than three lobes with the same body of attachment [Fig. 14(a)] requires more than three boundary self-energies in the simple dia-

FIG. 12. This figure is an aid in the proof that all interiors of the dual of a simple diagram are loop interiors with exactly three boundary vertices. The dotted circle is the loop which led to the vertex in its center as a result of the dual operation. The vertex on the dotted boundary is a vertex in the simple diagram which separates the two adjacent dual lines of the dual operation.

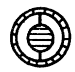

FIG. 13. Two loops which share at least two lines and thus create a self-energy.

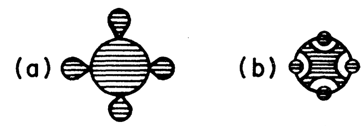

FIG. 14. (a) A simple dual with at least four lobes, (b) a simple diagram with at least four boundary self-energies. The antidual maps form (a) into form (b), the dual maps form (b) into form (a).

gram [Fig. 14(b)], which contradicts the definition of a simple diagram [Fig. 14(a) is necessarily the dual of Fig. 14(b)]. Consider a dual which has more than three lobes attached to the same body, and focus on one lobe. The vertex of attachment of that lobe corresponds to a loop in the simple diagram. We now show that that loop has two lines shared with the exterior, one on either side of what corresponds to the lobe. This is sufficient to prove our hypothesis, as it shows a separate boundary self-energy is required in a simple diagram for each lobe in the dual attached to the same body. The loop in the simple diagram which corresponds to the vertex of attachment is really a loop of attachment, that is, it connects two sets of loops which are otherwise unconnected: the loops corresponding to vertices of the lobe and the loops not corresponding to vertices of the lobe. The loop of attachment must have a line shared with the exterior on either side of the lines shared with loops corresponding to the lobe. This is because the loops corresponding to the lobe must be a nonzero length away from the loops corresponding to the body (excluding the loop of attachment), since they do not share a line and cannot share only a point.

To facilitate the proof of (2), we look more closely at the antidual operation. First consider the maze and blocking portions of the antidual operation on an arbitrary vertex of a simple dual (see Fig. 15). It is clear that (i) each vertex is transformed into a loop, and (ii) each line is transformed into a blocked road connecting the two loops that correspond to the endpoint vertices of that line. The sides of the roads result from the inserted loops or the surrounding boundary of the maze. Now shrink all inserted loops (which have three lines incident on them as a result of the blocking portion) to vertices. These are the internal vertices; each one belongs to the three loops that have arisen from the three vertices of the loop in the simple dual in which the inserted loop was inserted. All internal lines come from the lines of the blocking portion and are shared by two loops. All boundary lines are shared between a loop and the exterior and arise from vertices created by the blocking of roads which have a side provided by the surrounding boundary of the maze. This means that a loop of the antidual shares a line with the exterior if and only if the corresponding vertex of the simple dual touches the exterior on the corresponding side (e.g., assume region A in Fig. 15 is the exterior).

Just before the shrinking portion it is clear that the diagram is connected; one follows a path indicated by the corresponding connection in the simple dual, traversing loops or the lines of the blocking portion where necessary. The shrinking process cannot disconnect the diagram. Also the antidual is 1PI; cutting a line cannot disconnect the diagram because every line is part of a loop.

There must be exactly three lines per vertex because vertices arise only from blocking and shrinking. Blocking clearly leads to three lines per vertex; shrinking does also because the loops of the simple dual have three vertices (see Fig. 16).

Suppose the antidual of a simple dual leads to an internal self-energy (see Fig. 13). This is merely two loops which share at least two lines; thus the corresponding vertices of the simple dual must have been directly connected at least twice, which is a contradiction.

There are no more than three boundary self-energies in the antidual of a simple dual: [Fig. 14(b) is necessarily the antidual of Fig. 14(a)]. From the discussion above, a loop of the antidual shares a line with the exterior only if the corresponding vertex of the simple dual touches the exterior on

FIG. 15. The maze and blocking portions and the shrinking portion of the antidual process on an "arbitrary" simple-dual vertex.

FIG. 16. A diagrammatic illustration that internal vertices in the antidual of a simple dual have three lines per vertex.

the corresponding side. Thus the loop which houses the self-energy must correspond to a vertex of attachment, and for each boundary self-energy in the antidual there must be one lobe in the simple dual. Therefore, if the antidual has more than three boundary self-energies, it must have come from a simple dual with more than three lobes (attached to the same body), which is a contradiction.

APPENDIX B

Here we show

(1) the antidual-of-the-dual operation on simple diagrams is the identity operation, and

(2) the dual-of-the-antidual operation on simple duals is the identity operation.

(1). We know that the antidual-of-the-dual operation on an n-loop simple diagram is an n-loop simple diagram by Appendix A. We say two n-loop simple diagrams are equal (topologically equivalent) if there exists a 1-to-1 correspondence between the loops of each diagram such that for any loop of the first diagram, the line sharing with the exterior and with other loops and the order of the sharing correspond to the line sharing of the corresponding loop of the second diagram.

We define the 1-to-1 correspondence between the loops of a given n-loop simple diagram and its image under the antidual of the dual operation as follows: A loop in the given diagram corresponds to the loop it leads to in the image diagram; that is, the loop of the given diagram leads to a vertex in the dual operation, which leads to the corresponding loop in the antidual operation. From the discussions of Appendix A it is clear that this correspondence ensures the correspondence, in content and order, of the line sharing.

(2). Proven analogously.

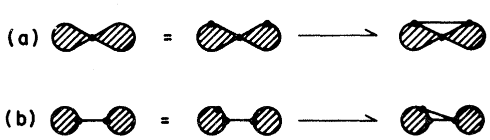

FIG. 17. Two types of boundaries not possible in a maximal simple dual, as one can add lines as indicated.

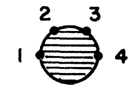

FIG. 18. A form used to show a maximal simple dual must have no more than three boundary vertices.

APPENDIX C

Here we prove prescription A generates all and only n-vertex maximal simple duals for $n>3$. The cases $n=1$, 2, and 3 are easily proven by construction from the definition.

We first show that the boundary of a maximal simple dual ($n>2$) must be topologically equivalent to the boundary of a circle. The only possible differences are (combinations of) pinches and lines [see Figs. 17(a) and (b)]. These cannot occur because one can add lines and still maintain n-vertex simple-dual properties in these cases. The equal signs in Fig. 17 come from the property that no vertex is directly connected to itself, the end points of a line are vertices, and $n>2$.

We now prove there are exactly three vertices on the boundary of a maximal simple dual when $n>2$. First we show there are no more than three, then we rule out 2, 1, and 0. Suppose there are at least four vertices on the boundary (see Fig. 18; we have used the topological equivalence to a circle proven above). If vertex 1 is directly connected to vertex 3, 2 cannot be directly connected to 4; thus we can connect them from the outside by adding a line, while maintaining the n-vertex simple-dual properties. If 1 is not directly connected to 3, we can again add a line. In either case this could not have been a maximal simple dual; this shows there can be no more than three boundary vertices. Two boundary vertices is ruled out by the topologically circular boundary ($n>2$) and no more than one direct connection between two vertices of a simple dual. One boundary vertex is ruled out by the circular boundary ($n>2$) and that no vertex is directly connected to itself. Zero is ruled out by the definition of a simple dual. Thus for $n>2$, maximal simple duals have exactly three boundary vertices.

The antidual of a simple dual with three boundary vertices is a diagram of the form of Fig. 4, where we are sure the boundary of the hatched region is continuous because of the nature of the antidual

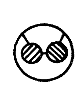

FIG. 19. A diagrammatic form used to argue that the hatched circle of Fig. 4 must be filled with a diagram whose boundary is topologically circular.

FIG. 20. The five different six-loop maximal simple diagrams.

FIG. 21. A diagram used to argue that internal lines cannot be added to a simple dual if the result is to be a simple dual.

operation. We can use the simple-diagram properties to show it must be topologically circular ($n>3$). The only possible differences are again (combinations of) pinches and lines. The pinch is ruled out by three lines per vertex. The line of the second possibility must have its end-point vertices. We must attach two of the incident lines to the two blobs (see Fig. 19), which contain at least the end-point vertices of the line, to make the diagram 1PI. Any attachment of the third line leads to a self-energy, ruling out this case.

The central circle must have all the properties of an $(n-3)$-loop simple diagram except for an academic exception: two-line vertices may be allowed on the boundary which are then converted into three-line vertices by the incident lines. The resulting maximal simple diagram could be obtained by the same central circle without the two-line vertices, because hitting a two-line vertex is equivalent to hitting no vertex, just hitting a boundary line. We have proven that the boundary of the central circle must be 1PI, and this must be true for internal lines also, if the maximal simple diagram is to be 1PI. It is clear that three lines per vertex is necessary inside and sufficient on the boundary of the central circle, and there can be no more than three boundary self-energies if we are to avoid internal self-energies, although as many as three are allowed, as we can destroy them with the incident lines. Finallv, it is clear that the central circle can have no internal self-energies and must be planar (and it must be a diagram). If the rules for attaching the incident lines as given in prescription A are followed, the central circle filled with an $(n-3)$-loop simple diagram will be a simple diagram. Also, filling the central circle with each simple diagram in all topological ways according to prescription A will generate all simple diagrams with three boundary loops ($n>3$), because we have seen that, except for an academic qualification, the central circle must be filled with a simple diagram if the whole diagram is to be simple. The duals of simple diagrams with three boundary loops must be maximal simple duals because Appendix A tells us they are simple duals (and thus by lemma 1 no internal line can be added if the result is to be a simple dual) and they have three boundary vertices on a topologically circular boundary (three boundary vertices on a topologically circular boundary implies no boundary; one may be added if the result is to be a simple dual). By Appendix B, the duals of all of the n-loop maximal simple diagrams are all of the n-vertex maximal simple duals (they are only maximal simple duals by definition), and we have seen that prescription A generates all and only maximal simple diagrams; thus prescription A generates all and only n-vertex maximal simple duals.

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[1]High-energy fermion-fermion scattering in SU_N gauge theory has been studied by L. Tyburski, Phys. Rev. D 13, 1107 (1976), and references therein.

[2]The large-momentum behavior of the elastic form factor in non-Abelian gauge theory has been studied by J. M. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976), and references therein.

[3]G. 't Hooft, Nucl. Phys. B72, 461 (1974); G. P. Canning, Phys. Rev. D 12, 2505 (1975).

[4]See Ref. 3. The "attachment of surfaces" mentioned in the first paper of Ref. 3 does not generally lead to a closed Eulerian surface. The second paper makes some progress by applying the commutation relations to untwist nonplanar diagrams. However, it is not proven that the commutation relations are sufficient to untwist *every* diagram to a sum of diagrams to which the Euler analysis applies. Notwithstanding, in the case of the scalar form factor of a quark which motivates us here, explicit low-order calculations show 't Hooft's conclusion to be correct for both U_N and SU_N [R. Cahalan and Dan Knight (unpublished)]. For the *vector* form factor of a quark, nonplanar graphs contribute to the leading power of N, but since the one-loop graph is $O(1/N)$, an exponential form (Ref. 2) would imply that the leading power cancels. For general graphical methods for the Lie algebras see P. Cvitanovic, Phys. Rev. D 14, 1536 (1976).

[5]Although the paper is completely self-contained, we list graphical references 6, 7, and 8 for the convenience of the reader.

[6]An excellent introduction to graph theory is given in

the book by O. Ore, *The Four Color Problem* (Academic, New York, 1967). Graphs similar to the ones constructed here are discussed in Section 9.6.

[7]An elegant and compact introduction to graph theory is given by J. W. Essam and M. E. Fisher, Rev. Mod. Phys. 42, 271 (1970). Note that our dual differs from the conventional one in that ours has no vertex corresponding to the exterior of the given diagram. When this vertex is included in the dual operation, the dual becomes its own inverse. See Theorem 7.7 of this reference.

[8]A nice physical application is given by M. J. Levine and R. Roskies, Phys. Rev. D 9, 421 (1974), who suggest that any graph in QED with exactly one nonzero external momentum (or all momenta zero) may be integrated analytically if its dual is a tree.

[9]See Table I for a dictionary of the terms used.

[10]Suppose there exists a vertex in a simple dual which is directly connected to itself. Then, if the shaded area of Fig. 1(b) is empty, there would be a loop with one vertex in the simple dual. If it is not empty, there is an interior which is not a loop interior in the simple dual. In either case, we have a contradiction. For simple diagrams, one uses the same figure and the properties of three lines per vertex and 1PI to disprove the presence of such vertices.

[11]Roughly, we are blowing up the simple dual like a two-dimensional balloon.

[12]After a line is removed, some interior lines may become boundary lines[9] and thus become candidates for the removal process.

[13]$n=6$ is the first case of more than one maximal simple dual. It is also the first case in which some of the simple diagrams are not reflection symmetric. We show the six-loop maximal simple diagrams in Fig. 20; the last two are reflections of each other.

[14]It is impossible to add a line and maintain the n-vertex simple-dual properties if one of the vertices to which the line is attached is an internal vertex. This is because an internal vertex must be at the center of a wheel (as we show in the next paragraph). The other vertex of attachment cannot be on the boundary of the wheel, as we would have two direct connections between the two vertices (see Fig. 21). Since both ends of a line cannot be attached to a single vertex, the only remaining alternative is a vertex outside the wheel, but this would lead to a nonplanar diagram.

To see that an internal vertex of a simple dual must be the center of a wheel, consider the lines emanating from the internal vertex. There must be at least three lines, as

(i) zero lines is ruled out by connectedness,

(ii) one is not possible for internal vertices of a planar diagram if all interiors are to be loops, and

(iii) two is not possible for internal vertices of a planar diagram if all interiors are to be loops with three vertices, and there is to be no more than one direct connection between any two vertices.

The outer end points of any two adjacent lines emanating from the internal vertex must be directly connected if the vertex is to be internal and all interiors are to be loops with three vertices. Thus an internal vertex of a simple dual is at the center of a wheel.

[15]P. Cvitanovic and T. Kinoshita, Phys. Rev. D 10, 3978 (1975); 10, 3991 (1975).